Last wk: spectral seqs of a double complex.

$X^{\cdot\cdot}$

$$\boxed{\begin{aligned} E_0^{pq} &= X^{pq} \\ E_1^{pq} &= H_v^q(X^{p\cdot}) \\ E_2^{pq} &= H_h^p(H_v^q(X^{\cdot\cdot})) \end{aligned}}$$

Let $\mathcal{C}$ ab. cat, $X^\cdot$ cochain cplx. $X^i = 0$ $i < 0$

<u>Def^n</u> A <u>Cartan–Eilenberg res^n</u> of $X^\cdot$ is a double cplx $(J^{pq})_{p,q\geq 0}$ with a cochain map $\varepsilon : X^\cdot \longrightarrow J^{\cdot 0}$

Such that:

- All J^{pq} are inj objects of $\mathcal{C}$ and each column $J^{p\cdot}$ is an inj res^n of X^p
- Each row $J^{\cdot q}$ is <u>split</u> in the sense that

$$J^{pq} \cong B^p(J^{\cdot q}) \oplus B^{p+1}(J^{\cdot q}) \oplus H^p(J^{\cdot q})$$

with differential $d_h = \begin{pmatrix} 0 & id & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

- for each p, $B_h^{p\cdot}(J^{\cdot\cdot})$, $Z_h^{p\cdot}(J^{\cdot\cdot})$, $H_h^{p}(J^{\cdot\cdot})$ are inj res^{n}s of $B^p(X^\cdot)$, $Z^p(X^\cdot)$, $H^p(X^\cdot)$ resp.

<u>Fact</u> If $\mathcal{C}$ has enough injectives, every $X^\cdot$ bounded-below complex has a CE res^n, + these are "functorial up to homotopy" in a suitable sense.

<u>Idea of pf</u> For each p, take (usual) res^{n}s of $B^p(X^\cdot)$ and $H^p(X^\cdot)$

Horseshoe Lemma $\Rightarrow$ $\exists$ inj res^n of $Z^p(X^\cdot)$ coming from SES $0 \to B^p \to Z^p \to H^p \to 0$

Repeat again using $0 \to Z^p X \to X^p \to B^{p+1} X \to 0$. "$\square$"

__Def^n__ Let $F : \mathcal{C} \to \mathcal{D}$ left-exact.

Assume $\mathcal{C}$ has enough injs, + choose a CE res^n for every bounded-below cplx in $Ch(\mathcal{C})$.

Set
$$\mathbb{R}^p(F)(X^\cdot)$$
$$= H^p(\mathrm{Tot}(F(J_X^{\cdot\cdot}))),$$
$J_X^{\cdot\cdot}$ CE res^n of $X^\cdot$.

Note if $X^\cdot = [Y]$ (Y in degree 0)
then $\mathbb{R}^p(F)(X^\cdot) = R^p(F)(Y)$.

From last lecture, $\exists$ __two__ sp seqs cvgt to $\mathbb{R}^{p+q}(F)(X^\cdot)$

① $E_0^{pq} = F(J^{pq})$

$E_1^{pq} = H_v^q(F(J^{p\cdot}))$

F of an inj res^n of X^p

$= R^q(F)(X^p)$

$E_2^{pq} = H^p((\mathbb{R}^q F)(X^\cdot))$.

② $E_0'^{pq} = F(J^{qp})$

$E_1'^{pq} = H^q(F(J^{\cdot p}))$

__Splitness__ $\Rightarrow$ this $= F(H^q(J^{\cdot p}))$

$= F(p^{th}$ term of an inj res^n of $H^q(X^\cdot))$

$E_2'^{pq} = H^p(\text{this})$

$= R^p(F)(H^q(X^\cdot))$.

__Thm__ In the above setting, $\exists$ functors
$$\mathbb{R}^p(F) : Ch^\cdot_{\geq 0}(\mathcal{C}) \longrightarrow \mathcal{D}$$
+ for every $X^\cdot$, two sp. seqs
$$E_2^{pq} = H_h^p((\mathbb{R}^q F)(X^\cdot))$$
$$E_2'^{pq} = R^p(F)(H^q(X^\cdot))$$
converging to $\mathbb{R}^{p+q}(F)(X^\cdot)$, naturally in $X^\cdot$.

__Example 1__ M compact complex manifold.

M is a top. space $\rightsquigarrow$ cohomology gps $H^i(M, \mathbb{C})$ — cohomology of constant sheaf $\underline{\mathbb{C}}$ on M.

Def^n: for $p \geq 0$, sheaf of holomorphic p-forms Ω^p_{hol} : sections over suffi. small opens look like

$$\sum_{0 < i_1 < i_2 < \cdots < i_p \leq \dim M} f_{i_1 \cdots i_p}(z_1 \cdots z_{\dim M}) \, dz_{i_1} \wedge \cdots \wedge dz_{i_p}$$

$z_1 \cdots z_{\dim M}$ coordinates.

Have maps $d : \Omega^p_{hol} \to \Omega^{p+1}_{hol} \to \cdots$ giving a complex of sheaves on M.

$$0 \to \underline{\mathbb{C}} \to \Omega^0 \to \Omega^1 \cdots$$

is exact ("holomorphic Poincaré lemma")

$\Omega^{\cdot}_{hol}$ is called the de Rham complex.

What is $\mathbb{R}^n \underbrace{(\Gamma)}_{\text{global sections functor.}} (\Omega^{\cdot}_{hol})$?

$$E_2^{pq} = H^p(R^q(\Gamma)(\Omega^{\cdot}_{hol}))$$

$$E_2^{pq} = R^p(\Gamma)(\underbrace{H^q \Omega^{\cdot}_{hol}}_{\text{zero if } q>0,\ \underline{\mathbb{C}} \text{ if } q=0})$$

E_2^{pq} has only one row

$$\Rightarrow \mathbb{R}^n(\Gamma)(\Omega^{\cdot}) = H^n(M, \underline{\mathbb{C}})$$

E_2^{pq} sp. seq must therefore converge to this

$$E_1^{pq}: H^p(M, \Omega^q_{hol}) \Rightarrow H^{p+q}(M, \underline{\mathbb{C}}).$$

(Hodge spectral sequence).

(Hard fact: differentials on E_2 page all zero.)

Example 2

Suppose $X^\cdot$ is a complex of F-acyclic objs. $(R^q(F)(X^p) = 0 \ \forall q \geq 1)$.

$$E_2^{pq} = H^p(\underbrace{(R^qF)(X^\cdot)}_{\text{zero for } q \geq 1})$$

Hence $\mathbb{R}^p(F)(X^\cdot) = H^p(F(X^\cdot))$.

+ we have a sp. seq.

$$E_2^{\prime pq} = R^p(F)(H^q X^\cdot) \Rightarrow H^p(F(X^\cdot)).$$

Example 2A

Universal coefficient formula for (co)homology in topology. X top. space

$C_\cdot(X, \mathbb{Z})$ complex of simplices in X

Let $Y^\cdot$ = this as a cochain complex in Ab^{op}.

$$F : Ab^{op} \longrightarrow Ab$$
$$G \longmapsto Hom_{Ab}(G, \mathbb{Z})$$

covariant + left exact.

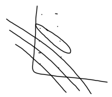
$Y^\cdot$ has injective, hence F-acyclic, terms.

$F(Y^\cdot)$ is the cochain complex $C^\cdot(X, \mathbb{Z})$.

$$H^p(Y^\cdot) = H_p(C_\cdot(X, \mathbb{Z})) = H_p(X, \mathbb{Z}).$$

$$E_2^{pq} = R^p(F)(H^q(Y^\cdot)) \Rightarrow H^{p+q}(F(Y^\cdot))$$

$$\underbrace{Ext^p_{\mathbb{Z}}(H_q(X, \mathbb{Z}), \mathbb{Z})}_{\text{zero for } p > 1, \text{ so get SESs}} \Rightarrow H^{p+q}(X, \mathbb{Z}).$$



$$0 \to Ext^1(H_{p-1}, \mathbb{Z}) \to H^p(X, \mathbb{Z}) \longrightarrow Hom(H_p, \mathbb{Z}) \longrightarrow 0.$$

(Slogan: Cohomology is "derived dual" of homology).

Example 2b

$$B \xrightarrow{G} C \xrightarrow{F} D$$

F, G both left-exact. B, C enough injs

$X \in B \rightsquigarrow I^\cdot$ inj res^n

$G(I^\cdot)$. What is $\mathbb{R}^n(F)(G(I^\cdot))$?

Assume For I inj obj of $\mathcal{B}$, $G(I)$ is F-acyclic. $X \in \mathcal{B}$, I inj res.

Then $\mathbb{R}^p(F)(G(I))$
$$= H^p((F \circ G)(I))$$
$$= R^p(F \circ G)(X).$$

Thm In above setting, $\exists$ sp seq
$$E_2^{pq} = R^p(F)(R^q(G)(X)) \Rightarrow R^{p+q}(F \circ G)(X).$$
("Grothendieck spectral sequence")

Pf of Hochschild–Serre sp. seq.

$H \lhd G$

STP that the functors
$$\mathbb{Z}G\text{-Mod} \xrightarrow[(-)^H]{} \mathbb{Z}(G/H)\text{-mod} \xrightarrow[(-)^{G/H}]{} Ab$$
satisfy hypotheses of Grothendieck sp seq.

In fact, if I is inj as a G-mod, I^H is even inj as a G/H mod, since
$$\mathrm{Hom}_{G/H}(A, I^H) = \mathrm{Hom}_G(A, I)$$
define G-action via $G \twoheadrightarrow G/H$.

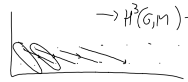
is exact in A. $\square$

Application ("Inflation–restriction seq")

For M a $\mathbb{Z}G$-mod, $\exists$ exact seq
$$0 \to H^1(G/H, M^H) \xrightarrow{inf} H^1(G, M)$$
$$\xrightarrow{res} H^1(H, M)^{G/H} \xrightarrow{tg} H^2(G/H, M^H)$$
$$\xrightarrow{inf} H^2(G, M).$$

Exercise: Suppose $H^j(H, M) = 0 \ \forall j > 1$

Then get LES $\to H^2(G, M)$
$$\to H^1(G/H, H^1(H, M)) \to H^3(G/H, M^H)$$
$$\to H^3(G, M) \to \cdots$$



Another cute application: derived functors of $\varprojlim$

$$\mathrm{Pro}(R\text{-Mod}) \longrightarrow R\text{-Mod}$$
$$(M_i)_{i \in \mathbb{N}} \longmapsto \varprojlim M_i$$

⚠ Left-exact but not right.

<u>Thm (Eilenberg)</u> (a) Derived functors $\varprojlim^{(p)}$ exist, & $\varprojlim^{(p)}$ is 0 for $p>1$.

(b) $\varprojlim^{(1)}$ vanishes on seqs satisfying the <u>Mittag-Leffler condⁿ</u>:

image $(M_{i+N} \to M_i)$ stabilizes as $N \to \infty$ for all i.

Classic example:

$$M_n = p^n \mathbb{Z} \qquad p \text{ (some prime)}$$

$\varprojlim M_n$ is 0.

$$0 \to (p^n\mathbb{Z})_{n\geq 1} \to (\mathbb{Z})_{n \geq 1} \to (\mathbb{Z}/p^n\mathbb{Z})_{n\geq 1} \to 0$$

$\varprojlim$: $\quad 0 \to \mathbb{Z} \to \mathbb{Z}_p \to$

$R^1\varprojlim$: $\quad \mathbb{Z}_p/\mathbb{Z} \to 0 \to 0 \to 0$

Conclude: $\varprojlim^1 (M_n) = \mathbb{Z}_p/\mathbb{Z}$.

G group

$$\begin{tikzcd}
\mathrm{Pro}(\mathbb{Z}G\text{-Mod}) \arrow[r, "\varprojlim"] \arrow[d, "(-)^G"'] & \mathbb{Z}G\text{-Mod} \arrow[d, "(-)^G"] \\
\mathrm{Pro}(Ab) \arrow[r, "\varprojlim"] & Ab
\end{tikzcd}$$

(commutes – easy check).

Considering Groth sp. seq. for $\underbrace{\varprojlim \circ (-)^G}_{F\text{, say}}$

$$\left.\begin{array}{l} H^p(G, \varprojlim^{(q)} M_n) \\ \varprojlim^{(p)} H^q(G, M_n) \end{array}\right\} \text{both converge to } R^{p+q}(F).$$

Assume M_n are Mittag-Leffler. then 1st collapses (but 2nd may not).

Since $\varprojlim^{(2)} = 0$ get

$$0 \to \varprojlim^{(1)} H^{q-1}(G, M_n) \to H^q(G, \varprojlim M_n)$$
$$\to \varprojlim H^q(G, M_n) \to 0.$$

If $H^{q-1}(G, M_n)$ is finite $\forall n$, then it must be Mittag-Leffler, so $\varprojlim^{(1)}$ vanishes + conclude that H^q commutes w. inverse limits.