Modular curves lecture 2.

<u>Convention</u> All Riemann surfaces are assumed <u>connected</u>.

<u>Recap</u> Want to find $g(X(\Gamma))$ $\forall \Gamma$.
Know $X(SL_2\mathbb{Z}) \cong \mathbb{P}^1(\mathbb{C})$ genus 0.

$\forall \Gamma$ have a map

$$X(\Gamma) \twoheadrightarrow X(SL_2\mathbb{Z})$$

<u>Def 1.3.3</u> (i) For $f: X \to Y$ non-const. morphism, $P \in X$, the <u>ramification degree</u> $e_P(f)$ is the unique integer $e \geq 1$ st f "looks like $z \mapsto z^e$ locally".
Note pts st $e_P(f) > 1$ are <u>isolated</u>.

(ii) If X, Y compact, the sum

$$\sum_{P \in f^{-1}(Q)} e_P(f)$$

is indep. of $Q \in Y$ + call this the <u>degree</u> of f.

The degree of $X(\Gamma) \to X(SL_2\mathbb{Z})$ is $[PSL_2(\mathbb{Z}) : \bar{\Gamma}]$. (= # of preimages of a generic pt of $X(SL_2\mathbb{Z})$)

<u>Thm 1.3.4</u> (Riemann–Hurwitz)
For $\underset{\text{degree } N}{f: X \to Y}$ non-const, X, Y compact,

$$2g(X) - 2 = N \cdot (2g(Y) - 2) + \sum_{P \in X} (e_P(f) - 1).$$

<u>Corollary 1.3.5</u> For any Γ, have

$$g(X(\Gamma)) = 1 + \frac{[PSL_2\mathbb{Z} : \bar{\Gamma}]}{12} - \frac{\varepsilon_2}{4} - \frac{\varepsilon_3}{3} - \frac{\varepsilon_\infty}{2}$$

ε_2 = # ell pts of order 2
ε_3 = " 3
ε_∞ = # cusps.

<u>Proof</u> Need to analyse ramification of $X(\Gamma) \xrightarrow{f} X(SL_2\mathbb{Z})$ at each $P \in X(\Gamma)$.

- $P \in Y(\Gamma)$ not in SL_2-orbit of i or ρ:

$$\begin{tikzcd} \mathcal{H} \arrow[d] \arrow[dd, bend left=60] \\ Y(\Gamma) \arrow[d] \\ Y(SL_2\mathbb{Z}) \end{tikzcd}$$

this map is unramified at any τ lifting a non-ell. pt of $Y(SL_2\mathbb{Z})$, so $e_P(f) = 1$.

- Pts of $Y(\Gamma)$ above $[i]$: all such P are either non-ell., or ell. of order 2.

If P is ell. of order 2, then $Y(\Gamma) \to Y(SL_2\mathbb{Z})$ is locally an isomorphism at P, $e_P = 1$.

If P non-elliptic, then local coordinate for $SL_2\mathbb{Z}$ is square of that for Γ, so $e_P = 2$.

$$N = 1 \cdot \varepsilon_2(\Gamma) + 2 \cdot \left(\begin{array}{l} \#\text{ of non-ell pts} \\ \text{of } Y(\Gamma) \text{ above } [i] \end{array} \right)$$

$$\Rightarrow \#\text{ of non-ell pts above } [i] = \frac{N - \varepsilon_2}{2}.$$

$$\Rightarrow \sum_{P \in f^{-1}([i])} (e_P - 1) = \frac{N - \varepsilon_2}{2}.$$

- P maps to $[\rho]$, $\rho = e^{2\pi i/3}$

Then $e_P(f) = \begin{cases} 1 & P \text{ elliptic} \\ 3 & P \text{ non-ell.} \end{cases}$

Def. of degree gives

$$\#\text{ non-ell. } P = \frac{N - \varepsilon_3}{3}$$

$$\Rightarrow \sum_{P \in f^{-1}([\rho])} (e_P - 1) = \frac{2 \cdot (N - \varepsilon_3)}{3}.$$

- P cusp: let h = width of cusp P = integer st $e^{2\pi i z/h}$ is local coord for $X(\Gamma)$ at P.

Local coord for $X(SL_2\mathbb{Z})$ at $[\infty]$ is $\left(e^{2\pi i z/h}\right)^h$. So $e_P(f) = h$.

Thus
$$\sum_{P \in f^{-1}([\infty])} e_P - 1$$
$$= \left(\sum_{P \in f^{-1}([\infty])} e_P \right) - \varepsilon_\infty$$
$$= N - \varepsilon_\infty.$$

$$\Rightarrow 2g(X(\Gamma)) - 2$$
$$= N \cdot (-2) + \frac{N - \varepsilon_2}{2} + \frac{2(N - \varepsilon_3)}{3} + (N - \varepsilon_\infty)$$
$$\Rightarrow g(X(\Gamma)) = 1 + \frac{N}{12} - \frac{\varepsilon_2}{4} - \frac{\varepsilon_3}{3} - \frac{\varepsilon_\infty}{2}. \quad \square$$

<u>Example</u> $\Gamma = \Gamma_0(11)$

Have $N = 12$, $\varepsilon_\infty = 2$ ($[0]$ and $[\infty]$)

$\varepsilon_2 = \varepsilon_3 = 0$ (exercise, cf. D&S)

$$\Rightarrow g = 1 + \frac{12}{12} - 0 - 0 - \frac{2}{2}$$
$$= 1.$$

<u>Exercise</u> (i) Verify $\varepsilon_2 = \varepsilon_3 = 0$ for $\Gamma_0(11)$.

(ii) Show that the only primes p st $g(X(\Gamma_0(p))) = 0$. ($\{2, 3, 5, 7, 13\}$)

<u>Remark</u> For any g, $\exists$ finitely many <u>congruence</u> subgps Γ of $PSL_2\mathbb{Z}$ of genus g. (J.G. Thompson)

ANNOUNCEMENTS

1. PhD students: email graduate.studies@ox.ac.uk to formally register.
2. There will be problem sets – first one next week.
3. Notes will be posted online.

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§1.4 Sheaves and Riemann–Roch.

Conjecture 1.4.1 Let X be a top. space. Then you already know what a sheaf on X is.

Now let X be a Riemann surf.

$\mathcal{O}_X$ ("structure sheaf")

def. by $\mathcal{O}_X(U)$ = holo. fcns $U \to \mathbb{C}$.

sheaf of rings, so can make sense of sheaves of $\mathcal{O}_X$-modules

Def^n An invertible sheaf on X is a sheaf of $\mathcal{O}_X$-mods that is locally free of rank 1.

($\Leftrightarrow$ has an inverse wrt tensor product of $\mathcal{O}_X$-mods).

Fact Invertible sheaves $\longleftrightarrow$ Line bundles with holomorphic structure.

(for geometers)

Now specialize to the case of X compact.

Have a notion of meromorphic sections of inv^{ble} $\mathcal{F}$

(= sections of $\mathcal{F} \otimes_{\mathcal{O}_X} \{$sheaf of meromorphic fcns$\}$.)

Thm 1.4.2 (Riemann Existence Thm)
Any invertible sheaf on a compact RS has a nonzero global meromorphic section.

$\Rightarrow$ a notion of degree of an inv^{ble} sheaf

= Sum of orders of vanishing of any nonzero global mero^c section.

(well-def., as sum of zeros + poles of a mero fcn is 0.)

Have $\deg(\mathcal{F} \otimes \mathcal{G}) = \deg \mathcal{F} + \deg \mathcal{G}$

$\deg(\mathcal{F}^{-1}) = -\deg \mathcal{F}$.

(Inv^{ble} sheaves are a group under $\otimes$, with $\mathcal{O}_X$ as identity, and deg is a homomorphism to $\mathbb{Z}$.)

Thm 1.4.3 (Riemann – Roch)
Let X cpct RS, $\mathcal{F}$ inv^{ble} sheaf on X

Then (i) $\underset{\mathcal{F}(X)}{H^0(X, \mathcal{F})}$ is finite – dim^l over $\mathbb{C}$

(ii)
$$\dim H^0(X, \mathcal{F}) - \dim H^0(X, \Omega \otimes \mathcal{F}^{-1}) = 1 - g + \deg \mathcal{F}.$$

where Ω is the sheaf of holomorphic differentials on X.

Note that if $\deg(\mathcal{F}) < 0$, $\mathcal{F}$ has no nonzero global sections; so if $\deg \mathcal{F}$ is <u>large</u>, $H^0(X, \Omega \otimes \mathcal{F}^{-1}) = 0$ + get a formula for $\dim H^0(X, \mathcal{F})$.

(Note that $\dim H^0(X, \Omega) = g(X)$ by taking $\mathcal{F} = \mathcal{O}_X$,
$\deg(\Omega) = 2g-2$ by taking $\mathcal{F} = \Omega$.)

(<u>Aside</u> $\exists$ a coho. theory for sheaves on Riemann surfaces for which $H^0(X, F)$ is sections. Then RR is combination of 2 things:

- a formula for $\chi(F) = \sum_{i \geq 0} (-1)^i \dim H^i(X, F)$
- <u>Serre duality</u>

$$H^i(X, \mathcal{F}) = H^{1-i}(X, \Omega \otimes \mathcal{F}^{-1})^*$$

"dualizing sheaf".)

<u>§1.5 The Katz sheaf</u>

Let $X = X(\Gamma)$ some Γ, and choose $k \in \mathbb{Z}$.

<u>Def^n 1.5.1</u> Let ω_k be the sheaf

$$\omega_k(V) = \{ \text{holo. functions on } \pi^{-1}(V) \subset \mathcal{H}^* \text{ satisfying } f(\gamma z) = j(\gamma, z)^k f(z) \ \forall \gamma \in \Gamma. \}$$

This is a sheaf of $\mathcal{O}_{X(\Gamma)}$-modules. If k odd and $-1 \in \Gamma$ it's the zero sheaf. Assume ~~this~~ this isn't the case.

<u>Thm 1.5.2</u> (i) ω_k is invertible.
(ii) $\omega_2 = \Omega_{X(\Gamma)}(\text{cusps})$.

<u>Proof</u> Part (i) is a case-by-case check. – show it on an open nbd of every $P \in X(\Gamma)$.
For P non-elliptic, not cusp, can find $V \ni P$ open st $\pi^{-1}(V) = \coprod_{\gamma \in \Gamma} \gamma U$ and $\omega_k(V) \cong \mathcal{O}_{\mathcal{H}}(U) \cong \mathcal{O}_X(V)$

Other cases: will come back to this next time

(ii): the isomorphism is
$$f \longmapsto f(z)dz.$$