My research:
Singularities in nonlinear PDEs and the Calculus of Variations

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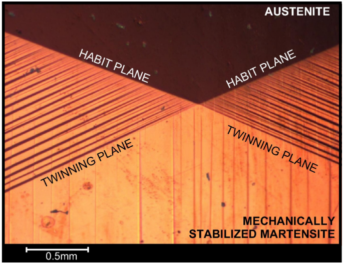
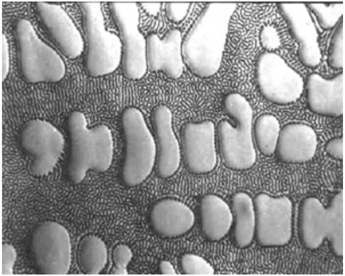
Aims of my research program: Understanding “singularities” – Examples:

Crystal microstructure



Multiphasic alloy (tin/lead)

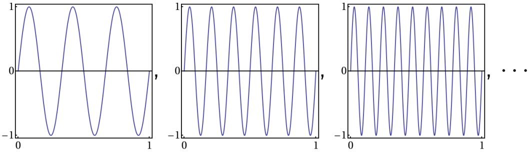


Wingtip vortex

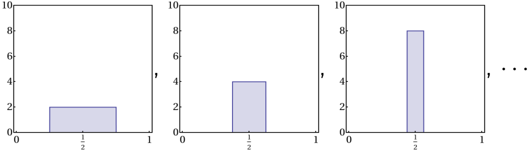
Fractal phenomena in nature (a broccoli)

Mathematical singularities: oscillations & concentrations

Oscillations:



Concentrations:



- Oscillations / concentrations are the *only* effects that can prevent a L^p-weakly converging sequence from converging strongly (Vitali's Thm.)!
- How to describe efficiently **location**, **value-distribution**, and **direction** of oscillations / concentrations?
- Sometimes, one can prove **restrictions** on them (e.g. only certain oscillation directions allowed) $\rightsquigarrow$ Compensated compactness theory.
- How do such singularities **appear** in time-dependent problems?

Singularities in lower semicontinuity theory

- Main question: *Which singularities can form in minimizing sequences?*
- **Lower semicontinuity in BV** ['10a, '12a] for quasi-convex integrands with linear growth:

$$\int_\Omega f(x, \nabla u)\,\mathrm{d}x + \int_\Omega f^\infty\left(x, \frac{\mathrm{d}D^s u}{\mathrm{d}|D^s u|}\right)\mathrm{d}|D^s u|,$$

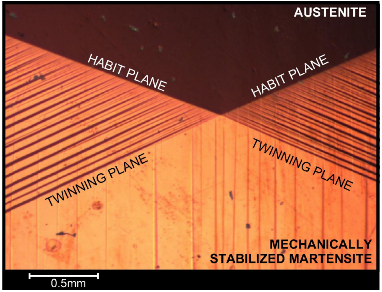
- **Lower semicontinuity in BD** ['11a] for *symmetric* quasi-convex functionals with linear growth depending on *symmetrized* gradients:

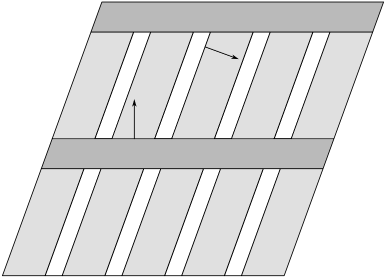
$$\int_\Omega f(x, \mathcal{E}u)\,\mathrm{d}x + \int_\Omega f^\infty\left(x, \frac{\mathrm{d}E^s u}{\mathrm{d}|E^s u|}\right)\mathrm{d}|E^s u|, \qquad Eu = \frac{Du + Du^T}{2}.$$

  $\leftrightarrow$ linearized elasticity.
- **"Good" approximation** of BV-functions with piecewise affine ones? ['12c]
- **Crucially important for these questions:** the shape of singularities
- **Methods:** Rigidity and ellipticity arguments, Harmonic Analysis, Geometric Measure Theory techniques, ...

Description of singularities: Young measures

- **Non-(quasi-)convex functionals**: possibly no classical minimizer.
- Description of oscillations/concentrations in (minimizing) sequences:
  $\rightsquigarrow$ **Young measures** & **generalized YM** (DiPerna–Majda) ['10b]
- *Which singularities are allowed in sequences of (symmetric) gradients?*
- **Microstructure**:

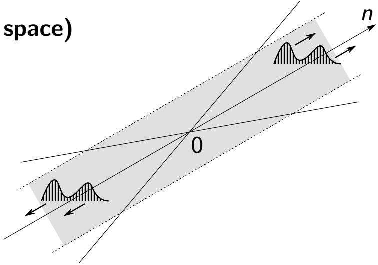




- **Characterization** of *generalized* Young measures generated by sequences of gradients and symmetric gradients ['10b, '11b]

Very recent work ['12b]: Microlocal compactness forms (MCFs)

Phase-space (hybrid real space / Fourier space)
approach to oscillations / concentrations:



In Fourier space: mass wanders out to ∞:

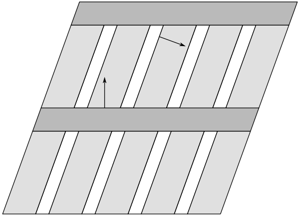
- Microlocal compactness forms (MCFs) are a new tool to study **localization**, **value distribution** and **directions** of oscillations / concentrations (contain Young measure and Tartar–Gerard H-measure).
- They measure the **difference between weak and strong compactness** $\rightsquigarrow$ "very weak regularity theory".
- Easy to "read off" **pointwise constraints** ($u(x) \in Z(x)$) and **differential constraints** (for example curl $u_j = 0$ or div $u_j = 0$) on generating sequence.
- Also for **time-dependent** problems (e.g. conservation laws).

Very recent work ['12b]: Microlocal compactness forms (MCFs), continued

- The theory can be considered a **microlocal analysis** for weak$\leftrightarrow$strong compactness (not C^∞-regularity): for example, there is an analogue of the **wavefront set**.

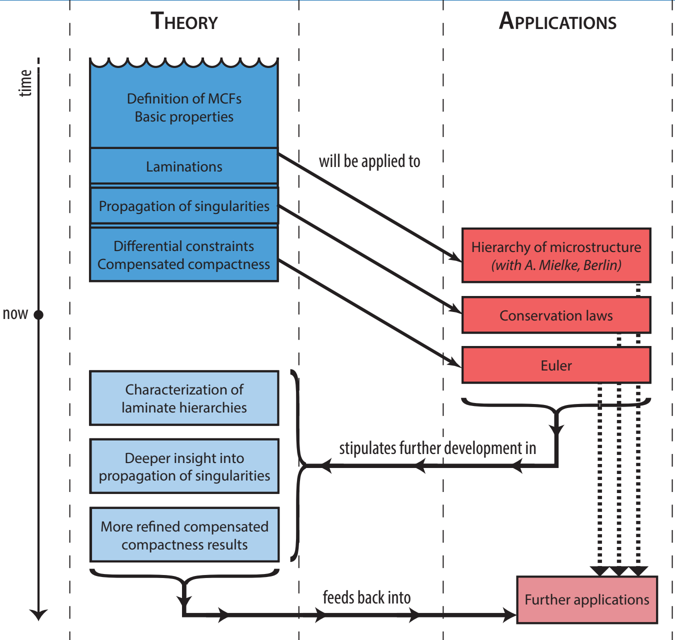
- Applications to **compensated compactness**
  $\rightsquigarrow$ weak-to-strong convergence principles

- MCFs reflect **hierarchy of microstructure** (e.g. laminates):



- Allow **relaxation** of **anisotropic** functionals.

The next five years...



Some references:

['10a] J. Kristensen and F. R., *Relaxation of signed integral functionals in BV*, Calc. Var. Partial Differential Equations **37** (2010), 29–62.

['10b] J. Kristensen and F. R., *Characterization of generalized gradient Young measures generated by sequences in $W^{1,1}$ and BV*, Arch. Ration. Mech. Anal. **197** (2010), 539–598.

['11a] F. R., *Lower semicontinuity for integral functionals in the space of functions of bounded deformation via rigidity and Young measures*, Arch. Ration. Mech. Anal. **202** (2011), 63–113.

['11b] F. R., *Characterization of Young measures generated by sequences in BV an BD*, submitted (December 2011), arXiv:1112.5613, 2011.

['12a] F. R., *Lower semicontinuity and Young measures in BV without Alberti's Rank-One Theorem*, Adv. Calc. Var. **5** (2012), 127–159.

['12b] F. R., *Directional oscillations, concentrations, and compensated compactness via microlocal compactness forms*, submitted.

['12c] J. Kristensen and F. R., *Piecewise affine approximations for functions of bounded variation*, submitted.