Problem sheet 1 now online

Deadline 14th.

Theorem 1.5.2

i) ω_k is loc. free of rank 1 (i.e. invertible)

ii) $\omega_2 \cong \Omega^1_{X(\Gamma)}(\text{cusps})$

Here, for $\mathcal{L}$ an $\mathcal{O}_X$-module sheaf and $D = \sum n_i P_i$ is a divisor (formal $\mathbb{Z}$-lin comb^n of pts)

we define $\mathcal{L}(D)(U) = \{$ meromorphic sections^x of $\mathcal{L}$ over U with $\mathrm{div}(x) + D \geq 0 \}$

$\mathcal{L}(P) =$ "allow simple poles at P"

$\mathcal{L}(-P) =$ "sections vanishing at P"

Conclusion of proof

(i) Want to show: for every $P \in X(\Gamma)$ $\exists$ nhd $V \ni P$ and $b \in \omega_k(V)$ st $\omega_k(V) = \mathcal{O}_X(V) \cdot b$.

Choose $\tau \in \mathcal{H}^*$ lifting P, $U \subset \mathcal{H}^*$ open st U is fixed by $\mathrm{Stab}_\Gamma(\tau)$ and

$$\pi^{-1}(V) = \coprod_{\gamma \in \bar{\Gamma}/\mathrm{Stab}_\Gamma(\tau)} \gamma U$$

where $V = \pi(U)$.

So $\omega_k(V) = \{ f : U \to \mathbb{C}$ hol. + wt k inv^t under $\mathrm{Stab}_\Gamma(\tau) \}$.

while $\mathcal{O}_X(V) = \{ f : U \to \mathbb{C}$ hol. + wt 0 inv^t — $\}$

So if $\mathrm{Stab}_\Gamma(\tau) = 1$ we're done (take $b = 1$).

or more generally if wt k action of $\mathrm{Stab}_\Gamma(\tau)$ is trivial.

(happens if τ elliptic + k div^{ble} by order of $\mathrm{Stab}_\Gamma(\tau)$).

- Ell pts: if τ is elliptic, conjugate it onto $z = 0$ as before, so $\mathrm{Stab}_\Gamma(\tau) =$ cyclic gp of rot^{n}s. order n, say.

  A fn $U \to \mathbb{C}$ is wt k inv^t under this gp iff

  $f(z) = z^a g(z^n)$ a = least nonnegative integer $\equiv k \bmod n$. some hol. g.

  So $z \mapsto z^a$ is our local basis.

- τ a cusp: if cusp is regular or weight is even, actions of $\mathrm{Stab}_\Gamma(\tau)$ in wt k and wt 0 coincide $\Rightarrow$ $b = 1$ works.

- τ an irregular cusp, k odd: (wlog $\tau = \infty$)

  $\mathcal{O}_X(V) = \{ f : U \to \mathbb{C}$ hol. st $f(z+h) = f(z) \}$

  $\omega_k(V) = \{ f : U \to \mathbb{C}$ hol: $f(z+h) = -f(z) \}$.

  and $z \mapsto e^{i\pi z/h}$ is a local basis

Part (ii):

clearly $f \longmapsto f(z)\,dz$

is a bij^n $\mathcal{O}_{\mathcal{H}}(\mathcal{H}) \to \Omega^1_{\mathcal{H}}(\mathcal{H})$

+ it commutes w Γ-action

if we put weight 2 action on $\mathcal{O}_{\mathcal{H}}$.

Passing to Γ-inv^{t}s,

$$\omega_2\big|_{Y(\Gamma)} = \Omega^1_{Y(\Gamma)}.$$

Need to show: sections of ω_2 extending to cusps

correspt. to differentials w. simple poles.

Suffices to treat $\tau = \infty$

Sections of Ω^1 near ∞ are $f(q)\,dq$

But $q = e^{2\pi i z/h}$

$\Rightarrow dq = \frac{2\pi i}{h} e^{2\pi i z/h}\,dz$

ie dz is a scalar times $\frac{dq}{q}$.

so $\mathcal{O}_X\, dz = \mathcal{O}_X \cdot \frac{dq}{q}$

$= \mathcal{O}_X(\infty) \cdot dq$ " (over a nhd V of ∞.) $\square$

Obviously

$$H^0(X(\Gamma), \omega_k) = M_k(\Gamma).$$

<u>Prop 1.5.3</u>

Let $r = \mathrm{LCM}\left(\text{orders of } \Gamma\text{-stabilizers of ell. pts}, \begin{cases} 2 & \text{if } \exists \text{ irreg cusps} \\ 1 & \text{otherwise} \end{cases}\right)$

$(1 \leq r \leq 12)$

Then $\omega_{k+r} \cong \omega_k \otimes \omega_r \quad \forall k \in \mathbb{Z}$.

and in particular if $r=1$ then $\omega_k = (\omega_1)^{\otimes k}\ \forall k$.

<u>Proof</u> For $k=r$ all the local bases b in Thm 1.5.2 were 1,

and bases of ω_k for general k only depended on $k \bmod r$.

So local basis for ω_{k+r} = product of ones for ω_k and ω_r. $\square$

<u>Def 1.5.4</u> If $r=1$ above (i.e. Γ has no ell pts, doesn't contain -1, and all cusps regular), say Γ is <u>neat</u>.

Then $\omega_k = \omega_1^{\otimes k}$, so $\omega = \omega_1$ is obviously rather important. Call this the <u>Katz sheaf</u>.

<u>1.5.5 Corollary</u> If Γ is neat,

then for $k \geq 2$ we have

$$\dim M_k(\Gamma) = (k-1)(g-1) + \frac{k}{2}\varepsilon_\infty.$$

<u>Proof</u> We have $\deg \omega = \frac{1}{2}\deg(\omega^{\otimes 2})$

$= \frac{1}{2}(\deg \Omega + \varepsilon_\infty)$

$= \frac{1}{2}(2g-2+\varepsilon_\infty)$

So if $k \geq 2$, $\deg \omega^{\otimes k} > 2g-2$ and R-R gives

$\dim H^0(X(\Gamma), \omega^{\otimes k}) = k\cdot\left(g-1+\frac{\varepsilon_\infty}{2}\right) - g + 1$

$= (k-1)(g-1) + \frac{k}{2}\varepsilon_\infty.$ $\square$

Similar (but messier) formulae in non-neat case

cf D+S ch 3

Example $\Gamma = \Gamma_1(5)$ is neat,
$g = 0$ and $c_\infty = 4$. ($\Gamma_1(N)$ neat iff $N \geq 5$)

So $\dim M_k(\Gamma) = k+1$ for $k \geq 2$.

For $k=1$ we need to worry about
$H^0(X(\Gamma), \Omega^1 \otimes \omega^{-1})$; but $\deg \Omega^1 = 2g-2 = -2$
$\deg \omega = 1$
so $\Omega^1 \otimes \omega^{-1}$ has deg -1.

If you try to do this for $\Gamma_1(23)$ it fails: $\deg \Omega \otimes \omega^{-1}$ is 0.
So dim^{n}s of wt 1 form spaces lie much deeper.

CHAP. 2 MODULAR CURVES AS ALGEBRAIC CURVES

§2.1 Modular curves over $\mathbb{C}$

Thm 2.1.1

(i) The $\mathbb{C}$-points of a smooth connected projective algebraic curve $/\mathbb{C}$ are canonically a Riemann surface.
$$X \longmapsto X^{an}$$
(ii) Every compact Riemann surface is X^{an} for a unique X.

(iii) $\exists$ equivalence of categories
$$\left(\text{loc. free sheaves of } \mathcal{O}_X\text{-mods}\right) \cong \left(\text{loc. free sheaves of } \mathcal{O}_{X^{an}}\text{-mods}\right)$$
preserving global sections.

(i) is basically the implicit fcn thm
We'll see later a bit about pf of (ii)
(iii) is Serre's "GAGA" thm.

The functors are on the one hand
$$\mathcal{F} \longmapsto \mathcal{O}_{X^{an}} \otimes_{\mathcal{O}_X} \mathcal{F}$$
and on the other hand
$$\mathcal{F} \longmapsto \left(\begin{array}{c}\text{subsheaf of } \mathcal{F} \text{ whose sections} \\ \text{over } U \text{ are elts of } \mathcal{F}(U) \text{ extending} \\ \text{to mero. sections on } X.\end{array}\right)$$

Hence for any Γ there's an alg variety $X(\Gamma)_{\mathbb{C}}$ and inv^{ble} sheaves ω_k on it st
$$M_k(\Gamma) = H^0(X(\Gamma)_{\mathbb{C}}, \omega_k).$$

Here's an alternative, nicer (IMHO) construction.

Thm 2.1.2
$$X(\Gamma)_{\mathbb{C}} = \operatorname{Proj}\left(\bigoplus_{k \geq 0} M_k(\Gamma)\right).$$

(Cf. Hartshorne §II.2 for def^n of Proj.)

Proof

One knows that for any Noetherian graded $\mathbb{C}$-algebra $S.$, with $S_0 = \mathbb{C}$.

$$\mathrm{Proj}(S.) = \mathrm{Proj}(S_{n.}) \quad \text{for any } n \geq 1$$

$$S_{n.} = \text{the subring } \bigoplus_{k \geq 0} S_{nk}.$$

Choose n = the r from last section,

so $$S_{n.} = \bigoplus_k H^0(X(\Gamma), \omega_n^{\otimes k})$$

We now quote a standard fact in alg. geom: inv^{ble} sheaves of +ve degree on curves are ample, so their sections give an embedding in proj. space. $\square$

(Remark: In fact the same argument can be used to prove Thm 2.1.1 (ii): take any ample inv^{ble} sheaf on a Riemann surf + get an embedding in $\mathbb{P}^N$, $N \gg 0$.)

§2.2 Descending the base field.

Question Does there exist an alg. curve over some number field K st we get $X(\Gamma)_{\mathbb{C}}$ by base ext^n?

Let's think a bit what this means.

- Clearly not all varieties $/\mathbb{C}$ are definable over number fields.

  $$Y^2 = X^3 + X + \pi.$$

  This isn't def $/$any number field, as its j-inv^t is
  $$\frac{6192}{27\pi^2 + 4}$$

  $\pi Y^2 = X^3 + X$ is definable $/\mathbb{Q}$ (it's isomorphic to $Y^2 = X^3 + X$.)

- Even if descents exist they might not be unique.

  Eg $\mathbb{P}^1_{\mathbb{Q}}$ and $\{X^2 + Y^2 + 2Z^2 = 0\} \subset \mathbb{P}^2_{\mathbb{Q}}$ become isomorphic over $\mathbb{C}$.

So we need to ask: is there a descent to a number field that "means something"?

The curves-fields correspondence

There is a bijection, for any field k,

$$\left\{\begin{array}{c}\text{smooth geometrically conn.}\\ \text{alg curves } /k\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{field ext}^n\text{s } K/k \\ \text{of transcendence degree 1} \\ \text{containing no algebraic} \\ \text{ext}^n \text{ of } k\end{array}\right\}$$

$$X \longrightarrow k(X), \text{ field of rat}^l \text{ fns on } X.$$

So for $X/\mathbb{C}$ curve

$$\left\{\begin{array}{c}\text{models of } X \text{ over} \\ k \subseteq \mathbb{C}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{subfields } L \text{ of } \mathbb{C}(X) \\ \text{generating it}/\mathbb{C} \text{ but with } L \cap \mathbb{C} = k\end{array}\right\}$$

So we want to look for nice subfields of $\mathbb{C}(X(\Gamma)_{\mathbb{C}}) = \{$meromorphic modular fcns of wt 0 + level $\Gamma\}$

<u>Thm 2.2.1</u> (i) $\mathbb{C}(X_0(N))$

$= \mathbb{C}(j(z), j(Nz))$

(ii) The minimal poly of $j(Nz)$ over $\mathbb{C}(j(z))$ lies in $\mathbb{Z}[j][Y] \subseteq \mathbb{C}(j)[Y]$.

In particular $X_0(N)_{\mathbb{C}}$ has a model$/\mathbb{Q}$ whose function field is $\mathbb{Q}(j(z), j(Nz))$.