MODULAR CURVES

Why?

1. More info on mod forms.
2. Friendly examples of moduli spaces.

§0 Waffle

Let $\mathcal{H} = \{ z \in \mathbb{C} \mid \operatorname{Im} z > 0 \}$

Group $SL_2(\mathbb{R}) \curvearrowright \mathcal{H}$

Take $\Gamma < SL_2(\mathbb{Z})$ finite index

$$Y(\Gamma) = {}_{\Gamma}\backslash^{\mathcal{H}}.$$

Will equip this with various interesting structures.

§0.1 Recap of modular forms

Fix · $\Gamma \leq SL_2\mathbb{Z}$ finite index ("level")

· $k \in \mathbb{Z}$ ("weight")

Then $\exists$ a space

$$M_k(\Gamma)$$

which is the functions

$f: \mathcal{H} \longrightarrow \mathbb{C}$, holomorphic, st

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$$

$\forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$.

and a growth cond^n on the boundary.

Subspace $S_k(\Gamma) \subset M_k(\Gamma)$ of cusp forms

Basic fact Both finite-dim^l / $\mathbb{C}$.

Any mod form has a q-expansion

$$f(z) = \sum_{n \geq 0} a_n q_h^n$$

$a_n \in \mathbb{C}$

$q_h = e^{2\pi i z/h}$ h = least integer st $\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \in \Gamma$.

§1 Modular curves as Riemann surfaces

§1.1 Mod curves as topological spaces.

$\mathcal{H}$ has a topology (obviously)
so $Y(\Gamma)$ gets quotient top.
(ie strongest top st
$\pi : \mathcal{H} \to Y(\Gamma)$ is cts)

Quotient tops can be pretty nasty
($\mathbb{Q}$ acting on $\mathbb{R}$ by translation)
— quotient can even be indiscrete top.

<u>Prop 1.1.1</u> For any $\tau_1, \tau_2 \in \mathcal{H}$. $\exists$ nhds $U_1 \ni \tau_1$, $U_2 \ni \tau_2$ st if $\gamma \in SL_2\mathbb{Z}$ satisfies $\gamma(U_1) \cap U_2 \neq \emptyset$, then $\gamma(\tau_1) = \tau_2$.

<u>Proof</u> See Prop 2.1.1 of Diamond + Shurman.

(We say $SL_2\mathbb{Z}$ acts <u>properly discontinuously on $\mathcal{H}$.</u>)

<u>Corollary 1.1.2</u> $Y(\Gamma)$ is <u>Hausdorff</u>.

<u>Proof</u> Let $P_1 \neq P_2$ be two pts of $Y(\Gamma)$.
Choose $\tau_1, \tau_2 \in \mathcal{H}$ lifting P_i.
Let U_1, U_2 be nhds of τ_i as in Prop 1.1.1.
I claim $V_i = \pi(U_i)$ are open nhds of P_i
st $V_1 \cap V_2 = \emptyset$.

Suppose $V_1 \cap V_2 \neq \emptyset$.
Then $\pi^{-1}(V_1) \cap \pi^{-1}(V_2) \neq \emptyset$

$$\bigcup_{\gamma \in \Gamma} \gamma U_1 \cap \bigcup_{\gamma' \in \Gamma} \gamma' U_2 = \emptyset$$

So $\gamma U_1 \cap \gamma' U_2 \neq \emptyset$ some $\gamma, \gamma' \in \Gamma$
So $(\gamma')^{-1} \gamma U_1 \cap U_2 \neq \emptyset$.
So $\underbrace{(\gamma')^{-1}\gamma}_{\in \Gamma} \tau_1 = \tau_2$, by our assumption on U_i
but $P_1 \neq P_2$, contradiction. $\square$

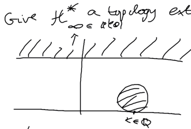
We're also interested in a slightly larger space $X(\Gamma)$ which is a compactification of $Y(\Gamma)$.

$X(\Gamma) = Y(\Gamma) \cup C(\Gamma)$ ("cusps" of Γ)

$C(\Gamma) = \Gamma \backslash \mathbb{P}^1(\mathbb{Q})$.

Let $\mathcal{H}^* = \mathcal{H} \cup \mathbb{P}^1(\mathbb{Q})$

Give $\mathcal{H}^*$ a topology extending that on $\mathcal{H}$



nhds of ∞ = $\{z : \operatorname{Im} z > R\}$ any R

nhds of $x \in \mathbb{Q}$ = circles tangent to $\mathbb{R}$ at x.

Action on $\mathcal{H}^*$ still properly discts so $X(\Gamma)$ is Hausdorff.

<u>Prop 1.1.3</u> $X(\Gamma)$ is compact.

<u>Proof</u> It suffices to find a compact subset of $\mathcal{H}^*$ mapping surjectively to $X(\Gamma)$.

Let $D^* = \{\infty\} \cup \{z \in \mathcal{H} : -\frac{1}{2} \leq \operatorname{Re} z \leq \frac{1}{2},\ |z| \geq 1\}$

Standard fact: D^* contains a pt of every $SL_2(\mathbb{Z})$ orbit on $\mathcal{H}^*$.

So if $\gamma_1 \dots \gamma_n$ coset reps for $\Gamma \backslash SL_2\mathbb{Z}$ then $\bigcup_{i=1}^{n} \gamma D^*$ surjects onto $X(\Gamma)$.

D^* is compact (easy) so done. $\square$

<u>§1.2</u> <u>Riemann surfaces : recap</u>

<u>Def 1.2.1</u> A <u>Riemann surface</u> consists of the following data:

- a topological space X (Hausdorff + second-countable)
- a collection $(U_i, V_i, \phi_i)_{i \in I}$, where $V_i \subset X$ are opens forming a cover of X

  U_i are opens in $\mathbb{C}$

  $\phi_i : U_i \to V_i$ is a homeomorphism

such that if $V_i \cap V_j \neq \emptyset$, the map
$$U_i \cap \phi_i^{-1}(V_i \cap V_j) \xrightarrow{\phi_j^{-1} \circ \phi_i} U_j \cap \phi_j^{-1}(V_i \cap V_j)$$
is <u>holomorphic</u>.

Roughly: A Riemann surface is the least amount of structure on X needed to make sense of a fn $X \to \mathbb{C}$ being holomorphic.

We'll now show that $Y(\Gamma)$ and $X(\Gamma)$ have <u>natural Riemann surface structures</u>.

<u>Def 1.2.2</u> We say $P \in Y(\Gamma)$ is an <u>elliptic pt</u> if for some (+ hence any) $\tau \in \mathcal{H}$ lifting P, $\operatorname{Stab}_{\overline{\Gamma}}(\tau) \neq \{1\}$.
($\overline{\Gamma}$ = image of Γ in $PSL_2(\mathbb{Z}) = \Gamma / \Gamma \cap \{\pm 1\}$).

If P is elliptic for Γ, then it maps to an elliptic pt of $Y(SL_2\mathbb{Z})$ + there are only 2 of these (orbits of i and $\rho = e^{2\pi i/3}$)

If P isn't elliptic (or a cusp), we can easily find a chart around P.

Let τ be a lifting of P and apply Prop 1.1 with $\tau_1 = \tau_2 = \tau$. Let $\mathcal{U} = \mathcal{U}_1 \cap \mathcal{U}_2$.
Then $\mathcal{U}$ is a nhd of τ st $\gamma\mathcal{U} \cap \mathcal{U} = \emptyset$ for any $\gamma \neq 1 \in \overline{\Gamma}$.
Let V = image of $\mathcal{U}$ in $Y(\Gamma)$; then $\varphi = \pi|_{\mathcal{U}}$ is a homeomorphism $\mathcal{U} \xrightarrow{\sim} V$.

If P is elliptic, need to be a bit cleverer.

Prop 1.1.1 gives us a $\mathcal{U} \ni \tau$ st $\mathcal{U} \cap \gamma\mathcal{U} \neq \emptyset$ iff $\gamma \in \underbrace{\mathrm{Stab}_{\overline{\Gamma}}(\tau)}_{\text{order 2 or 3}}$

Choose $\delta \in SL_2\mathbb{C}$ shifting $\begin{cases} \tau \text{ to } 0. \\ \overline{\tau} \text{ to } \infty \end{cases}$
Then $\mathrm{Stab}_{\overline{\Gamma}}(\tau)$ goes to a (finite) cyclic gp of Mobius transfmⁿs fixing 0 and ∞, hence multⁿ by $e^{2\pi i/n}$ $n = 2$ or 3.

$$\begin{tikzcd} \delta\cdot\mathcal{U} \arrow[r, "\sim", "z \mapsto z^n"'] \arrow[d, "\mathrm{Stab}_{\delta\Gamma\delta^{-1}}(0)"', "\int"] & \mathcal{U}' \arrow[dl, dashed, "\sim"] \\ V \subset Y(\Gamma) & \end{tikzcd}$$

this is our coord. chart.

Lastly, if P is a cusp, we argue similarly: choose δ mapping P to ∞, $\mathrm{Stab}_{\delta\Gamma\delta^{-1}}(\infty)$ is a group of <u>translations</u> + $z \mapsto e^{2\pi i z/h}$ gives a local coordinate. $\square$

We've proved: $\exists$ Riemann surf. structures on $Y(\Gamma)$ and $X(\Gamma)$ st $\pi : \mathcal{H} \longrightarrow Y(\Gamma)$ is holomorphic.

(clearly the unique such structure.)

<u>§1.3</u> <u>Genus, ramification, Riemann–Hurwitz.</u>

<u>FACT</u> Riemann surfaces are orientable smooth 2-manifolds, + there aren't very many of these. (Compact, connected) ones all look like doughnuts

Formally: define <u>genus</u> as unique integer $g = g(M)$ st

$$H^1(M, \mathbb{Z}) \cong \mathbb{Z}^{2g}.$$

Genus is closely related to Euler characteristic

$$\chi(M) = \sum_{i \geq 0} (-1)^i \operatorname{rk} H^i(M, \mathbb{Z})$$

If M as above, $H^0 \cong H^2 \cong \mathbb{Z}$
and $H^i = 0$ for $i \geq 3$

so $\chi(M) = 2 - 2g$.

($\chi =$ "$V - E + F$" for graphs)

Prop 1.3.2 For $\Gamma = SL_2 \mathbb{Z}$, the space $X(\Gamma)$ is isomorphic (as a Riemann surf, so in particular as 2-mfld) to $\mathbb{P}^1(\mathbb{C}) \cong S^2$.

Proof The "j-invariant"

$$j(z) = q^{-1} + 744 + 196884q + \dots$$

is $SL_2(\mathbb{Z})$-inv^t + descends to a holomorphic map

$$X(SL_2 \mathbb{Z}) \to \mathbb{P}^1(\mathbb{C}).$$

It's bijective (count zeros using contour integration) so it has a holo. inverse.

$\square$