This is the main reason why we care about derived functors: $R^iF(x)$ measures how much F fails to be exact, and $R^{i+1}F(x)$ measures how much $R^iF(x)$ fails to be exact.

Proof: use horseshoe lemma to assume that $I_Y = I_X \oplus I_Z$ without loss of generality. Then

$F(I_Y) = F(I_X) \oplus F(I_Z)$ so $0 \to F(I_X) \to F(I_Y) \to F(I_Z) \to 0$ is exact in $Ch(\mathcal{D})$

because split short exact sequences are respected by additive functors (regardless of exactness). The thesis follows applying the long exact sequence for cohomology.

$\square$

A similar picture works for L_iF:

$$\to L_2F(X) \to L_2F(Y) \to L_2F(Z) \to$$
$$\to L_1F(X) \to L_1F(Y) \to L_1F(Z) \to FX \to FY \to FZ \to 0$$

Tohoku viewpoint (Grothendieck 1957): for $\mathcal{C}, \mathcal{D}$ as above a δ-functor $\mathcal{C} \to \mathcal{D}$ is a sequence of functors

$$T^i : \mathcal{C} \to \mathcal{D}, \qquad i \geq 0$$

with natural transformations $T^iZ \to T^{i+1}X$ for s.e.s. $0 \to X \to Y \to Z \to 0$ such that

$$0 \to T^0X \to T^0Y \to T^0Z \to T^1X \to T^1Y \to T^1Z \to \dots \quad \text{is exact.}$$

Is there a unique way to extend a given left-exact T^0 to a δ-functor? No. Is there a universal one? Yes.
If (T^i) is a δ-functor, $\exists!$ natural transformation

$$R^i(T^0) \to T^i \qquad \text{(if } \mathcal{C} \text{ has enough injectives)}$$

Some examples of derived functors

(i) Let $\mathcal{C}$ be any abelian category with enough injectives. $A \in Ob(\mathcal{C})$

$$Hom(A, \cdot) : \mathcal{C} \to Ab \quad \text{is a left exact functor}$$

We set $Ext^i_{\mathcal{C}}(A,B) = R^i(Hom_{\mathcal{C}}(A,\cdot))(B)$ this is a functor $\mathcal{C}^{op} \times \mathcal{C} \to Ab$
If $A_1 \xrightarrow{f} A_2$ we get a natural transformation

$$Hom(A_2, \cdot) \to Hom(A_1, \cdot)$$

and then a natural transformation between right derived functors $Ext^i(A_2, B) \to Ext^i(A_1, B)$.

General fact: a natural transformation $F_1 \to F_2$ between left exact functors yields a natural transformation

$$R^i(F_1) \to R^i(F_2) \qquad \forall i$$

Prop: TFAE for $A \in Ob(\mathcal{C})$

(i) A is projective
(ii) $Ext^i(A, \cdot)$ is the zero functor $\forall i > 0$
(iii) $Ext^1(A, \cdot)$ is the zero functor

(Ext detects projectives)

General statement: TFAE (i) F is exact
(ii) $R^i(F) = 0 \quad \forall i > 0$
(iii) $R^1(F) = 0$

Proof: $i) \Rightarrow ii) \to iii)$ are clear

$iii) \Rightarrow i)$ If this holds, then for every $0 \to A \to B \to C \to 0$ we have the following exact sequence:

$$0 \to F(A) \to F(B) \to F(C) \to \underset{0}{\underset{\|}{R^1F(A)}} \to \underset{0}{\underset{\|}{R^1F(B)}} \to \dots \qquad \Rightarrow F \text{ is exact}$$

□

For example let $\mathcal{C} = Ab$, $n \geq 2$. What is $Ext^i(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z})$?

We have $Ext^0(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) = Hom(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) = 0$. The sequence $\mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ is an injective resolution of $\mathbb{Z}$.

$$Hom(\mathbb{Z}/n\mathbb{Z}, \mathbb{Q}) = 0$$

$$Hom(\mathbb{Z}/n\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) = ? \qquad 1 \in \mathbb{Z}/n\mathbb{Z} \text{ must go to } \tfrac{a}{n} \text{ for } a \in \mathbb{Z} \Rightarrow Hom(\mathbb{Z}/n\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) \simeq \mathbb{Z}/n\mathbb{Z}$$

we get $0 \to \mathbb{Z}/n\mathbb{Z} \to 0$ then $Ext^1(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) = \mathbb{Z}/n\mathbb{Z}$ $\quad Ext^i(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) = 0 \ \forall i \neq 1$

We can consider $Hom(\cdot, B)$ as $\mathcal{C}^{op} \to Ab$. If $\mathcal{C}^{op}$ has enough injectives we can derive this

Prop: if $\mathcal{C}$ has both enough injectives and projectives, then

$$Ext^i(A,B) = R^i(Hom(\cdot, B))(A)$$

Equivalently: we can compute Ext using either an injective resolution of B or a projective resolution of A.
For example in the above example we can use the projective resolution of $\mathbb{Z}/n\mathbb{Z}$ given by $\mathbb{Z} \xrightarrow{n} \mathbb{Z}$. This result is called "balancing of Ext"

Sketch of proof: choose $P_\cdot \to A$ projective resolution
$B \to I^\cdot$ injective resolution

We want to show $H^i(Hom(P_\cdot, B)) \simeq H^i(Hom(A, I^\cdot))$

Consider the group $X^{pq} = Hom(P_p, I^q)$ (double complex) and $T^i = \bigoplus_{p+q=i} X^{pq}$ (total complex)

Fact: $T^\cdot$ is a cochain complex and the maps $P_0 \to A$, $B \to I^0$ give quasi-isomorphisms

$$Hom(A, I^\cdot) \xrightarrow{\sim} T^\cdot \xleftarrow{\sim} Hom(P_\cdot, B)$$

by composition $\quad P_0 \to A \to I^i \Rightarrow$ element of $X^{0i} \subseteq T^i$ $\qquad$ composing $\quad P_i \to B \to I^0 \Rightarrow$ element of $X^{i0} \subseteq T^i$

□

Corollary: B is injective $\iff Ext^i(A,B)$ vanishes $\forall A \ \forall i \geq 1$
$\iff Ext^1(A,B)$ vanishes $\forall A$

We can use Ext to check both injectives and projectives.

12: Group cohomology

Let G be a group, the category of $\mathbb{Z}[G]$-Mod is abelian. Consider the functor

$$(\cdot)^G : \mathbb{Z}[G]\text{-Mod} \to Ab$$

which is naturally isomorphic to $Hom_{\mathbb{Z}[G]}(\mathbb{Z}, \cdot)$

Def: $H^i(G,M) = R^i((\cdot)^G)(M) = Ext^i_{\mathbb{Z}[G]}(\mathbb{Z}, M)$

We can use 2 approaches to computing this:
- injective resolutions of M $\quad \forall M$
- projective resolution of $\mathbb{Z}$

It is clear that computing a projective resolution of $\mathbb{Z}$ is considerably less work and will allow us to make Ext explicit in this case.

$$\mathbb{Z}[G] \xrightarrow{g-1} \mathbb{Z}[G] \quad \text{is a projective resolution of } \mathbb{Z} \text{ in } \mathbb{Z}[G]\text{-Mod}$$

So for any M in $\mathbb{Z}[G]$-Mod $\qquad H^i(G,M)$: i-th cohomology of Hom(resolution, M)

We have

$$\text{Hom(resolution}, M) \implies M \xrightarrow{(g-1)} M \quad \text{so} \quad H^0(G,M) = M^{g=1}$$

$$H^1(G,M) = M/_{(g-1)M}$$

For any G, there exists a systematic way to build a projective resolution of $\mathbb{Z}$ in $\mathbb{Z}[G]$-Mod ("bar resolution")

Def: let X_n = free $\mathbb{Z}[G]$-module on set of symbols $(g_1, \dots, g_n)$ where $g_i \in G$.
$X_0 = \mathbb{Z}[G]$

$d: X_n \to X_{n-1}$ defined as $\sum_{i=0}^{n} (-1)^i d^i$

$$d^0(g_1, \dots, g_n) = [g_1](g_2, \dots, g_n)$$
$$d^i(g_1, \dots, g_n) = (g_1, \dots, g_i g_{i+1}, g_{i+2}, \dots, g_n)$$
$$d^n(g_1, \dots, g_n) = (g_1, \dots, g_{n-1})$$

For example if $X_0 = \mathbb{Z}[G]$
$X_1 = \bigoplus_{g \in G} \mathbb{Z}[G](g)$

$$d: X_1 \to X_0$$
$$(g) \mapsto g-1$$

Fact: X is a projective (indeed free) resolution of $\mathbb{Z}$ in $\mathbb{Z}[G]$-Mod.

Hence
$$H^i(G,M) = H^i(\text{Hom}(X_\bullet, M))$$

$\text{Hom}(X_i, M)$ is called the group of ~~iths~~ i-cochains of G with values in M, which equals the M-valued functions on $G \times \dots \times G$ (i-times) by construction of the free $\mathbb{Z}[G]$-module X_i. If we set $C^i(G,M) = \text{Hom}_{\mathbb{Z}[G]}(X_i, M)$ then

$$H^i(G,M) = H^i(C^\bullet(G,M))$$

We have done a rather unusual thing: we have given a canonical choice of a resolution that allows to compute the derived functor for any M. This doesn't usually happen as we compute derived functors ~~saying~~ saying "let us choose a resolution of this object".

General fact: in many cases we can compute derived functors using a much wider class of resolutions.
Let $F: \mathcal{C} \to \mathcal{D}$ left exact and $\mathcal{C}$ has enough injectives we define

Def: $Y \in Ob(\mathcal{C})$ is F-acyclic if $R^i(F)(Y) = 0 \ \forall i \geq 1$.

Injective objects are F-acyclic for every functor F as above.

Prop: let $X \in Ob(\mathcal{C})$ and $[X] \to Y^\bullet$ an F-acyclic right resolution of X. Then
$$R^i(F)(X) \simeq H^i(F(Y^\bullet))$$

Proof: we have, for all $n \geq 0$, short exact sequences

$$0 \to Z^n(Y) \to Y^n \to Z^{n+1}(Y) \to 0$$

because $Y^\bullet$ is exact at the n-th spot. We deduce a long exact sequence

$$0 \to F(Z^nY) \to F(Y^n) \to F(Z^{n+1}Y) \to R^1F(Z^nY) \to \underset{\overset{\|}{0}}{R^1F(Y^n)} \to R^1F(Z^{n+1}Y) \to \dots$$

Hence for all $n \geq 0$, $j \geq 1$
$$R^jF(Z^{n+1}Y) \simeq R^{j+1}(F)(Z^nY)$$

Then
$$R^mF(X) \simeq R^m(F)(Z^0Y) \simeq R^{m-1}(F)(Z^1Y) \simeq \dots \simeq R^1F(Z^{m-1}Y) = \mathrm{coker}(F(Y^m) \to F(Z^mY)) =$$
$$= \mathrm{coker}(F(Y^m) \to Z^m(FY)) = \text{(by exactness)}$$
$$= H^m(FY)$$
□

Notational point: if $(Y^\cdot, d)$ is a cochain complex

$$Z^iY = \ker(d^i : Y^i \to Y^{i+1}) \quad i\text{-cocycles} \qquad B^iY = \mathrm{Im}(d^{i-1} : Y^{i-1} \to Y^i) \quad i\text{-coboundaries}$$

This recalls the topological setting: if $\mathcal{X}$ is a simplicial complex

$$C^i(\mathcal{X}, \mathbb{Z}) = \{\mathbb{Z}\text{-valued functions on } i\text{-simplices of } \mathcal{X}\}$$

and cocycles and coboundaries are defined using a differential d and looking at $(i-1)$-simplices.

EXAMPLE: if $\mathcal{C} = \mathbb{Z}[G]\text{-Mod}$ and $F(M) = M^G$ then $C^\cdot(G, M)$ is not an injective resolution of M, but it is an F-acyclic one. This gives an alternative proof of the fact that $H^i(G,M) = H^i(C^\cdot(G,M))$.

(3) Sheaves

Let X be a topological space. An abelian presheaf on X is a collection of abelian groups $\mathcal{F}(U)$, $U \subseteq X$ open, together with restriction maps

$$\mathcal{F}(U) \to \mathcal{F}(V) \quad \text{if } V \subseteq U \text{ opens}$$

It is called an abelian sheaf if whenever $\mathcal{U} = \bigcup_{i \in I} U_i$

$$0 \to \mathcal{F}(\mathcal{U}) \xrightarrow{\text{restr.}} \prod_{i \in I} \mathcal{F}(U_i) \xrightarrow{\mathrm{res}^{U_i}_{U_i \cap U_j} - \mathrm{res}^{U_j}_{U_i \cap U_j}} \prod_{i,j \in I} \mathcal{F}(U_i \cap U_j) \quad \text{is an exact sequence.}$$

The category of sheaves on X $Sh(X)$ is an abelian category having enough injectives.

The functor

$$\Gamma : Sh(X) \to Ab$$
$$\mathcal{F} \mapsto \mathcal{F}(X)$$

is not exact.

(Recall that as Ab^{T-F} is a subcategory of Ab where cokernels don't agree, the same happens in $Sh(X) \subseteq PSh(X)$

However Γ is left exact so we can compute its right derived functors.

Def: $H^i(X, \mathcal{F}) = R^i(\Gamma)(\mathcal{F})$ in particular $H^0(X, \mathcal{F}) = \mathcal{F}(X)$

Fact: if X is nice enough (e.g. paracompact manifolds), $H^i(X, \underline{\mathbb{Z}})$ coincides with cohomology defined using singular cochains, where

$$\underline{\mathbb{Z}} = \text{"constant sheaf"} \qquad \underline{\mathbb{Z}}(U) = \text{constant functions } U \to \mathbb{Z}$$
(is the sheafification of the constant presheaf $\mathbb{Z}$)

Advantage of sheaf viewpoint: there are many rational operations on schemes which don't preserve constant sheaves.

If $f: X \to Y$ is a morphism, for any $\mathcal{F} \in Ob(Sh(X))$ we define its direct image $f_*\mathcal{F} \in Ob(Sh(Y))$ defined by

$$f_*\mathcal{F}(U_Y) = \mathcal{F}(f^{-1}(U_Y))$$

This construction respect composition: $(f_1 \circ f_2)_* = (f_2)_* \circ (f_1)_*$. Hence we can write

$$X \xrightarrow{f_1} Y \xrightarrow{f_2} \{pt\} \qquad\qquad \underset{\overset{\wedge}{\Gamma}}{(f_2 \circ f_1)_*} = (f_2)_* \circ (f_1)_*$$

and thus we can reduce the computation of the derived functors of Γ to the computation of derived functors of f_*, possibly mapping X in "simpler" spaces.

This fits into a general approach – started by Grothendieck in algebraic geometry – of studying morphisms (or varieties together with morphisms to a base space) rather than varieties alone. The emphasis shifts from spaces to maps between them.