Groups, graphs and magic

Katerina Hristova

University of Warwick

October 19, 2016
Women in Maths

- Hypatia
- Maria Agnesi
- Sophie Germain
- Mary Somerville
- Ada Lovelace
- Sofia Kovalevskaya
- Emmy Noether

many, many more..
Augusta Ada Byron

- Born on 10 December 1815 in London, daughter of Lord Byron
- Got her education in science and mathematics by private tutors amongst which was Mary Somerville
- Married William King on 8 July 1835
- Her husband received the title Earl of Lovelace. Thus, she became Countess of Lovelace
- Died on 27 November 1852 at the age 36
Ada’s Achievements

- Worked together with Charles Babbage, who called her “enchantress of numbers”
- Developed what is believed to be the first computer algorithm, the idea of which was to compute Bernoulli numbers
- Realised that the engine can not only be used as a computational machine, but also it can be programmed to do other things, i.e. produce music

**Figure:** Reproduction of “The Analytical Engine” designed by Babbage
Emmy Noether

- Born on 23 March 1882 in Erlagen, Germany
- Got her education in mathematics at the University of Erlagen
- Taught there for 7 years, moved to University of Gottingen and then to Bryn Mawr College, Pennsylvania
- Died on 14 April 1935 at the age 53
The significance of Emmy Noether’s work

- Called “the most significant creative mathematical genius thus far produced since the higher education of women” by Albert Einstein
- Worked on algebraic invariant theory, Galois theory (in particular on the Inverse Galois problem where she made significant progress)
- Gave the first definition of a commutative ring in her book “Theory of Ideals in Ring Domains” (1921)
- Introduced the concepts of ideals in commutative rings and analysed the ascending chain condition on ideals which led to the discovery of the so called Noetherian rings
- Worked on representation theory, key discoveries in the representation of an algebra. Also worked noncommutative algebra, topology, physics etc...
Let $G$ be a group and $\Gamma$ an oriented graph with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$.

**Definition**

$G$ acts on $\Gamma$ if $G$ acts on $V(\Gamma)$ and $E(\Gamma)$. In other words, an action of a group on a graph is a homomorphism $\phi : G \rightarrow \text{Isom}(\Gamma)$.

**Example**

$C_2$ acting on a segment by permuting the vertices.

We assume throughout that the actions are with no inversion.
Two important graphs connected to the group action:

- The quotient graph $\Delta$ is the graph $\Gamma \mod G$.
- The fundamental domain of the action is a subgraph $X$ of $\Gamma$ such that $X \cong \Gamma \mod G$. 
An amalgam of two groups

Let \( G_1, G_2 \) be groups with presentations \( G_1 = \langle S_1 \mid R_1 \rangle \) and \( G_2 = \langle S_2 \mid R_2 \rangle \). Suppose \( A \) is another group such that \( \iota_1 : A \to G_1 \) and \( \iota_2 : A \to G_2 \) are monomorphisms.

**Definition**

The *amalgamated product* of \( G_1 \) and \( G_2 \) over \( A \) is the group defined by the following presentation:

\[
\langle S_1, S_2 \mid R_1, R_2, \iota_1(a) = \iota_2(a), \text{ for all } a \in A \rangle
\]

denoted \( G_1 *_A G_2 \).

**Example**

- free products are amalgams over the trivial subgroup
- van Kampen’s theorem
Main Theorem

There exists a one to one correspondence between the following sets:

\[
\left\{ \text{Groups acting on trees with fundamental domain a segment} \right\} \leftrightarrow \left\{ \text{Groups of the form } G_1 \ast_A G_2 \right\}
\]
Examples: The infinite dihedral group $D_\infty$

The infinite dihedral group is defined in the following way:

$\langle a, b | a^2 = 1, aba = b^{-1} \rangle.$

A simple map sending $a \mapsto x$ and $ab \mapsto y$ enables us to rewrite the presentation above as:

$D_\infty = \langle x, y | x^2 = y^2 = 1 \rangle.$

In particular, we have 2 generators with no relations between them and each of them generates a group of order two. What this means is that:

$D_\infty \cong \mathbb{Z}/2 \ast \mathbb{Z}/2 = \mathbb{Z}/2 \ast \mathbb{Z}/2.$
There exists a tree $\Gamma$ on which $D_\infty$ acts with fundamental domain a segment. Let us find $\Gamma$.

We know that $\text{Stab}(v_1) = \mathbb{Z}/2$, $\text{Stab}(e) = \{1\}$ $\text{Stab}(v_2) = \mathbb{Z}/2$.

Thus $\Gamma$ is the bi-infinite line:
Examples: The modular group $\text{SL}_2(\mathbb{Z})$

$\text{SL}_2(\mathbb{Z})$ acts on the upper-half plane $\mathbb{H}$ on the left by Mobius transformations:

$$
\begin{pmatrix}
    a & b \\
    c & d 
\end{pmatrix} \cdot z = \frac{az + b}{cz + d}, \quad z \in \mathbb{H}.
$$
Take the segment $\gamma \in \mathbb{H}$ with endpoints $v_1 = e^{\pi/3i}$ and $v_2 = i$. Using the action above we find:

- $\text{Stab}(v_1) = \mathbb{Z}/6$
- $\text{Stab}(e) = \mathbb{Z}/2$
- $\text{Stab}(v_2) = \mathbb{Z}/4$

Thus, $\text{SL}_2(\mathbb{Z}) \cong \mathbb{Z}/6 \ast_{\mathbb{Z}/2} \mathbb{Z}/4$. 
Generalisations

This can be generalised to taking trees instead of segments, and amalgamated products of $n$ groups. The theory behind this beautiful result is called Bass-Serre theory.
Thank you for your attention!