Thm 3.7.5 $\mathcal{P}$ is representable
$\iff$ it's relatively rep^{ble} and rigid.

Sketch of proof

Start from 2 basic moduli problems.

- "naive level $\Gamma(3)$" over $\mathbb{Z}[\frac{1}{3}]$
- "Legendre moduli problem" ($\Gamma(2)$ + choice of differential) over $\mathbb{Z}[\frac{1}{2}]$

Both have group actions ($GL_2(\mathbb{F}_3)$ and $GL_2(\mathbb{F}_2) \times \{\pm 1\}$.)

Given $\mathcal{P}$ rel. rep^{ble} + rigid, construct one object by taking $\mathcal{E}/Y(3)$ — relative representability gives us a scheme $/Y(3)$ + this has a $GL_2(\mathbb{F}_3)$-action. Take inv^{ts} (OK because $\mathcal{P}$ is rigid) & this gives an object $\mathcal{E}/S$ representing $\mathcal{P}$ on $Ell/R[\frac{1}{3}]$.

Legendre gives an object over $R[\frac{1}{2}]$ similarly.

By rigidity these agree $/R[\frac{1}{6}]$ so we get a representing obj. over R. $\square$

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§ 3.8 General Level Structures

Fix N and a subgp $H \subset GL_2(\mathbb{Z}/N)$.

Fact 3.8.1 $\exists$ a moduli problem $\mathcal{P}_H$ on $Ell/\mathbb{Z}[\frac{1}{N}]$ st if $\bar{k}$ alg. closed $E/\bar{k} \in Obj(Ell/\mathbb{Z}[\frac{1}{N}])$

$$\mathcal{P}_H(E/\bar{k}) = \left\{ H\text{-orbits of isomorphisms } (\mathbb{Z}/N)^2 \xrightarrow{\sim} E[N] \right\}.$$

For $H = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$, this is $\Gamma(N)$,

$E/S \longmapsto$ (pairs of sections $P, Q \in E[S]$ generating $E[N]$ in every fibre)

$H = \left\{ \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \right\}$ this is $\Gamma_1(N)$

$H = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\}$ it's $\Gamma_0(N)$.

(**Remark** If k is a field, E/k, then image of $\mathcal{P}_H(E/k)$ in $\mathcal{P}_H(E/\bar{k})$ is

$\{$ H-orbits of bases of $E[N](\bar{k})$ in which image of $Gal(\bar{k}/k)$ lands in H $\}$.)

Prop 3.8.2 $\mathcal{P}_H$ is relatively rep^{ble} and "étale over $Ell/\mathbb{Z}[\frac{1}{N}]$"

(this means: $\forall\, E/S \in Obj(Ell/\mathbb{Z}[\frac{1}{N}])$, the functor $T \longmapsto \mathcal{P}_H(E \times_S T)$ is represented by an étale S-scheme.)

Proof For $H = \{1\}$, for $E/S \in Obj(Ell/\mathbb{Z}[\frac{1}{N}])$ we can find an explicit S-scheme representing $\mathcal{P}_H$ on Sch/S: it's an open subscheme of $E[N] \times_S E[N]$ given by non-vanishing of Weil pairing.

For general H just take the quotient of this by H. $\square$

(So it's easier to relatively represent $\mathcal{P}_H$ than it is to define it.)

Prop 3.8.3 $\mathcal{P}_H$ is rigid on $Ell/R[\frac{1}{6}]$ if + only if the preimage in $SL_2(\mathbb{Z})$ of $H \cap SL_2(\mathbb{Z}/N)$ contains no elts of finite order (i.e. has no elliptic pts + doesn't contain -1).

Proof (sketch) Over $\mathbb{C}$ this is routine.

To prove general statement suffices to check it on objects $E/\bar{k}$, $\bar{k}$ alg. closed + not too large.

If $\bar{k}$ has char. 0 can embed it into $\mathbb{C}$.

Can show: if k has finite char. ≥ 5, E/k ell. curve, $\phi \in Aut(E)$, then the pair (E, ϕ) lifts to char. 0.

Cf. somewhere in ch. VI of Deligne–Rapoport. $\square$

This gives a complete classification of modular curves + their associated moduli problems.

Remarks

(i) As in case of $Y_0(N)$, for $\underbrace{H \text{ non-rigid}}_{\text{(i.e. preimage in } SL_2\mathbb{Z} \text{ has torsion.)}}$ we can still construct a $\mathbb{Z}[\frac{1}{N}]$-scheme Y which is "the best approximation" to representing $\mathcal{P}_H$: have a map

$$\widetilde{\mathcal{P}}_H(S) \longrightarrow Y(S)$$

which is surjective for S field, bijective for S alg. closed.

(ii) If Γ = preimage$(H) \subset SL_2\mathbb{Z}$, then $Y_{\mathcal{P}_H}(\mathbb{C})$ isn't quite $\Gamma \backslash \mathcal{H}$. It's a union of such things corresponding to quotient $\dfrac{(\mathbb{Z}/N)^\times}{\det H}$.

In particular our version of $Y(N)$ is not geometrically connected.

Can write $Y_{\mathcal{P}_H}(\mathbb{C})$ more intrinsically as

$$GL_2(\mathbb{Q}) \backslash GL_2(\mathbb{A}) / \mathbb{R}_{>0} \cdot SO_2(\mathbb{R}) \cdot U$$

U = preimage$(H) \subset GL_2(\hat{\mathbb{Z}}) \subset GL_2(\mathbb{A}_{fin})$

Chapter 4. Leftovers

4.1 Katz Modular Forms

Recall we defined, for E/S ell. curve, $\underline{\omega}_{E/S} = \pi_*(\Omega^1_{E/S})$

Prop 4.1.1 If $S = Y_{P_H}$ for some H as before, E/S universal ell. curve, then $\underline{\omega}_{E/S}$ is the "Katz sheaf" from Chapter 2.

Proof Just unravel the def$^{\text{n}}$s.
Will show that both have same pullback to $\mathcal{H}$ + and actions of Γ agree.
By def$^{\text{n}}$, pullback of ω_{Katz} is $\underline{\mathbb{C}}$.
Pullback of $\underline{\omega}_{E/S}$ is $\pi_*\left(\begin{matrix}\text{relative differentials} \\ \text{on } \mathbb{C}/\mathbb{Z}+\mathbb{Z}\tau\end{matrix}\right)$ $= \underline{\mathbb{C}} \cdot (2\pi i\, dz)$
But the isomorphism $\mathbb{C}/\mathbb{Z}+\mathbb{Z}\tau \cong \mathbb{C}/\mathbb{Z}+\mathbb{Z}\gamma\tau$ for $\gamma \in SL_2\mathbb{Z}$ is mult$^{\text{n}}$ by $(c\tau+d)^{-1}$ on $\mathbb{C}$, so it multiplies dz by this constant. So action coincides with the one we defined in construction of ω_{Katz}.

Def 4.1.2 For Γ torsion free cong. subgp of level N, R a $\mathbb{Z}[\frac{1}{N}]$-algebra, define

$$KM_k(\Gamma, R) = H^0(Y(\Gamma)\times R, \underline{\omega}^k_{E/-})$$

(an R-module).

Concretely: a Katz mod form of wt k over R is a rule attaching to each triple

$(E/S, \alpha, \omega)$:
- S an R-sch.
- E/S ell. curve
- $\alpha \in P_H(E/S)$
- ω a basis of $\Gamma(E, \Omega^1_{E/S})$.

an elt of $\Gamma(S, \mathcal{O}_S)$, st
- compatible with base change in S
- homogeneous of wt k in ω.

((Cf. Katz "P-adic properties of modular schemes + modular forms", Springer LNM #330.)

Fun thing: over $R = \mathbb{Z}[\frac{1}{6}]$, for any ell. curve E/R and $\omega \in \Omega^1$, $\exists!$ short Weierstrass eq$^{\text{n}}$ st $\omega = \frac{dx}{Y}$, and E_4 (resp. E_6) are the maps $(E,\omega) \longmapsto a_4$ coeff of this eq$^{\text{n}}$ (resp. a_6).

§4.2 Cusps & the Tate curve

Consider the ring

$$\mathbb{Z}((q)) = \left\{ \sum_{n=-N}^{\infty} a_n q^n \;\middle|\; a_n \in \mathbb{Z} \right\}$$

We'll define an ell. curve / this, & a differential st evaluating at this pair gives q-exp$^{\text{n}}$ of a Katz MF.

Def 4.2.1 Tate(q) = the ell. curve

$$y^2 + xy = x^3 + a_4 x + a_6$$

$$\left.\begin{aligned} a_4 &= -\sum_{n\geq 1} \frac{5n^3 q^n}{1-q^n} \\ a_6 &= -\sum_{n\geq 1} \frac{(7n^5+5n^3)/12 \cdot q^n}{1-q^n} \end{aligned}\right\} \in \mathbb{Z}[[q]]$$

Find that discriminant (Tate(q)) is exactly q-expansion of Δ (wt 12 cusp form) $\in q + q^2\mathbb{Z}[[q]] \subset \mathbb{Z}((q))^\times$.
Hence Tate(q) is an ell. curve.
Tate(q) = "the q-expansion of $\mathbb{C}/\mathbb{Z}+\mathbb{Z}\tau$"
= "$\mathbb{C}^\times / q^{\mathbb{Z}}$"

Prop 4.2.2 If $\tau \in \mathcal{H}$, then series defining Tate(q) converge at $q = e^{2\pi i \tau}$ and define a curve $\cong \mathbb{C}/\mathbb{Z}+\mathbb{Z}\tau$.

(Convergence is easy, + we check $j(\text{Tate}(q))$ is the q-exp$^{\text{n}}$ of $j(\tau)$.)

Prop 4.2.3 $\exists$ series $X(u,q), Y(u,q) \in \mathbb{Z}[u, u^{-1}, (1-u)^{-1}][[q]]$
st $(X(u,q), Y(u,q)) \oplus (X(v,q), Y(v,q))$ $= (X(uv,q), Y(uv,q))$
gp law on Tate(q)
(interpret $(X(u,q), Y(u,q))$ as ∞ if $u=1$.)

Proof Take

$$X(u,q) = \frac{u}{(1-u)^2} + \sum_{d\geq 1}\left(\sum_{m|d} m(u^m + u^{-m} - 2)\right)q^d$$

$$Y(u,q) = \frac{u^2}{(1-u)^3} + \sum_{d\geq 1}\left(\sum_{m|d}\left\{\frac{m(m-1)}{2}u^m - \frac{m(m+1)}{2}u^{-m} + m\right\}\right)q^d.$$

Sneaky part: $\exists$ straightforward change of coords from Tate(q) to

$$y^2 = 4x^3 - g_4(\tau)x - g_6(\tau)$$

which is $\mathbb{C}/\mathbb{Z}+\mathbb{Z}\tau$ via $(\wp(z,\tau), \wp'(z,\tau))$.
X and Y are just $\wp$ and $\wp'$ as power series in $u = e^{2\pi i z}$, $q = e^{2\pi i \tau}$.

So identity $(X(u,q), Y(u,q)) \oplus \ldots = \ldots$ holds for all u, q in an open subset of $\mathbb{C}\times\mathbb{C}$ so it holds as an identity of power series.

Prop 4.2.4 Cusps of Y_{P_H}
$\leftrightarrow \{ P_H$ level structures on Tate(q) over $\mathbb{Z}[[q^{\frac{1}{N}}, \zeta_N]]$, modulo automorphisms $q^{\frac{1}{N}} \overset{b}{\mapsto} \zeta_N^a \cdot q^{\pm\frac{1}{N}} \}$
(+ we thus get an action of $Gal(\mathbb{Q}(\mu_N)/\mathbb{Q})$).

Example $Y_1(5)$.

Points of order 5 on Tate(q) over $\mathbb{Z}[\frac{1}{5}, \zeta_5][[q^{\frac{1}{5}}]]$ are (images of) $q^{a/N}\zeta_N^b$, $a,b \in (\mathbb{Z}/5)^2 - \{(0,0)\}$