The geometry and topology of automorphism groups of free groups

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Abstract

This is an introduction to the study of the group $Out(F_n)$ of outer automorphisms of free group via its on Outer space.

Contents

1 Introduction

Automorphisms of finitely-generated free groups were first systematically studied in the early part of the last century. Important contributions were made by W. Magnus, J. H. C. Whitehead and J. Nielsen. In the 1970's J. Stallings reproved many of those results and established new ones by considering a free group as the fundamental group of a finite graph and modeling automorphisms of the free group by homotopy equivalences of the graph. Some of these results are covered in the course [18], available on Open Math Notes. (Those notes also contain more information about some of the topics in the present notes.)

The present notes begin with the definition of Outer space, which was introduced in the mid 1980's to study the group $Out(F_n)$ of outer automorphisms of the free group F_n [8]. Outer space is a contractible space on which $Out(F_n)$ acts properly. It should be thought of as an analog of the Teichmüller space of a surface S_g, with its action of the mapping class group $Mod(S_g)$ or of a symmetric space with the action of an arithmetic group. The idea is to use the geometry and topology of Outer space and its quotient to study the group $Out(F_n)$.

2 Graphs and Outer space

Marc Culler, who was a student of Stallings, was interested in automorphisms of free groups and proved a 'realization theorem' which said:

Theorem 2.1. *(Culler [6])*
Let $F \leq Out(F_n)$ be a finite subgroup. Then F can be realized on a marked graph.

To make sense of this statement, we make the following definitions:

Definition 2.2. Let R_n be the graph with a single vertex v and n oriented edges labeled $a_1, \dots a_n$, i.e. a *rose* with n petals. A *marking* of an unlabeled graph G is a homotopy eqivalence $g : R_n \to G$. The pair (G, g) is a *marked graph.*

We will permanently identify $F_n = \langle a_1, \dots a_n \rangle$ with $\pi_1(R_n, v)$, so that a marking $g\colon R_n \to G$ induces an isomorphism $g_*\colon F_n \to \pi_1(G, g(v))$.

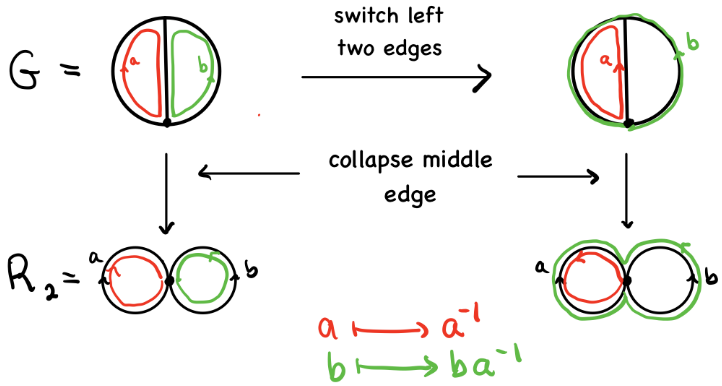
Now suppose that (G, g) is a marked graph, $b = g(v)$ and $f\colon G \to G$ is a graph automorphism. The map f induces an isomorphism $f_*\colon \pi_1(G, b) \to \pi_1(G, f(b))$, and we can form the diagram

$$\begin{array}{ccccc} \pi_1(G,b) & \xrightarrow{f_*} & \pi_1(G,f(b)) & \xrightarrow{\cong} & \pi_1(G,b) \\ {\scriptstyle g_*}\uparrow & & & & {\scriptstyle g_*}\uparrow \\ F_n & & \xrightarrow{\quad\varphi\quad} & & F_n \end{array}$$

The second isomorphism on the top line depends on a choice of path from $f(b)$ to b. Different choices of this path result in automorphisms of $\pi_1(G, b)$ that differ by an inner automorphism, so the automorphism φ on the bottom line is well-defined only up to inner automorphism, i.e. f determines an *outer automorphism* of F_n.

Definition 2.3. In the diagram above we say that (G, g) *realizes* $\varphi \in Out(F_n)$. We say $F \leq Out(F_n)$ is *realized* on (G, g) if (G, g) realizes all elements of F.

In the figure below we illustrate a marked graph realizing the automorphism of $F_2 = \langle a, b \rangle$ that sends $a \to a^{-1}$ and $b \to ba^{-1}$. (For the marking choose any homotopy inverse to the collapse map indicated in the figure.)



Since homotopic maps $f \simeq f^{'} : R_n \to R_n$ give the same map on π_1, we have a map

$$\pi_0(HE(R_n)) \twoheadrightarrow Out(\pi_1(R_n)) = Out(F_n)$$

from the group of *homotopy classes of homotopy equivalences* of R_n to $Out(F_n)$. This map is surjective: each automorphism $\alpha\colon F_n \to F_n$ is induced by the map $f_\alpha\colon R_n \to R_n$ that sends the oriented loop a_i to the loop spelled by $w_i = \alpha(a_i)$.

Exercise 1. Show this map is an isomorphism.

We are now ready to define Outer Space. Recall that in the Teichmuller space of a surface S_g we move around by deforming the metric on S_g. In Outer Space, denoted CV_n, we will move around by varying the lengths of edges in a graph. Here is the formal definition.

Definition 2.4. A *metric graph* is a graph with the path metric induced by assigning a positive real length $l_e > 0$ to each edge e.

Points in Outer Space are equivalence classes of marked metric graphs (G, g) where:

- G is a finite connected graph with no bivalent vertices, no leaves, and edges of positive length
- scaling all edge lengths l_e by a constant λ gives the same point. Alternatively, we can normalize the edge lengths so that $\sum l_e = 1$
- $g\colon R_n \to G$ is a marking.

The equivalence relation is given by $(G, g) \sim (G', g')$ if there exists an isometry $h : G \to G'$ such that $h \circ g \simeq g'$.

$$\begin{array}{ccc} G & \xrightarrow{h} & G' \\ {\scriptstyle g}\uparrow & \nearrow_{g'} & \\ R_n & & \end{array}$$

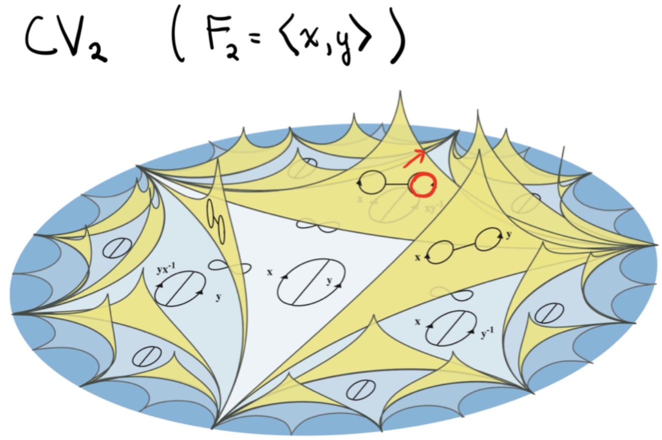
Next we need to define a topology on CV_n. For each combinatorial marked graph (G, g) with k edges, we assign an open $(k-1)$-simplex $\sigma(G, g)$, consisting of all metrics on G whose edge-lengths sum to 1. If $F \subset G$ is a forest, then the *forest collapse* $c_F\colon G \to G/\!\!/F$ is the map that collapses each edge of F to a point. We say $\sigma(G', g')$ is an *interior face* of $\sigma(G, g)$ if there exists a forest collapse c_F such that $(G', g') \sim (G/\!\!/F, c_f \circ g)$:

$$\begin{array}{ccccc} G & \xrightarrow{c_F} & G/\!\!/F & \xrightarrow{\cong} & G' \\ {\scriptstyle g}\uparrow & \nearrow & & \nearrow_{g'} & \\ R_n & & & & \end{array}$$

Interior faces of $\sigma(G, g)$ in this sense are naturally identified with (open) faces of the closed simplex $\overline{\sigma}(G, g)$. Faces which are not interior are said to be *at infinity*. Let $\widehat{\sigma}(G, g)$ be the subspace of $\overline{\sigma}(G, g)$ consisting of $\sigma(G, g)$ and all of its interior faces. If two different simplices have a common interior face, we glue the two simplices along this face; we call this a *face relation*. We now define CV_n as

$$CV_n = \big(\coprod \widehat{\sigma}(G, g)\big)/\text{face relations}$$

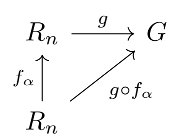
with the quotient topology, Below is an an illustration of CV_2 with this topology:



The letters and arrows on each graph indicate the markings. The same graph appears infinitely often, with different markings. Shrinking the red subgraph to a point gives a face which is “at infinity,’ i.e. this face is not in CV_2.

Exercise 2. Show $\dim(CV_n) = 3n - 4$.

The group $Out(F_n)$ acts on CV_n on the right by changing the marking: realize $\alpha \in Out(F_n)$ by a homotopy equivalence $f_\alpha : R_n \to R_n$. Then $(G, g) \cdot \alpha = (G, g \circ f_\alpha)$.



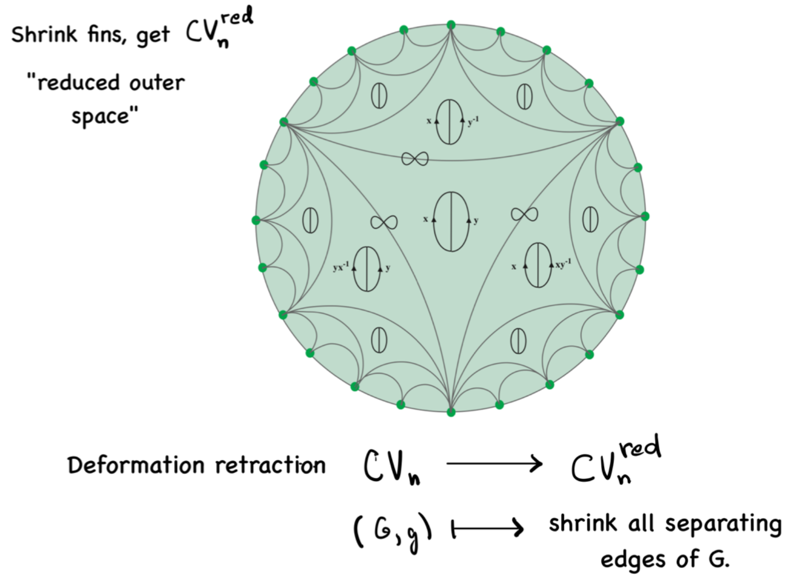
Exercise 3. Prove $Stab(G, g) \cong Isom(G)$, the group of isometries of the metric graph G.

Theorem 2.1 can now be re-interpreted as saying that any finite subgroup of $Out(F_n)$ fixes a point of CV_n. In fact a stronger statement is true: the fixed point set of a finite subgroup is contractible.

2.1 Reduced Outer Space

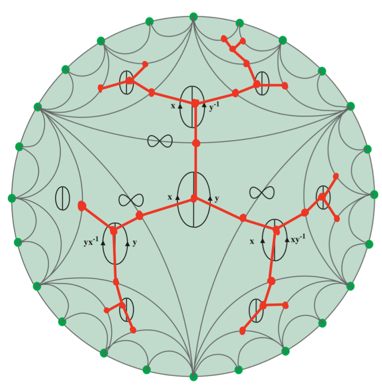
We can define a deformation retraction of Outer Space CV_n onto a subspace, denoted CV_n^{red}, which we call *reduced Outer Space*. The deformation retraction linearly shrinks all separating edges of G to points. The result for $n = 2$ is illustrated below. Note that CV_2^{red} is a manifold.

Exercise 4. Prove CV_n^{red} is not a manifold for $n \geq 3$.



2.2 Spine of Outer Space

The set of open simplices $\sigma(G, g)$ in CV_n is partially ordered by the face relation. The geometric realization of this partially ordered set (poset) is called the *spine of* CV_n, which we denote by K_n. There is a natural inclusion map $K_n \hookrightarrow CV_n$ which sends each vertex of K_n to the barycenter of the corresponding simplex in CV_n. The entire space CV_n deformation retracts onto K_n; the image of CV_n^{red} is denoted K_n^{red}. Here is an image of K_2.



Exercise 5. Show $\dim(K_n) = 2n - 3$.

The original proof that Outer Space is contractible proceeded by showing that the spine is contractible. An picture of the spine in dimension 2 is given below the following theorem.

Theorem 2.5. *(Culler-Vogtmann [8])*
K_n^{red} is contractible, and hence CV_n^{red}, CV_n and K_n are also contractible.

3 Trees in Outer space

Let $(G, g) \in CV_n$. Then $\pi_1(G)$ acts freely on the universal cover $\widetilde{G}$ of G, which is a metric tree, by isometries. The assummption that G is finite and all vertices are at least trivalent implies that this action is *minimal*, i.e. $\widetilde{G}$ has no invariant subtrees. Since the marking g identifies $\pi_1(G)$ with F_n, we have a free minimal isometric action of F_n on a metric simplicial tree.

Let $\rho_1 : F_n \to Isom(T_1)$ and $\rho_2 : F_n \to Isom(T_2)$ be two such actions on trees T_1 and T_2. We define ρ_1 and ρ_2 to be *equivalent* if there exists an isometry $h : T_1 \to T_2$ such that the following diagram commutes, for every $g \in F_n$:

$$\begin{array}{ccc} T_1 & \xrightarrow{h} & T_2 \\ {\scriptstyle \rho_1(g)}\downarrow & & \downarrow{\scriptstyle \rho_2(g)} \\ T_1 & \xrightarrow[h]{} & T_2 \end{array}$$

With this definition, equivalent marked graphs correspond to equivalent actions of F_n, so we have the following alternate definition of CV_n:

Definition 3.1. Outer space CV_n is the set of equivalence classes of free, minimal, isometric actions of F_n on metric simplicial trees.

An automorphism $\alpha \in Out(F_n)$ acts on this set by twisting the actions:

$$\begin{array}{ccc} F_n & \xrightarrow{\rho} & Isom(\tilde{G}) \\ {\scriptstyle \alpha}\uparrow & \nearrow_{\rho\circ\alpha} & \\ F_n & & \end{array}$$

Exercise 6. Prove that an inner automorphism acts trivially on CV_n, i.e. if α is inner, then there exists an isometry $\widetilde{G} \to \widetilde{G}$ which commutes with the action of $\pi_1(G)$.

We will give two ways to define a topology on this set of actions. The first is the *equivariant Gromov-Hausdorff topology*. Actions ρ_1 and ρ_2 are close in this topology if, for every finite set $\{g_i\}$ of elements of F_n and every finite set $\{x_j\}$ in T_k, there is a matching set $\{x'_j\}$ of points in T_ℓ $(k, \ell = 1, 2)$ such that the distances between the $g_i x_j$ and $g_i x'_j$ are within ε for all i and j.

The second way to define a topology depends on the notion of length functions. Given a free action $\rho : F_n \to Isom(T)$ on a metric tree, we can define a length function as follows.

$$\begin{aligned} l_\rho : F_n &\to \mathbb{R}_{>0} \\ g &\mapsto \inf_{x \in T} d(x, gx) \end{aligned}$$

If T is a simplicial tree and the action is free the infimum is realized, and $l_\rho(g)$ is non-zero for all $g \neq 1$.

Exercise 7. Show g and hgh^{-1} have the same length.

We let $\mathcal{C}$ denote the set of conjugacy classes of non-trivial elements of F_n (this can be identified with the set of non-trivial cyclically reduced words). By the above exercise, our length function is actually a map from $\mathcal{C}$ to $\mathbb{R}_{>0}$.

Theorem 3.2. *(Chiswell, Culler-Morgan, Alperin-Bass)*
A free action ρ of F_n on a metric simplicial tree T is determined by its length function

$$l_\rho : \mathcal{C} \to \mathbb{R}_{>0}$$

Since we consider two actions the same if we scale all lengths by a positive constant, this theorem implies that CV_n actually embeds into the infinite-dimensional real projective space $\mathbb{P}^{\mathcal{C}}$. Culler and Morgan also proved

Theorem 3.3. *(Culler-Morgan) The closure of the image of CV_n in $\mathbb{P}^{\mathcal{C}}$ is compact.*

See [7] for the proof of these theorems and further references for the first one.

We now have 3 ways to think of Outer Space, namely as

1. a space of marked metric graphs, which decomposes into a disjoint union of open simplices $\sigma(G, g)$
2. a subspace of $\mathbb{P}^{\mathcal{C}}$, where $\mathbb{P}^{\mathcal{C}}$ has the weak topology
3. a space of free minimal actions on metric trees, with the equivariant Gromov-Hausdorff topology

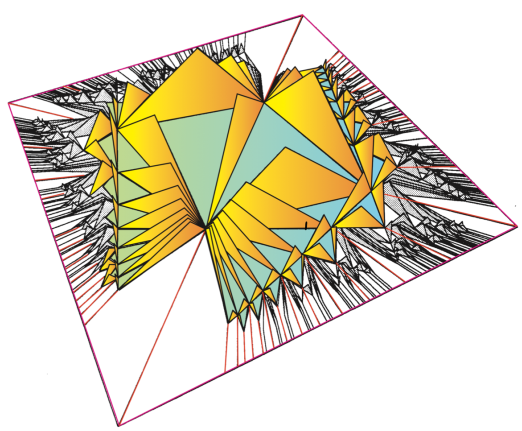
Paulin proved that all of these topologies on CV_n are equivalent. Warning: this is not true for several generalizations of Outer Space which have appeared since!

3.1 Length function compactification

The idea of embedding CV_n into the space of projective length functions on F_n was inspired by Thurston's embedding of the Teichmüller space for S_g into the space of projective length functions on $\pi_1(S_g)$. Thurston showed that there is a finite set of elements of $\pi_1(S_g)$ whose lengths determine the entire length function, so that the image actually projects onto a finite-dimensional projective space, whose closure is therefore automatically compact. Culler and Morgan's theorem that the closure of CV_n is compact does not assume the existence of such a finite-dimensional projection. For $n = 2$, however, there is one:

Lemma 3.4. *Let $F_2 = \langle a, b\rangle$, and let $\rho : F_2 \to Isom(T)$ be an action in CV_2. Then the lengths of $a, b, ab, ab^{-1}, aba^{-1}b^{-1}$ completely determine this action. In particular, $CV_2 \hookrightarrow \mathbb{P}^4$.*

In [9] the embedding $CV_2 \hookrightarrow \mathbb{P}^4$ was constructed explicitly, so that one can draw the following picture of its closure. The boundary square and the pink "hairs" sticking in are in the closure, but not in CV_2.

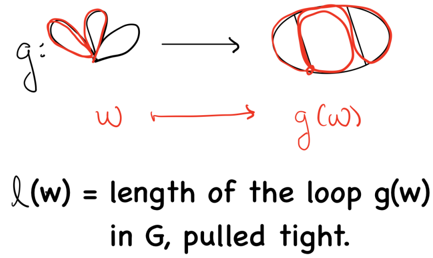


It turns out that $n = 2$ is the only rank for which some projection of $\mathbb{P}^{\mathcal{C}}$ onto a finite-dimensional subspace $\mathbb{P}^k$ restricts to an embedding of CV_n.

Theorem 3.5. *(Smillie-Vogtmann [17])*
Let $n > 2$, and let $W \subset \mathcal{C}$ be a finite set of conjugacy classes in F_n. Then there exist arbitrarily many different actions with the same lengths on every element of W.

In fact, there exists a $(2n - 5)$-parameter family of such actions.

Proof. We will prove this result for the case when $n = 3$. To think about the length of a conjugacy class, we can think about the graph (G, g). Recall $w \in \mathcal{C}$ is a cyclically reduced word.



We let $W = w_1, \ldots, w_k$ be a finite set in $\mathcal{C}$, and let $F_3 = \langle a, b, c \rangle$. Let m be greater than the largest power of c in any w_i. Define an automorphism

$$\begin{aligned} f : F_n &\to F_n \\ a &\mapsto c^m a c^m \\ b &\mapsto b \\ c &\mapsto c \end{aligned}$$

Now in the reduced words $w_i' = f(w_i)$, the letter a is always preceded and followed by a power of c:

$$\ldots cac \ldots$$

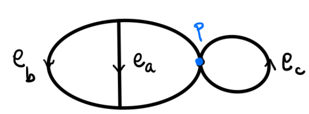
Now apply a new automorphism

$$\begin{aligned} f' : a &\mapsto b^{-k} a b^{-k} \\ b &\mapsto b \\ c &\mapsto c \end{aligned}$$

Now in the words $w_i'' = f'(w_i')$ the letter a is always preceded by cb^k and followed by $b^{-k}c$:

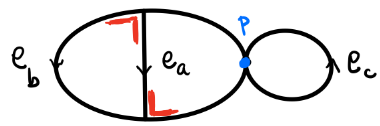
$$\begin{aligned} w' &= \ldots cac \ldots \\ &\downarrow \\ w'' &= \ldots cb^k a b^{-k} c \ldots \end{aligned}$$

Consider the metric graph

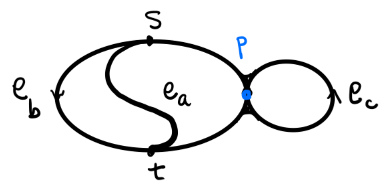


where the unmarked edges have the same length. Mark this graph to get a point in CV_3 by sending a to the loop that uses e_a and the unmarked edges, b to the loop that uses e_b and the unmarked edges. and c to the loop labeled c,

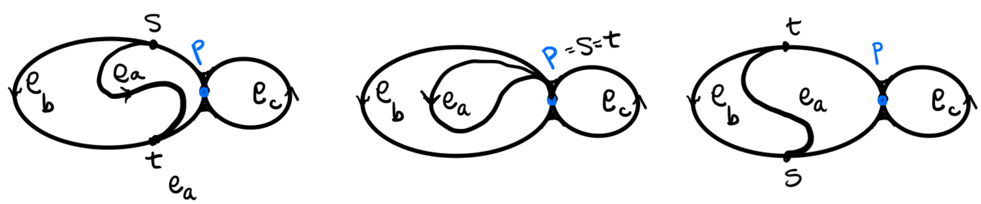
If a does not appear in w_i, then $w_i'' = w_i$, and the loop representing w_i'' must pass through p. If a does appear in w_i, then the loop representing w_i'' also passes through p. Furthermore, the loops representing the w_i'' never make the turns indicated in red below:



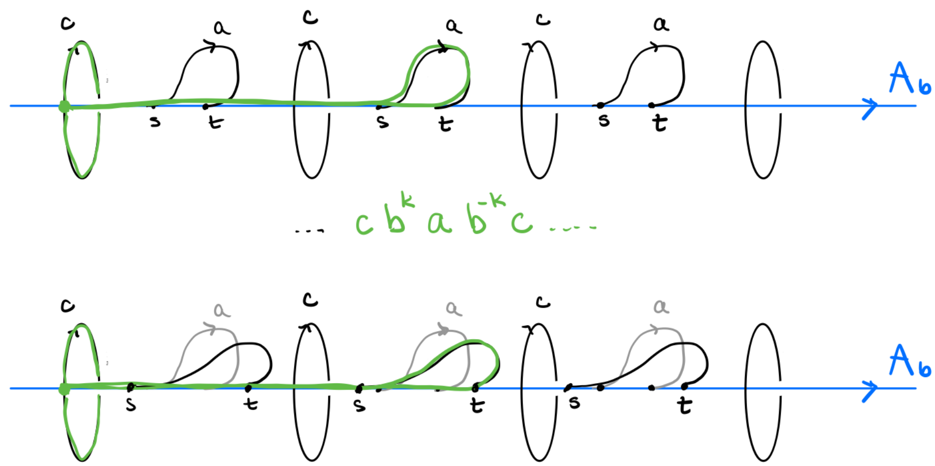
i.e. they must stay on the 'train track' below:



We can then move s and t towards p by ε to get a different metric graph, in which all of the words w_i'' have the same length. We can even move s and t to the point p (at which point the graph is a rose) or past p (which will change the marking).



The process may be easier to understand by looking at the covering space obtained by unwrapping b, i.e. look at the covering space of G corresponding to the subgroup normally generated by a and c. In this covering space b has an axis A_b, and there are loops labeled a and c in every fundamental domain for b on this axis:



Let $\alpha = f' \circ f$, so $w_i'' = \alpha(w_i)$. Since the length of w_i'' is the same in all of these marked metric graphs (G, g), the length of $w_i = \alpha^{-1}(w_i'')$ is the same in all of the graphs $(G, g) \cdot \alpha$. $\square$

3.2 Points in the closure of CV_n

In general, points in the closure of $CV_n \subset \mathbb{P}^{\mathcal{C}}$ are limits of length functions of free simplicial actions, and limits of such limits. A limit of free minimal actions may may develop non-trivial vertex stabilizers, non-trivial edge stabilizers and dense orbits. The trick for proving the closure is compact is that length functions of free simplicial actions are determined by certain inequalities, so limits are also given by inequalities, though strict inequalities may become non-strict. In the limit, one still gets an action on a metric space, but this metric space may not be a simplicial tree. In general the metric space is an $\mathbb{R}$-*tree*:

Definition 3.6. An $\mathbb{R}$-*tree* is a geodesic metric space such that any geodesic triangle is a tripod.

Unlike a simplicial tree, an $\mathbb{R}$-tree may have dense branch points.

In addition, a limit of free actions may not be free, i.e. it may develop non-trivial (and therefore infinite) stabilizers. However, Cohen and Lustig [4] proved that limits of free minimal simplicial actions are always *very small* actions, where

Definition 3.7. An action of F_n on an $\mathbb{R}$-tree is *very small* if

- all arc stabilizers are cyclic
- For all $g \in G \setminus \{1\}$, $Fix(g)$ is an interval, and
- $Fix(g^n) = Fix(g)$

Bestvina and Feign [1] then proved that there are no additional points in the closure $\overline{CV_n}$, i.e. the closure is precisely the set of very small actions. Finally, Gaboriau and Levitt [11] filled out the picture by showing that the dimensions of the closure $\overline{CV_n}$ and its "boundary" $\partial\overline{CV_n} = \overline{CV_n} \setminus CV_n$ are what you expect, namely $\dim(\overline{CV_n}) = 3n-4$, and $dim(\partial\overline{CV_n}) = 3n-5$.

4 Spheres and Outer space

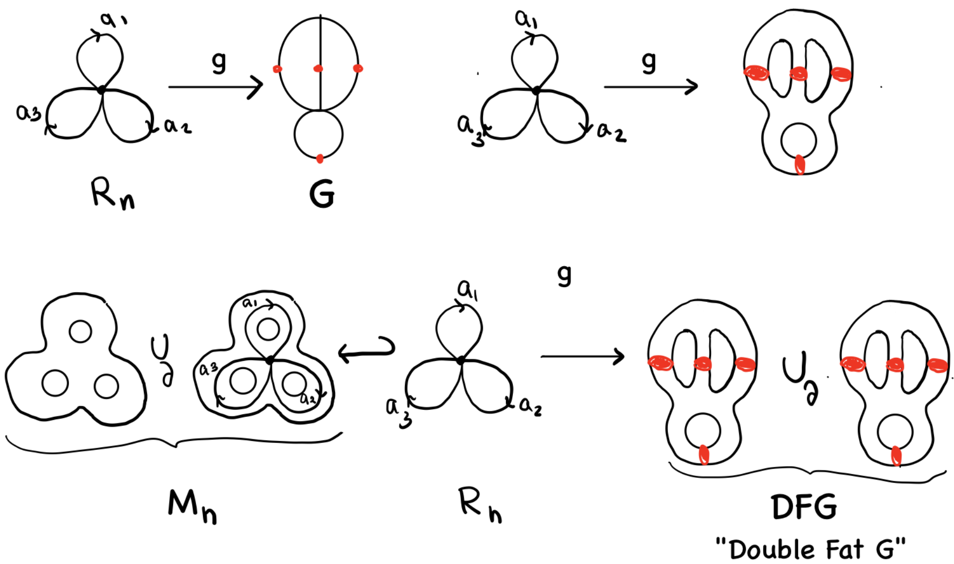
Let us reconsider our first definition of Outer space CV_n as a union of open simplices $\sigma(G,g)$, modulo face relations. We would like to define its simplicial completion CV_n^* as the union of the closures $\overline{\sigma}(G,g)$ of these simplices:

$$CV_n^* = \coprod \overline{\sigma}(G,g)/\text{face relations}$$

The problem with this definition is that it is not clear how to identify two faces that are not in CV_n, i.e. faces at infinity. One way to fix this is to reinterpret points in CV_n as systems of 2-spheres in a particularly nice 3-manifold. This was first explained by Hatcher in [13]. We refer the reader to Hatcher's paper for details of the following construction.

Let (G,g) be a point in CV_n. We associate a system of 2-spheres in a doubled handlebody to (G,g) by:

- Put a red dot in each edge in G
- Fatten G to make a handlebody. The red dots become red disks
- Double the handblebody, i.e. glue together two copies by the identity on the boundary to get a "Double Fat G" (DFG). Red disks now become red 2-spheres, and the inclusion $G \hookrightarrow DFG$ identifies $\pi_1(G)$ with $\pi_1(DFG)$.
- Double the standard rose R_n to obtain $M_n \approx \#_n(S^1 \times S^2)$, and identify $F_n \equiv \pi_1(R_n)$ with $\pi_1(M_n)$ via the inclusion map $R_n \hookrightarrow M_n$.

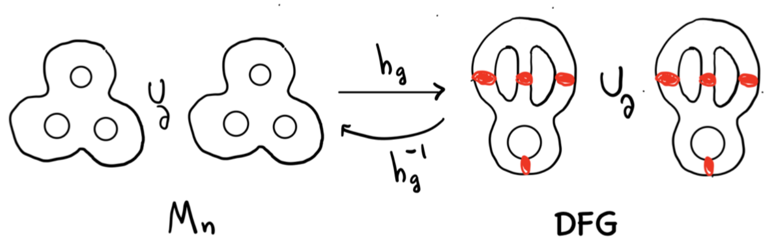


Theorem 4.1. *Let (G, g) be a point of CV_n. Then there is a homeomorphism $h_g : M_n \to DFG$ making the following diagram of isomorphisms commute:*

$$\begin{array}{ccc} \pi_1(R_n) & \xrightarrow{g_*} & \pi_1(G) \\ \downarrow & & \downarrow \\ \pi_1(M_n) & \xrightarrow{(h_g)_*} & \pi_1(DFG) \end{array}$$

where the vertical arrows are induced by inclusion.

Now use the homeomorphism h_g to pull the 2-spheres in DFG back to M_n, thus obtaining a collection of disjoint 2-spheres in M_n. Note that none of these 2-spheres bounds a ball. Equivalent marked graphs give isotopic sets of spheres, so this gives a map from CV_n to isotopy classes of collections of non-trivial disjointly embedded 2-spheres in M_n.

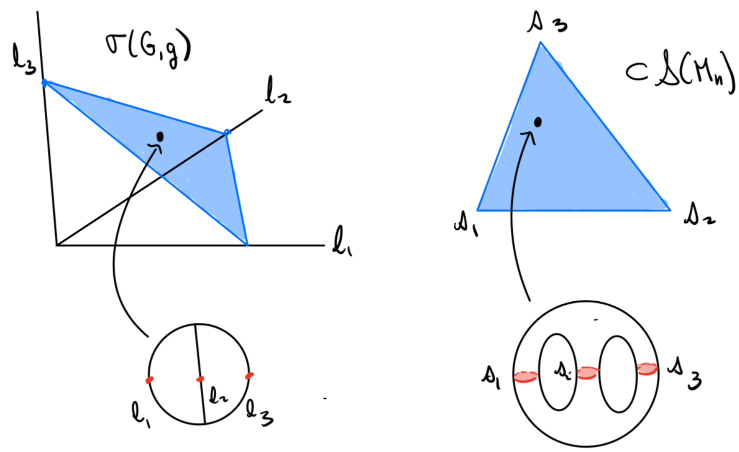


Pull the red 2-spheres back to get a collection of 2-spheres in M.

People familiar with the curve complex of a surface will be not be surprised by the next definition.

Definition 4.2. A sphere in a 3-manifold M is *trivial* if it bounds a ball. A *sphere system* in M is a set of distinct isotopy classes of non-trivial 2-spheres in M that have a set of disjoint representatives. The *sphere complex* $\mathcal{S}(M)$ is the simplicial complex whose k-simplices are sphere systems with $k + 1$ elements.

If the graph G has m edges, then the construction above associates to (G, g) a sphere system in M_n with m spheres. Recall that we have normalized the metric on G so that the sum of the lengths of its edges is 1. If we assign a *weight* to each corresponding sphere equal to the length of the edge, then these weights give barycentric coordinates on the simplex in $\mathcal{S}(M_n)$ corresponding to this sphere system. Thus we have a map from CV_n to $\mathcal{S}(M_n)$.



Since a homeomorphism sends a set of disjoint spheres to another set of disjoint spheres, there is a natural action of the group of homeomorphisms of M_n on $\mathcal{S}(M_n)$. Since homotopic homeomorphisms of M_n are isotopic, we in fact get an action of the mapping class group $\pi_0(Homeo(M_n))$ on $\mathcal{S}(M_n)$. By a theorem of Laudebach [15] the natural map $\pi_0(Homeo(M_n)) \to Out(\pi_1(M_n)) = Out(F_n)$ is surjective, and the kernel of this map is a finite product of $\mathbb{Z}/2\mathbb{Z}$'s, generated by Dehn twists in 2-spheres. The effect of a Dehn twist on a 2-sphere in M_n is minimal: the image of a 2-sphere is isotopic to the 2-sphere we start with. Thus the kernel acts trivially, and we get an action of $Out(F_n)$ on $\mathcal{S}(M_n)$. The map from CV_n to $\mathcal{S}(M_n)$ that we have constructed is equivariant with respect to this action.

We call a sphere system *complete* if each component of its complement in M_n is simply-connected; otherwise it is *incomplete*. If you remove some spheres from an incomplete system it is still incomplete, so the incomplete systems form a subcomplex of $\mathcal{S}(M_n)$. On the other hand, if you remove too many spheres from a complete system it becomes incomplete, so the complete systems do not form a subcomplex. Hatcher identified the image of CV_n under the map to $\mathcal{S}(M_n)$ as follows.

Theorem 4.3. *(Hatcher [13]) Let $S^\infty(M_n) \subset S(M_n)$ be the subcomplex of incomplete sphere systems. Then $S(M_n) \setminus S^\infty(M_n)$ is homeomorphic to CV_n.*

The full complex $\mathcal{S}(M_n)$ is the simplicial completion CV_n^* that we were looking for. Hatcher also proved that CV_n^* is contractible, and his contraction restricts to the subspace CV_n, giving another proof that CV_n is contractible.

A single sphere s in M_n is *separating* if it cuts M_n into two pieces A and B. By Van-Kampen's Theorem, we can deduce that

$$F_n = \pi_1(\text{A}) *_{\pi_1(s)} \pi_1(\text{B}).$$

If the complement $M_n \setminus s$ of s has only one component, then by Van Kampen's Theorem we have an HNN extension

$$F_n = \pi_1(M_n \setminus s)) *_{\pi_1(s)} .$$

In either case, a sphere can be viewed as a way to "split" F_n over the trivial group $\pi_1(s)$. This is why $S(M)$ is also known as the *free splitting complex*. Bass-Serre theory tells us that a free splitting of F_n is equivalent to an action on a (simplicial) tree with trivial edge stabilizers. It was from this point of view that Handel and Mosher proved the following theorem:

Theorem 4.4. *(Handel-Mosher [12]) $S(M_n)$ is Gromov hyperbolic.*

Hillion and Horbez [14] gave a proof soon afterwards in the language of sphere complexes.

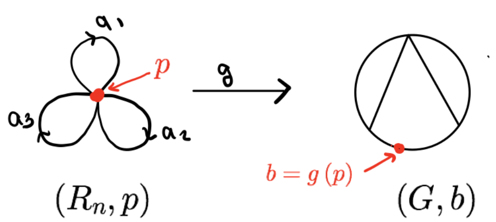
In geometric group theory we like to have spaces which are quasi-isometric to the group we are studying. The group $Out(F_n)$ contains free abelian subgroups, so is not hyperbolic; in particular we observe

Corollary 4.5. *$Out(F_n)$ is not quasi-isometric to $S(M_n)$.*

However, we do have a space quasi-isometric to $Out(F_n)$, namely the spine K_n of CV_n. This is because $Out(F_n)$ acts on K_n with finite stabilizers and compact quotient, so by the Svarc-Milnor lemma K_n is quasi-isometric to $Out(F_n)$. The simplicial completion $CV^* = \mathcal{S}(M_n)$ gives a nice way of describing K_n. Namely, let $\mathcal{S}'(M_n)$ be the barycentric subdivision of $\mathcal{S}(M)$. Then K_n is equal to the subcomplex of $\mathcal{S}'(M_n)$ spanned by vertices *not* in $(S^\infty)'(M_n)$.

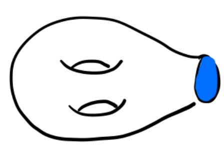
5 Related groups and spaces

So far we've focused on $Out(F_n)$ and Outer space, but similar constructions can be made for related groups. For example we could ask - what about $Aut(F_n)$? The group $Aut(F_n)$ can identified with the group of homotopy classes of homotopy equivalences of a *base pointed* rose, and we can use the same construction to make a space of basepointed marked metric graphs, where all maps and homotopies are required to respect basepoints:



This space was christened *Autre espace* by F. Paulin, and anglicized to "Auter space."

We can also describe Auter space as a subspace of a sphere complex, where we replace M_n by the manifold obtained by punching a 3-ball B^3 out of M_n. We draw $M_n \setminus B^3$ schematically by putting a blue patch on the boundary of the fat rose handlebody; the idea is that when we double this handlebody we should leave the blue patches unglued, so that they form the boundary sphere of $M_n \setminus B^3$.



Double this to get $M_2 \setminus B^3$

We now define a 2-sphere to be trivial if it either bounds a ball or is parallel to the boundary sphere we just created. Note that spheres that were isotopic before we removed the ball may no longer be isotopic. As before, we let $\mathcal{S}(M_n \setminus B^3)$ be the sphere complex for $M_n \setminus B^3$ and $\mathcal{S}^\infty(M_n \setminus B^3)$ the subcomplex of incomplete sphere systems. Then Auter space can be identified with the complement of $\mathcal{S}^\infty(M_n \setminus B^3)$ in $\mathcal{S}(M_n \setminus B^3)$, and it has a cocompact spine that is quasi-isometric to $Aut(F_n)$. Hatcher's proof that $\mathcal{S}(M_n) \simeq pt$ also works for $\mathcal{S}(M \setminus B^3)$, and Laudenbach's proof that the mapping class group of M_n modulo Dehn twists is isomorphic to $Out(F_n)$ also shows that the mapping class group of $M_n \setminus B^3$ modulo Dehn twists is isomorphic to $Aut(F_n)$

$$\pi_0(Homeo(M_n \setminus B^3))/\text{Dehn twists} \cong Aut(F_n)$$

Why stop at punching out one 3-ball? We could punch out any number $s \geq 1$ of holes in M_n, and define

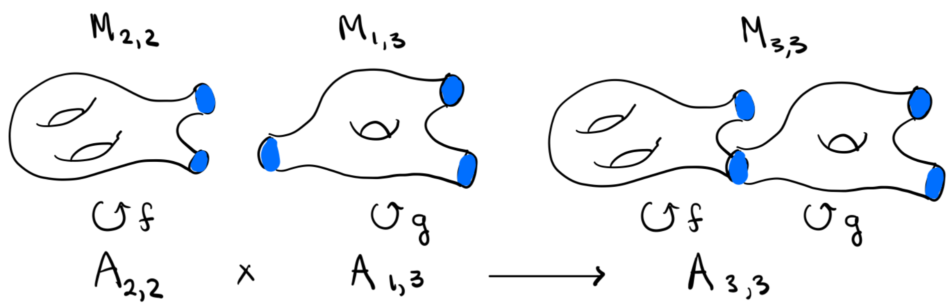
$$M_{n,s} = M_n \setminus \sqcup_s B^3$$

The group

$$A_{n,s} = \pi_0(Homeo(M_{n,s}, \partial))/\text{Dehn twists}$$

consisting of homotopy classes of homotopy equivalences that fix the boundary, modulo Dehn twists, acts on the sphere complex sphere complex $\mathcal{S}(M_{n,s})$, and there is a subcomplex $\mathcal{S}^\infty(M_{n,s})$ of incomplete sphere systems, There is a spine $K_{n,s}$, whose vertices correspond to complete sphere systems, that is quasi-isometric to $A_{n,s}$. The spine can be described as the subcomplex of the barycentric subdivision of $\mathcal{S}(M_{n,s})$ spanned by incomplete sphere systems.

The manifolds $M_{n,s}$ can be glued together along boundary components to form new manifolds $M_{N,S}$ of the same type. Since elements of $A_{n,s}$ are represented by homeomorphisms that fix the boundary, they can be combined to give an element of $A_{N,S}$. This gives a map from the product of the groups $A_{n,s}$ to $A_{N,S}$ called an *assembly map*.



6 Homology

We have the following spaces with $A_{n,s}$ actions, where $A_{n,0} = Out(F_n)$ and $A_{n,1} = Aut(F_n)$.

$$\begin{aligned} CV^*_{n,s} &: \text{contractible, action is cocompact but not proper} \\ &\cup \\ CV_{n.s} &: \text{invariant, contractible, action is proper but not cocompact} \\ &\cup \\ K_{n.s} &: \text{invariant, contractible, action is both proper and cocompact} \end{aligned}$$

Now we want to use these to learn about the groups $A_{n,s}$.

A classical result of Hurewicz says

Theorem 6.1. *Let X be a contractible space on which G acts freely and properly. Then $H_i(X/G)$ is an invariant of G.*

Many of our actions are proper, but none are free. For example, for $(G, g) \in CV_{n,s}$, $Stab_{A_{n,s}}(G, g)$ is isomorphic to the group of isometries of G that fix its leaves. Although this may not be trivial, it *is* at least a finite group, and the cohomology of a finite group H with coefficients in a field $\Bbbk$ of characteristic 0 (such as $\Bbbk = \mathbb{Q}, \mathbb{R}$ or $\mathbb{C}$), is trivial

$$H_*(H; \Bbbk) = 0 :$$

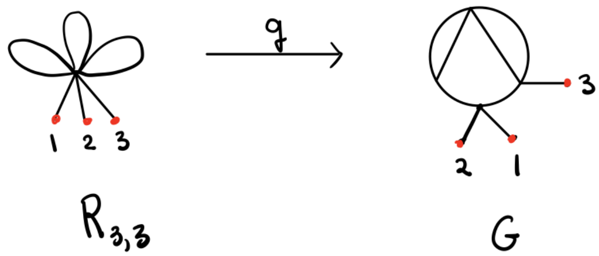
This fact can be used to show that

$$H_*(G; \Bbbk) \cong H_*(X/G; \Bbbk).$$

From now on we will assume all homology is with trivial rational coefficients. We are interested in the homology of the quotient spaces $CV_{n,s}/A_{n,s}$.

6.1 $CV_{n,s}$ as a space of graphs

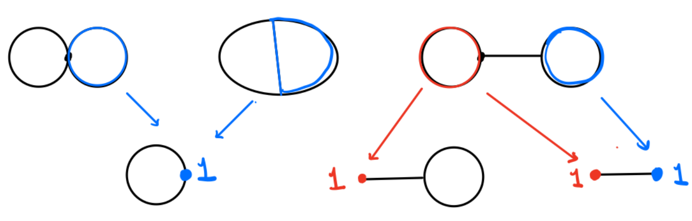
The space $CV_{n,s}$ and group $A_{n,s}$ for $s > 0$ can also be described in terms of graphs. Let $R_{n,s}$ be the the "rose with n petals and s leaves," and define $\partial R_{n,s}$ to be the univalent vertices of $R_{n,s}$, which we assume are numbered $\{1, \ldots s\}$. Then $A_{n,s} = \pi_0(HE(R_{n,s}, \partial R_{n,s}))$, the group of homotopy classes of homotopy equivalences of $R_{n,s}$ that fix $\partial R_{n,s}$; here all maps and homotopies must fix $\partial R_{n,s}$. A point in $CV_{n,s}$ is then a pair (G, g) where G is a finite metric graph with s numbered leaves and no bivalent vertices, and $g\colon R_{n,s} \to G$ is a homotopy equivalence that sends the i-th leaf of $R_{n,s}$ to the i-th leaf of G.



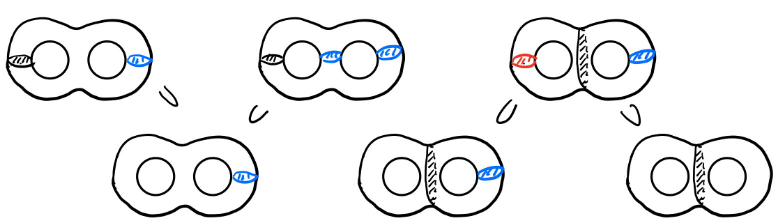
An element of the group $A_{n,s}$ can be modeled by a homotopy equivalence of $R_{n,s}$, and $A_{n,s}$ acts on $CV_{n,s}$ by changing the marking.

6.2 Tropical moduli spaces

Algebraic geometers call metric graphs "tropical curves" and the quotient space $CV_{n,s}/A_{n,s}$. the *moduli space of tropical curves of genus n with s marked points*, denoted $\mathcal{MG}_{n,s}$. Algebraic geometers also prefer to compactify this space by adding what they call "*stable*" tropical curves. These are obtained by collapsing each component of a subgraph a point, and labeling this point by the genus of the subgraph that produced it. It is easy to see that "stable tropical curves" correspond to points in faces of the sphere complex that are at infinity:



Tropical curves



Corresponding sphere systems

Thus the algebraic geometers' compactification $\overline{\mathcal{MG}}_{n,s}$ is just the quotient of the sphere complex $\mathcal{S}(M_{n,s})$, i.e. the simplicial completion $CV^*_{n,s}$, modulo the action of $A_{n,s}$.

In 2021 Chan, Galatius and Payne [3] observed that

$$\overline{\mathcal{MG}}_g \cong \mathcal{C}(S_g)/Mod(S_g)$$

where $\mathcal{C}(S_g)$ is the curve complex of the closed surface S_g of genus g. They then used this to relate $H_*(Mod(S_g))$ to Kontsevich's *commutative graph complex*, which can be identified with the relative chains

$$C_*(CV_n^*/Out(F_n), CV_n^\infty/Out(F_n))$$

Willwacher proved that the homology of the commutative graph complex contains the Grothendiek-Teichmüller Lie algebra $\mathfrak{grt}_1$. Francis Brown showed that $\mathfrak{grt}_1$ contains a free Lie algebra on infinitely many generators. Combining these results gives many new cohomology classes in $Mod(S_g)$.

6.3 Finiteness results

In the exercises you proved $\dim(K_n) = 2n-3$, so in particular $\dim(K_n/Out(F_n)) = 2n-3$. Since $K_n/Out(F_n)$ has the same homology as $Out(F_n)$ we have proved

Proposition 6.2. *$H_i(Out(F_n)) = 0$ for $i > 2n-3$.*

Furthermore, we know that $K_n/Out(F_n)$ is a finite cell complex (the vertices correspond to the isomorphism classes of admissible graphs of rank n). Thus we can conclude

Proposition 6.3. *$H_i(Out(F_n))$ is finitely-generated for all i and n.*

Since the homology is finitely generated the Euler characteristic of $K_n/Out(F_n)$ is defined, and is equal to the alternating sum of the Betti numbers of $Out(F_n)$. Since K_n is the realization of a partially ordered set of marked graphs, the Euler characteristic of $K_n/Out(F_n)$ can be computed by essentially counting isomorphism classes of graphs. However, be warned that the number of isomorphism classes grows extremely fast with n, so this soon becomes impractical for concrete calculations.

6.4 More on the groups $A_{n,s}$

Even if you are only interested in $Out(F_n)$ it is still useful to understand the groups $A_{n,s}$ and the spaces $CV_{n,s}$ and $\mathcal{MG}_{n,s}$ for $s > 0$. Here are a few places they have shown up in the literature so far.

- Proofs of homology stability use all of these spaces and groups. Homology stability says that the groups $H_i(Out(F_n))$ are independent of n for $n >> i$.
- The spaces $CV_{n,s}$ encode the local structure of CV_n.
- The groups $A_{n,s}$ occur in proofs that $Out(F_n)$ is a virtual duality group. Virtual duality is a generatlization of Poincaré duality that works for groups that are not necessarily the fundamental groups of acyclic manifolds. It gives a relation between homology and cohomology in a complimentary dimension.
- The groups $A_{n,s}$ play a role in recent results on the asymptotic behavior of the Euler characteristicof $Out(F_n)$.

Finally, we can use the homology of the groups $A_{n,s}$ for small n and s, where it is relatively easy to calculate, to construct cycles for larger n and s, where almost nothing is known. This uses the assembly maps we defined earlier, such as

$$A_{n,s} \times A_{m,t} \to A_{n+m+k-1,s+t-2k}$$

induced by gluing k boundary spheres of $M_{n,s}$ to boundary spheres of $M_{m,t}$. Since we are using homology with trivial rational coefficients the induced map on homology is

$$H_i(A_{n,s}) \otimes H_j(A_{m,t}) \to H_{i+j}(A_{n+m+k-1,s+t-2k})$$

where we have used the Künneth formula on the domain. For $n = 1$ and $n = 2$ the homology of $A_{n,s}$ has been completely calculated (see [5]). For $n = 1$ and $s = 2k + 1$ odd, the homology in dimension $2k$ is one-dimensional, generated by a class α_k. The image of $\alpha_k \otimes \alpha_k$ under the map

$$H_{2k}(A_{1,2k+1}) \otimes H_{2k}(A_{1,2k+1}) \to H_{4k}(A_{2k+2,0}) = H_{4k}(Out(F_{2k+2}))$$

is called the k-th *Morita class* μ_k. Morita conjectured that μ_k non-trivial for all k. This has been proved for $k \leq 4$ but the question remains open for $k > 4$.

From the asymptotic behavior of the Euler characteristic [2] we know that assembling homology from the groups $A_{1,s}$ cannot give all of the homology of $Out(F_n)$, even if we find a way to prove that the images are non-trivial. We can use the homology of the groups $A_{2,s}$, which has also been computed, to get a significantly larger number of potential classes, but again the problem of proving that they are non-trivial is still still open.

6.5 Outer space and symmetric space

Abelianization $F_n \to Z^n$ induces a map on automorphism groups $Aut(F_n) \to Aut(\mathbb{Z}^n) = Out(\mathbb{Z}^n) = GL(n, \mathbb{Z})$. Since inner automorphisms map to the identity, we get a map

$$\alpha\colon Out(F_n) \to GL(n, \mathbb{Z}).$$

This map is mirrored on the level of spaces by a natural map, called the *Jacobian* from Outer space CV_n to the symmetric space $Q_n = SL(n, \mathbb{R})/SO(n)$. Here's how you define it.

A point in CV_n is a marked graph (g, G). A point in Q_n is a positive definite quadratic form on $\mathbb{R}^n$. So we need to get a positive definite quadratic form from a marked graph. To do this, look at the chain complex for G with coefficients in $\mathbb{R}$:

$$0 \to C_1(G) \xrightarrow{\partial} C_0(G)$$

$C_1(G)$ is a vector space with basis the edges of G. Equip this with the diagonal form, where the diagonal entries are the squares of the lengths of the edges. $H_1(G)$ is the kernel of the boundary map, i.e. it is a subspace of $C_1(G)$, so we can restrict this quadratic form to $H_1(G)$. The marking $g : R_n \to G$ identifies $\mathbb{R}^n \equiv H_1(R_n)$ with $H_1(G)$, giving the desired quadratic form on $\mathbb{R}^n$.

We have $Out(F_n)$ acting on CV_n and $GL(n, \mathbb{Z})$ acting on Q_n. The Jacobian map $J_n\colon CV_n \to Q_n$ is compatible with these actions, i.e. for every $\varphi \in Out(F_n)$ the following square commutes:

$$\begin{array}{ccc} CV_n & \xrightarrow{J_n} & Q_n \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \alpha(\phi)} \\ CV_n & \xrightarrow{J_n} & Q_n \end{array}$$

Therefore we get an induced map on the quotient spaces

$$T_n\colon \mathcal{MG}_n \to SL(n, \mathbb{Z})\backslash Q_n).$$

Algebraic geometers call T_n the "tropical Torelli map" by analogy with the Torelli map on the moduli space of curves. Very recent work by Francis Brown shows how to pull back equivariant differential forms from Q_n to CV_n and proves a type of "Stokes theorem" for these forms [10]. In particular, the forms that generate the cohomology of $GL(n, \mathbb{Z})$ can be used to detect classes in the relative homology of CV_n^* modulo CV_n^∞, which can be identified with Kontsevich's commutative graph homology.

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