Assignment #3 & sol^ns for #2 now online

## Notes on last lecture

$(X_n)_{n\geq 0}$ res^n of $\mathbb{Z}$ in $\mathbb{Z}G$-Mod.

nasty formula for $d: X_{n+1} \to X_n$.

$\tilde{X}_n$ = free $\mathbb{Z}$-mod on symbols $\langle g_0, \dots, g_n \rangle$, $g_i \in G$

$g \cdot \langle \dots \rangle = \langle g g_0, \dots, g g_n \rangle$.

$d: \tilde{X}_{n+1} \to \tilde{X}_n$:

$\langle g_0 \dots g_{n+1} \rangle \mapsto \sum_i (-1)^i \langle g_0, \dots, \hat{g}_i, \dots g_{n+1} \rangle$

In fact $\tilde{X}_n \cong X_n$ (so in particular $\tilde{X}_n$ is $\mathbb{Z}G$-free)

$(g_1, \dots g_n)$ basis vect of $X_n$

$\mapsto \langle 1, g_1, g_1 g_2, \dots, g_1 \cdots g_n \rangle \in \tilde{X}_n$.

· A correction: I claimed last time that

$$C^i(G, M) := \mathrm{Hom}_{\mathbb{Z}G}(X_i, M)$$

were a resolution of M acyclic for the G-inv^ts functor.

What I should have said:

$$V^i(G, M) = \mathrm{Hom}_{\mathbb{Z}}(X_i, M)$$

$$(g \cdot \phi)(x) = g \phi(g^{-1} x)$$

& $C^i(G, M)$ is G-inv^ts of $V^i$

so $H^i(C^\cdot(G, M))$ computes $H^i(G, M)$

---

## Spectral Sequences

"Tool for combining multiple sources of cohomology."

Eg $G$ group $H \triangleleft G$.

$H^i(H, M)$, for $M \in \mathbb{Z}G$-Mod, naturally a $G/H$ module.

(Either argue that any $[g] \in G/H$ gives a nat^l transf^n from $(-)^H$ to itself + hence propagates to derived functors; or observe that $(-)^H: \mathbb{Z}G\text{-Mod} \to \mathbb{Z}[G/H]\text{-Mod}$ is well-def. + so has derived functors.)

Can we recover $H^i(G,M)$ from $G/H \curvearrowright H^j(H,M)$?

## Toy example

$G = \mathbb{Z}^2 = \langle g, h \rangle$

$H = \langle h \rangle \cong \mathbb{Z}$

We know a proj res$^n$ of $\mathbb{Z}$ as $\mathbb{Z}G$-module:

$$0 \to \mathbb{Z}G \xrightarrow{\begin{pmatrix} g-1 \\ -(h-1) \end{pmatrix}} \mathbb{Z}G \oplus \mathbb{Z}G \xrightarrow{(h-1,\ g-1)} \mathbb{Z}G \to 0$$

So $H^0(G,M) = M^{g=1,h=1}$

$$H^1(G,M) = \frac{\{(m_1,m_2) \in M^2 : (h-1)m_1 = (g-1)m_2\}}{\{((g-1)m, (h-1)m) : m \in M\}}$$

$$H^2(G,M) = M/_{(g-1)M+(h-1)M}$$

H-cohomology: $H^0(H,M) = M^{h=1}$

$H^1(H,M) = M/_{(h-1)M}$

$$H^0(G/H, H^0(H,M)) = H^0(G,M)$$

(unsurprising!)

$$H^1(G/H, H^1(H,M)) = H^2(G,M)$$

What about $H^1(G,M)$?

$m \longmapsto (m,0)$

gives a map

$$M^{h=1}/_{(g-1)M^{h=1}} \hookrightarrow H^1(G,M)$$

$(m_1, m_2) \longmapsto m_2 \mod (h-1)M$

gives a map $H^1(G,M) \twoheadrightarrow (M/_{(h-1)M})^{g=1}$

$\rightsquigarrow$ SES

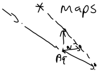
$$0 \to H^1(G/H, H^0(H,M)) \to H^1(G,M) \to H^0(G/H, H^1(H,M)) \to 0.$$

Gen$^l$ picture: $H^n(G,M)$ "built up from" $H^i(G/H, H^j(H,M))$, $i+j=n$.

**Def$^n$** Let $r_0 \in \mathbb{Z}_{\geq 0}$, $\mathcal{C}$ ab. cat.

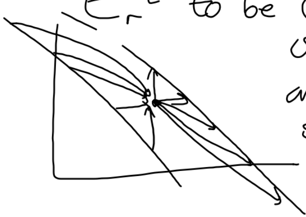
A <u>first quadrant cohomological spectral sequence</u> in $\mathcal{C}$ <u>starting at the $r_0$ sheet</u> consists of the following data:

- for each $r \geq r_0$,
  - a collection of objects $E_r^{pq}$, $p,q \geq 0$, of $\mathcal{C}$
  - maps $d_r^{pq} : E_r^{pq} \to E_r^{p+r,\, q-r+1}$ st $d_r^2 = 0$



- for each $r \geq r_0$, isomorphisms between $E_{r+1}^{p,q}$ + cohomology at $(p,q)$ spot of $E_r^{pq}$ wrt maps $d_r$.

If $p < 0$ or $q < 0$, we understand $E_r^{pq}$ to be $0$. Thus incoming + outgoing arrows at $(p,q)$ are both $0$ for $r >> 0$ so $E_r^{pq} = E_{r+1}^{pq} = E_{r+2}^{pq} = \cdots$ $=: E_\infty^{pq}$.



Def$^n$ We say $(E_r^{pq})$ converges to some collection of objects $(L^n)_{n \geq 0}$ if for every $n \geq 0$, $\exists$ filtration

$$0 = F^{n+1}L^n \subseteq F^n L^n \subseteq F^{n-1} L^n \subseteq \cdots \subseteq F^0 L^n = L^n$$

st $E_\infty^{pq} = \dfrac{F^p L^{p+q}}{F^{p+1} L^{p+q}}$ $\forall p,q$.

Thm Let $H \trianglelefteq G$ groups, $M$ a $G$-module.
Then $\exists$ sp. seq. starting at $E_2$

$$E_2^{pq} = H^p(G/H, H^q(H,M))$$

converging to $H^n(G,M)_{n\geq 0}$.

[Notation: "$E_2^{pq} = H^p \ldots \Rightarrow H^{p+q}(G,M)$"]

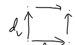
"Hochschild–Serre spectral seq."

If you haven't seen this before, write out $E_2$ terms for $G = \mathbb{Z}+\mathbb{Z}$, $H = \mathbb{Z}$ + compare with example.

---

The sp. seq. of a double complex

Suppose we have $X^{pq}$, $p,q \geq 0$
+ differentials $d_h : X^{pq} \to X^{p+1,q}$
$d_v : X^{pq} \to X^{p,q+1}$



$d_v^2 = 0$, $d_h^2 = 0$, $d_v d_h + d_h d_v = 0$.

(This is close to an obj of $Ch^\circ(Ch^\circ(\mathcal{C}))$, modulo different signs.)

$T = \mathrm{Tot}(X^{\cdot\cdot})$

$$T^n = \bigoplus_{p+q=n} X^{pq}, \quad d = d_h + d_v.$$

$X^{\cdot\cdot}$ is a double complex, and $T$ its total (single) complex.

Will see that $\exists$ sp. seq $E_0^{pq} = X^{pq}$ converging to $H^*(T^\cdot)$.

Imagine we have $x \in X^{n0}$ ($\mathcal{C} = Ab$ or $R$-Mod)

How does $x$ contribute to coho. of $T$?

$x \rightsquigarrow \tilde{x} = (x, 0, \ldots, 0) \in T^n$

$d(\tilde{x}) = (d_h x, d_v x, 0 \ldots 0) \in T^{n+1}$

So $\tilde{x} \in Z^n(T)$ iff $d_h x = d_v x = 0$.

How might it happen that $\tilde{x}$ is a coboundary?

- "level 0": if $x = 0$ then $\tilde{x}$ is a coboundary (duh)
- $\tilde{x} \overset{?}{=} d((y, 0, 0 \ldots))$, $y \in X^{n-1,0}$

This happens if $\begin{cases} x = d_h y \\ d_v y = 0 \end{cases}$

ie if $x$ is $0$ in $\dfrac{\ker(d_v^{n,0})}{d_h(\ker d_v^{n-1,0})}$

"level 1" condition.

What if $\tilde{x} = d(\,(y_0, y_1, 0 \cdots)\,)$?

$d_v(y_1) = 0$

$d_v(y_0) + d_h(y_1) = 0$

$d_h(y_0) = x$.

Then $y_1 \in \ker(d_v^{n-2,1}) \cap \{d_h^{n-2,1} \in \operatorname{Im}(d_v)\}$ (*)

If $y_1 \in$ this, choose $y_0$ st $d_v y_0 = -d_h y_1$

+ consider image of $d_h y_0$ in $X^{n,0}$

⇝ well-def as an elt of

(**) $\dfrac{\ker d_v^{n0}}{d_h(\ker d_v^{n-1,0})}$ So get a map from (*) to (**)

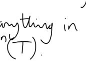
+ anything in image goes to 0 in $H^n(T)$.

map from a sub of $X^{n-2,1}$ to a sub-quotient of $X^{n,0}$.

Repeat: get a map from some subquot of $X^{n-3,2}$ to cokernel of the last map, .

+ these are exactly the incoming differentials at $(n,0)$ spot;

eventually get image of $X^{n,0}$ in $H^n(T)$.




Upshot Whenever you have a double complex $(X^{pq})$, get $E_r^{pq} \Rightarrow H^{p+q}(T)$

$E_0^{pq} = X^{pq}$ w. $d_v$

$E_1^{pq} = H_v^q(X^{p\bullet})$ w. maps induced by $d_h$

$E_2^{pq} = H_h^p H_v^q(X^{\bullet\bullet})$.

Can also flip $p, q$ + get a sp. seq.

${'E}_2^{pq} = H_v^p H_h^q(X^{\bullet\bullet})$

with same limit $H^*(T)$

(generally w. different filt$^n$s.)

## Edge maps

Let $(E_r^{pq})$ any 1st quad sp. seq.

$E_r^{p,0}$ spot: all outgoing differentials are 0 after $E_1$

+ similarly for incoming differentials at $(0,q)$

So get maps

$$E_r^{p0} \to E_{r+1}^{p0} \to \cdots \to E_\infty^{p,0}$$

$$E_\infty^{p,0} \hookrightarrow L^p$$

"edge map": $E_r^{p0} \to L^p$

+ similarly

$$E_r^{0q} \hookleftarrow E_{r+1}^{0q} \hookleftarrow E_{r+1}^{0q} \cdots \hookleftarrow E_\infty^{0q}$$

$$\uparrow$$

$E_r^{0q} \leftarrow L^q$ edge map.

Eg in double complex setting edge map $E_2^{p0} = H_h^p H_v^0(X^{\cdot\cdot}) \to H^p(T^\cdot)$ is given by inclusion of $(\ker d_v^{p0})_{p\geq 0}$ as a subcomplex of $T$

+ other edge map $H_h^0(H_v^q(X^{\cdot\cdot})) \leftarrow H^q(T^\cdot)$ comes from projecting $T^n$ onto $(0,n)$ degree part.

More interestingly, we'll see later that in HS sp seq

$$H^p(G/H, H^0(H,M)) \longrightarrow H^p(G,M)$$

$$H^q(G,M) \longrightarrow H^0(G/H, H^q(H,M))$$

given by inflating cocycles on $G/H$ to $G$, + restricting to $H$, respectively.

(Can make $d_2^{01}: H^0(G/H, H^1(H,M)) \to H^2(G/H, H^0(H,M))$ explicit in terms of the $C^i(G,M)$ but it's painful!)

---

## A cute example from algebraic topology

$X$ top. space (Hausdorff, locally contractible, paracompact)

$G \circlearrowright X$ acting properly

(every $g \in G$ acts via a homeomorphism, every $x \in X$ has a neighbourhood not intersecting any of its $G$-translates)

$X/G$ Hausdorff.

$H^i(X, \mathbb{Z})$ are $\mathbb{Z}G$-mods.

<u>Thm</u> $\exists$ sp. seq. $E_2^{pq} = H^p(G, H^q(X,\mathbb{Z})) \Rightarrow H^{p+q}(X/G, \mathbb{Z})$

Eg if $X$ is contractible, $H^q(X,\mathbb{Z}) = 0 \ \forall q \geq 1$, $H^0(X,\mathbb{Z})$ is trivial $G$-mod.

$$\rightsquigarrow H^n(X/G, \mathbb{Z}) = H^n(G, \mathbb{Z})$$