CONTINUOUS EIGENFUNCTIONS OF THE TRANSFER OPERATOR FOR THE DYSON MODEL

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Abstract. In this article we prove that there exists a continuous eigenfunction for the transfer operator corresponding to potentials for the classical Dyson model in the subcritical regime for which the parameter α is greater than $3/2$, and we conjecture that this value is sharp.

This is a significant improvement on previous results where the existence of a continuous eigenfunction of the transfer operator was only established for general potentials satisfying summable variations, which would correspond to the parameter range $\alpha > 2$. Moreover, this complements as result by Bissacot, Endo, van Enter and Le Ny [8], who showed that there is no continuous eigenfunction at low temperatures.

Our approach to obtaining these new results involves a novel approach based on random cluster models.

1. Introduction

It is well-known [30] that there exists a continuous and strictly positive eigenfunction of a transfer operator defined on a symbolic shift space with a finite number of symbols such that the potential has summable variations. Here we prove the existence of a continuous eigenfunction for the important special class of Dyson potentials in the subcritical regime up to at least the critical line for Bernoulli percolation, i.e., in particular when the potential does not satisfy the condition of summable variations.

For the Dyson model a continuous eigenfunction means that there is a continuous Radon-Nikodym derivative between the two-sided equilibrium measure (a translation invariant Gibbs measure) and the one-sided Gibbs measure.

A complementary paper to ours is the one by Bissacot, Endo, van Enter, and Le Ny [8], where they show that there is no continuous eigenfunction in the context of the Dyson model for low enough temperatures, although they phrase their result in terms of the lack of the g-measure property. We will describe the connection further below.

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1.1. **Preliminaries and results.** Let T be the left shift on the space $X = \mathcal{A}^{\mathbb{N}}$, where $\mathcal{A} = \{-1, +1\}$. We denote by $C(X)$ the space of continuous functions and by $\mathcal{M}(X)$ the space of probability measures on X. Let $\phi : X \to \mathbb{R}$ be a continuous function which we refer to as the *one-point potential*. The transfer operator $\mathcal{L} = \mathcal{L}_\phi$ is a positive operator $\mathcal{L} : C(X) \to C(X)$ defined by

$$\mathcal{L} f(x) = \sum_{y \in T^{-1}x} e^{\phi(y)} f(y). \tag{1}$$

From the Ruelle-Perron-Frobenius theorem, we can deduce the existence of an *eigenmeasure* $\nu \in \mathcal{M}(X)$ to the dual operator $\mathcal{L}^* : \mathcal{M}(X) \to \mathcal{M}(X)$ that satisfies $\mathcal{L}^*\nu = \lambda\nu$, or equivalently

$$\int \mathcal{L} f \, d\nu = \lambda \int f \, d\nu, f \in C(X),$$

for the maximum eigenvalue $\lambda > 0$.

In this paper, we want to establish the existence of a corresponding continuous *eigenfunction* $h(x)$, $0 < h < \infty$, for the *long-range Dyson model* where

$$\phi(x_0, x_1, \ldots) = x_0 \cdot \beta \sum_{j=1}^{\infty} \frac{x_j}{j^\alpha}, \tag{2}$$

for parameters $\alpha > 1$ and $\beta > 0$.

An *equilibrium measure* $\mu \in \mathcal{M}(X)$ corresponding to ϕ is a minimiser of the free energy $P(\mu; \phi) = \mu(\phi) - H(\mu)$ among the set $\mu \in \mathcal{M}_T(X)$ of all translation invariant probability measures, that is measures such that $\mu = \mu \circ T^{-1}$. If there is an eigenfunction h with $\int h \, d\nu = 1$ then the equilibrium measure μ can be written as $\mu = h\nu$.

It is well-known ([18, 1]) that the long-range Dyson model with potential (2) has a critical value of the parameter β, $\beta_c = \beta_c(\alpha)$, such that we have a unique equilibrium state μ and a unique eigenmeasure ν for $0 < \beta < \beta_c$ and two ergodic states for $\beta_c < \beta$. This $\beta_c(\alpha)$ is also the critical β for *percolation* in the corresponding *random cluster* model with $q = 2$. There is also the critical parameter $\beta_c^1(\alpha)$ for percolation in the Bernoulli random graph model.

We can now present our main result.

Theorem 1. *For $3/2 < \alpha \leq 2$ there exists a continuous eigenfunction of $\mathcal{L}$ whenever $0 < \beta \leq \beta_*$. Here, $\beta_* = \beta_*(\alpha)$ is a critical value satisfying $0 < \beta_c^1 \leq \beta_* \leq \beta_c$.*

We can define β_* as the supremum of $\beta \leq \beta_c$ for which the corresponding random cluster model has a cluster size distribution with an exponentially decreasing tail, see (28). Note that β_* could well be equal to β_c.

Let $\mathrm{var}_n f = \sup\{|f(x) - f(y)| : x_i = y_i, 0 \le i \le n-1\}$. If one assumes *summable variations* $\sum_{n=1}^{\infty} \mathrm{var}_n(\phi) < \infty$ then the existence of a continuous eigenfunction $h(x)$ for $\mathcal{L}$ follows from a classical "cone-argument" used in, for example, Walters [30]. For the Dyson model, summability of variations means that $\alpha > 2$ and that the eigenfunction $h(x)$ is Hölder continuous.

In Theorem 1, we have a continuous eigenfunction in a context when $\alpha > 3/2$ and $0 < \beta < \beta_*$, and thus with the variations $\mathrm{var}_n \phi = O(1/n^{\alpha-1})$, as $n \to \infty$. Note that the condition $\alpha > 3/2$ means that $\sum_{n=1}^{\infty} (\mathrm{var}_n \phi)^2 < \infty$.

If we have a strictly positive continuous eigenfunction h of the transfer operator, then we can recover an equilibrium measure μ ([29, 30]) as the *Doeblin measure* ([7]; g-measure in Keane's terminology [27]) for the Doeblin function (g-function) $g(x)$ defined by

$$g(x) = \frac{e^{\phi(x)}}{\lambda} \cdot \frac{h(x)}{h(Tx)}, \tag{3}$$

since $\sum_{y \in T^{-1}x} g(y) = 1$ for all x. The corresponding transfer operator $\mathcal{L}_g$ is a Markov transition operator and $\mu \in \mathcal{M}(X)$ is a Doeblin measure for $g(x)$ if $\mathcal{L}_g^* \mu = \mu$, i.e., it is the invariant distribution for a stationary Markov process on the state space X. We refer to [23, 25, 16, 6, 21, 19]) for results on Doeblin measures.

In Bissacot, Endo, van Enter and Le Ny [8], they show that at low temperatures for the Dyson model, there is no continuous Doeblin function $g(x)$ that represents the Gibbs measure, and this gives a counterexample to the existence of a continuous eigenfunction of the transfer operator for the Dyson model. Fernandez and Maillard [17] proved that a Gibbs measure can be represented by a Doeblin measure in the Dobrushin regime.

There were some extensions made to establish the existence of a measurable eigenfunction bounded away from zero and infinity [10]. Walters proved some regularity (but not continuity) for an eigenfunction ([30], Theorem 5.1) under the so-called Bowen condition.

We conjecture that the condition $\alpha > 3/2$ is sharp in the sense that we do not have a continuous eigenfunction h, $0 < h < \infty$, for the transfer operator for the Dyson model when $\alpha < 3/2$. We are grateful to Aernout van Enter for pointing out that there is support in the mathematical physics literature for such a conjecture, see Endo, van Enter, and Le Ny [14]. One possible approach to prove sharpness might be to use the theory of Gallesco, Gallo, and Takahashi [19] in combination with the observation that $\alpha > 3/2$ means that $\sum_{n=1}^{\infty} (\mathrm{var}_n \phi)^2 < \infty$.

1.2. **The method of proof.** Assume that ν is an eigenmeasure for $\mathcal{L}^*$ and that there is some translation invariant μ which is absolutely continuous with respect to ν, i.e. such that $\mu = h\nu$. We use that the Radon-Nikodym

derivative $h(x) = d\mu/d\nu$ is an eigenfunction of the transfer operator $\mathcal{L}$. This follows from the identities

$$\int g \cdot h \, d\nu = \int (g \circ T) \cdot h \, d\nu = \int \frac{1}{\lambda} \mathcal{L}(g \circ T \cdot h) \, d\nu = \int g \cdot \left(\frac{1}{\lambda} \mathcal{L} h \right) d\nu,$$

where the last equality follows from the definition of $\mathcal{L}$. This hold for all $g \in C(X)$ if only if $\mathcal{L}h = \lambda h$ as elements of $L^1(\nu)$.

Starting with the translation invariant μ and the eigenmeasure ν, we can try to construct the Radon-Nikodym derivative $h(x)$ as the limit of the likelihood ratios $d\mu|_{\mathcal{F}_n}/d\nu|_{\mathcal{F}_n}$, i.e.

$$h(x) = \lim_{n \to \infty} h_n(x) \quad \text{where} \quad h_n(x) = \frac{\mu[x_0, \dots x_n]}{\nu[x_0, \dots, x_n]}. \tag{4}$$

The limit (4) is well-defined ν-almost everywhere by the martingale convergence theorem. If it exists in $L^1(\nu)$ then it equals $h = d\mu/d\nu$ and we can deduce the existence of an eigenfunction h in $L^1(\nu)$. By studying the associated *random cluster model*, we can prove that the limit in (4) is actually *continuous* for the relevant Dyson models.

The proof goes roughly as follows. Let $\nu(x)$ be the eigenmeasure of $\mathcal{L}_\phi^*$ and let $\mu(\bar{x})$, $\bar{x} \in \mathcal{A}^{\mathbb{Z}}$, denote the natural extension of the equilibrium measure μ. We can represent $\nu(x)$ and $\mu(\bar{x})$ as Gibbs measures for the Ising model corresponding to potentials $\Phi(x)$ and $\Phi(\bar{x})$, respectively. Let $\Gamma(V)$ denote the space of graphs on vertex set V. We lift $\nu(x)$ to a spin-cluster model $\nu(\gamma_+, x)$ and $\mu(\bar{x})$ to a spin-cluster model $\mu(\gamma, \bar{x})$, where $\gamma_+ \in \Gamma(\mathbb{N})$ and $\gamma \in \Gamma(\mathbb{Z})$ are random graphs. The bipartition $\mathbb{Z} = \mathbb{Z}_{<0} \uplus \mathbb{N}$ decomposes the graph γ as $\gamma = (\gamma_-, \gamma_0, \gamma_+)$. It follows from the properties of the random cluster model that we can factorise the distribution of γ as

$$\mu(\gamma_-, \gamma_0, \gamma_+) = e^{R(\gamma)} \cdot \nu(\gamma_-) \otimes \eta(\gamma_0) \otimes \nu(\gamma_+),$$

where we prove in Lemma 2 that e^R is an element of L^1. This factorisation gives us an opening to compute the likelihood ratios in (4) and to prove the continuity of the limit using the dominated convergence theorem.

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2. The proof of Theorem 1

2.1. **Configurations, graphs and potentials.**

2.1.1. *Configurations and potentials.* By a configuration space we mean a set of form $\mathcal{A}^S = \{x : S \to \mathcal{A}\}$, where $\mathcal{A}$ is a finite set (the "alphabet") and S (the "sites") is a countable set. We refer to elements $x \in X$ as configurations and we give the space X the usual product topology and sigma-algebra $\mathcal{F}$. For $G \subset S$, we write x_G for the restriction $x|_G : G \to A \in \mathcal{A}^G$ of x to G and $\mathcal{F}_G$ for the σ-algebra generated by x_G.

We use $\bar{F}$ to denote the complement $S \setminus F$ of F. For all F we can decompose $x \in X$ as $x = (x_F, x_{\bar{F}})$. We express that F is a finite subset of S by writing $F \Subset S$. Writing $F \uparrow S$ signifies that $F \Subset S$ runs through an arbitrary increasing sequence of finite sets with limit S. We denote $[x]_F$ the cylinder set $[x]_F = \{y \mid y_F = x_F\}$ at F containing x.

Implicitly, we assume all functions introduced are measurable. For a function $f : X \to \mathbb{R}$ the *variation* at $F \subset S$ is $\operatorname{var}_F f = \sup\{|f(x) - f(y)| : x_F = y_F\}$. A function f is *local* at F if f is $\mathcal{F}_U$-measurable for some finite subset $U \Subset S$ and it is *continuous* if $\lim_{F\uparrow S} \operatorname{var}_F f = 0$.

For two sequences $x, y \in X$, let $\Delta(x, y) \subset S$ denote the set where they are different, i.e. $\Delta(x, y) := \{i : x(i) \neq y(i)\}$. With a *potential* $\phi(x)$ on X, we mean a formal limit $\phi(x) = \lim_{F\uparrow S} \phi_F(x)$ of local functions, $\operatorname{var}_F \phi = 0$, such that the difference

$$\Delta\phi(x, y) := \lim_{F\uparrow S} \phi_F(x) - \phi_F(y)$$

is finite and well defined for any pair of configurations x and y such that $\Delta(x, y)$ is a finite set. We can formally add potentials as long as it is clear that the underlying limits give a well defined equality $\Delta(\phi + \psi)(x, y) = \Delta\phi(x, y) + \Delta\psi(x, y)$ for the differences. Note that, we can consider two potentials $\phi = \lim \phi_F$ and $\psi = \lim \psi_F$ to be equal when it holds for all $F \Subset S$ that $\phi_F(x) - \psi_F(x)$ does not depend on x.

2.1.2. *Graphs.* Let $V^{(2)}$ denote the set of unordered pairs ij of elements $i, j \in V$, i.e. $V^{(2)}$ is $V \times V$ modulo the equivalence relation with equivalence classes $ij = \{(i, j), (j, i)\}$. We consider a *graph* on the vertex set $V = V(G)$ to be a map $G : E \to V$ from a set of edges $E = E(G)$ to $V^{(2)}$. Edges e of the form $G(e) = ii$, $i \in V$, are *loops*. We thus allow for multiple edges and loops. The *complete graph* on V, $K(V)$, is the inclusion map of the non-loops in $V^{(2)}$. Given a bipartition $V = V_+ \uplus V_-$ of V the *complete bipartite* graph $K(V_+, V_-)$ is the inclusion of $V_+ \times V_-$ in $V^{(2)}$.

A spanning subgraph γ of G is a restriction of G to a subset $E(\gamma) \subset E(G)$. We denote by $\Gamma(G)$ the space of spanning subgraphs γ of G and we can represent $\gamma \in \Gamma(G)$ with a configuration $\gamma \in \{0, 1\}^{E(G)}$ or, equivalently, a subset $\gamma \subset E(G)$. If G is the complete graph $K(V)$ or the complete

bipartite graph $K(V_+, V_-)$, we write $\Gamma(V)$ or $\Gamma(V_+, V_-)$ instead. We denote by $G[F]$ the subgraph *induced* on $F \subset V$, which means the restriction $G[F] : G^{-1}(F^{(2)}) \to F^{(2)}$ of G in both the domain and codomain. All spanning subgraphs $\gamma \in \Gamma(G)$ we consider will have finite degrees, i.e.

$$D(F, \gamma) := \sum_{i \in F} \sum_{j \in V} \gamma(ij) < \infty,$$

for all $F \Subset V$.

Consider an equivalence relation $\sim$ on V or, equivalently, a partition $\tilde{V} = V/\!\sim$ into equivalence classes and identifying projection $\pi : V \to \tilde{V}$. A *contraction* $G/\!\sim$ of G along $\sim$ is the graph $G/\!\sim : E(G) \to \tilde{V}^{(2)}$ obtained from G by identifying pairs in the codomain along $\sim$. Note that $E(G) = E(G/\!\sim)$ so $\Gamma(G)$ and $\Gamma(G/\!\sim)$ are equal as sets. If $F \Subset V$ then we write G^F for the local contraction at F obtained from the equivalence relation "$x, y \in F$ or $x = y$", i.e. by contracting all vertices in F.

2.1.3. *Clusters and decomposition along a cut* (V_+, V_-). For a graph $\gamma \in \Gamma(V)$, let $\mathcal{C}(\gamma) = \{C\}$ denote the partition of V into connected components ("clusters"). If V is countably infinite, we consider the number of clusters $\omega(\gamma) = |\mathcal{C}(\gamma)|$ as a potential: The difference

$$\Delta\omega(\gamma, \gamma') = \lim_{F \uparrow V} \omega(\gamma[F]) - \omega(\gamma'[F])$$

for induced subgraphs along $F \uparrow V$ is well defined and finite. We have $|\Delta\omega(\gamma, \gamma')| \leq |\Delta(\gamma, \gamma')|$ for any two graphs γ and γ' with a finite symmetric difference and the limit is eventually constant. Moreover, the potential $\omega(\gamma)$ is continuous at γ unless γ contains more than one infinite cluster.

The analysis of the likelihood ratios in (4) leads us to consider contractions γ^F of a random graphs γ at the finite sets $F = [0, n-1]$. We see that, given $F \Subset V$, the number of clusters $\omega(\gamma) = |\mathcal{C}(\gamma)|$ satisfies the potential equality

$$\omega(\gamma) = \omega_F(\gamma) + \omega(\gamma^F) - 1 \tag{5}$$

where $\omega_F(\gamma)$ is the number of clusters intersecting F and $\omega(\gamma^F)$ is the number of clusters in the contraction γ^F. The constant difference of one is irrelevant for equality between potentials.

We will on occasion consider a bipartition (or "cut") $V = V_+ \uplus V_-$ of the vertex set in two sets. Such a cut decomposes a given graph $\gamma \in \Gamma(V)$ into three graphs

$$\gamma = (\gamma_+, \gamma_0, \gamma_-) \in \Gamma(V_+) \times \Gamma(V_+, V_-) \times \Gamma(V_-) \tag{6}$$

and the analysis of the two-sided model with the one-sided rely on analysing the decomposition for random graphs γ. The graphs $\gamma_\pm = \gamma[V_\pm]$ are γ induced on the two vertex parts and the subgraph γ_0 is the bipartite subgraph $\gamma \cap K(V_+, V_-)$ connecting vertices $i \in V_-$ with vertices $j \in V_+$.

A cut gives the potential identity

$$\omega(\gamma) = \omega(\gamma_+) + \omega(\gamma_-) - |\gamma_0| + R(\gamma), \tag{7}$$

where the cardinality $|\gamma_0|$ of γ_0 is taken as a linear potential and where $R(\gamma)$ is a correction term. Since $\omega(\gamma \setminus \gamma_0) = \omega(\gamma_+) + \omega(\gamma_-)$, we obtain the following expression for R

$$R(\gamma) = |\gamma_0| - (\omega(\gamma \setminus \gamma_0) - \omega(\gamma)) = |\gamma_0| - (\omega(\gamma_+) + \omega(\gamma_-) - \omega(\gamma)). \tag{8}$$

We could treat $R(\gamma)$ as a potential, but, in the cases of our interest, we shall see that $R(\gamma)$ is a regular function which is small in a certain sense.

The corank of a graph γ is the maximum number of edges that one can remove from γ without increasing the number of components $\omega(\gamma)$. It is also the minimum number one must remove in order to make the graph acyclic. If we consider the bipartite (multi-)graph $\tilde{\gamma}_0 \in \Gamma(V_+/\mathcal{C}(\gamma_+), V_-/\mathcal{C}(\gamma_-))$ obtained from contracting γ_0 along the clusters in $\mathcal{C}(\gamma_+)$ and $\mathcal{C}(\gamma_-)$, then $R(\gamma)$ in (8) equals the corank of $\tilde{\gamma}_0$.

If we contract the terms in the equality (7) at a finite set $F \Subset V_+$, we obtain

$$\omega(\gamma^F) = \omega(\gamma_+^F) + \omega(\gamma_-) - |\gamma_0| + R_F(\gamma). \tag{9}$$

Here $R_F(\gamma)$ equals $\omega(\gamma^F) - \omega(\gamma^F \setminus {\gamma_0}^F) = \operatorname{corank} \tilde{\gamma}_0^F$, where $\tilde{\gamma}_0^F$ denotes the bipartite graph γ_0 contracted first at F and then along $\mathcal{C}(\gamma_+) \uplus \mathcal{C}(\gamma_-)$.

For some fixed chosen order on the edges in γ_0, we say an edge ij is *irrelevant* if there is an edge in γ_0 from the same cluster C in $\mathcal{C}(\gamma_-)$ that precedes it in the chosen order. We define $Q(\gamma_-, \gamma_0)$ as the total number of irrelevant edges. That is,

$$Q(\gamma_-, \gamma_0) = \sum_{C \in \mathcal{C}(\gamma_-)} (D(C, \gamma_0) - 1)_+, \tag{10}$$

where $D(C, \gamma_0) = \sum_{j \in C} \sum_{j \in V_+} \gamma_0(ij)$ is the degree of $C \subset V_-$ in γ_0. Note that Q does not depend on γ_+.

From the interpretation of R_F as the co-rank in $\tilde{\gamma}_0^F$, it follows that for all F

$$R_F(\gamma) \leq Q(\gamma_-, \gamma_0) \tag{11}$$

since, the graph where all vertices in one part has degree 1, is acyclic. Assuming $Q < \infty$, we also have

$$R_F(\gamma) \to Q(\gamma_-, \gamma_0) \quad \text{as } F \uparrow V_+, \tag{12}$$

since all cycles in $\tilde{\gamma}_0^F$ eventually become 2-cycles when $F \uparrow V_+$.

2.2. **Random configurations and random graphs.** Let $\mathcal{M}(X)$ denote the space of probability distributions of configurations. Elements $\alpha \in \mathcal{M}(X)$ are usually written as $\alpha(x)$ in order to make it clear that α is a distribution of the random configuration $x \in X$. We also use $x \sim \alpha$ to denote that x has distribution α. When we need to specify a parameter p of a distribution α we use the form $\alpha(x;p)$. We denote the marginal distribution of x_A by $\alpha(x_A)$. For a partition $A \uplus B = S$, we denote by $\alpha(x_A) \otimes \beta(x_B)$ the product measure $(\alpha \otimes \beta)(x)$ of $\alpha(x_A)$ and $\beta(x_B)$. A measure η is a *Bernoulli measure* if and only if $\eta(x) = \eta(x_A) \otimes \eta(x_B)$ for every bipartition $S = A \uplus B$. To parametrise a general Bernoulli distribution, we use a function $p : S \to \mathcal{M}(\mathcal{A})$ such that $p(s)$ is a probability distribution on $\mathcal{A}$. The product distribution $\bigotimes_{s \in S} p(s)(x_s)$ is the corresponding Bernoulli measure $\eta(x;p)$. We write $\mu(x) \prec \nu(x')$ to state *stochastic domination* between elements in $\mathcal{M}(X)$, which means that we can couple $\mu(x)$ and $\nu(x')$ so that with probability one $x \leq x'$ with respect to the partial order $\cdot \leq \cdot$.

By a *gibbs measure* (small g), we mean Gibbs measures, sufficiently generalised to cover the random cluster setting below. It is a probability distribution $\mu(x) \in \mathcal{M}(X)$ consistent with an associated *specification*. That is, for every finite set $F \Subset S$ and for every (instead of just for μ-almost every) exterior configuration $x_{\bar{F}}$, we have a well defined conditional probability $\mu(x_F \mid x_{\bar{F}})$ of $x_F \in \mathcal{A}^F$ given $x_{\bar{F}} \in \mathcal{A}^{\bar{F}}$. Hence, for each finite set $F \subset S$, the map $x_{\bar{F}} \mapsto \mu(\cdot \mid x_{\bar{F}})$ is a well defined function of the exterior configuration $x_{\bar{F}}$, satisfying the obvious consistency conditions for conditional probabilities. An immediate class of unique gibbs measures are the Bernoulli measures. The corresponding specification $\eta(x_F \mid x_{\bar{F}}; p) = \eta(x_F; p|_F)$ is independent of the boundary $x_{\bar{F}}$.

Note that we do not require the specification of a gibbs measure to be continuous and we are thus talking about gibbs measures in an extended sense. The same specification may have multiple consistent gibbs measures, but, in our context, we can consider all gibbs measure of this form to be *unique* by the subcriticality assumption on β. The reference [15] gives a more thorough and rigourous introduction of Gibbs measures and some of their extensions.

We can *modulate* a gibbs measure $\alpha(x)$ with an "exponential of a potential" $e^{\phi(x)}$ to obtain a set of new gibbs measures denoted by $e^{\phi} \ltimes \alpha$. Let $\mu = {\gamma_0}^{\phi} \ltimes \alpha$ denote an element of this set. For all finite subsets $F \Subset S$, the specification of $e^{\phi} \ltimes \alpha$ at F, i.e. $\mu(x_F \mid x_{\bar{F}})$, is well-defined by the relation

$$\frac{\mu(x_F \mid x_{\bar{F}})}{\mu(y_F \mid x_{\bar{F}})} = \exp\left(\Delta\phi(x,y)\right) \cdot \frac{\alpha(x_F \mid x_{\bar{F}})}{\alpha(y_F \mid x_{\bar{F}})}, \tag{13}$$

where $y = (y_F, x_{\bar{F}})$ and $\alpha(\cdot|x_{\bar{F}})$ is the specification of α. We write $e^{\phi} \ltimes \alpha$ to denote the set of weak limit points, as $F \uparrow S$, of $\mu(x_F \mid x_{\bar{F}}) \cdot \alpha(x_{\bar{F}})$ where $\alpha(x_{\bar{F}})$ denotes the marginal distribution of $x_{\bar{F}}$. In our context, this limit

will be a unique measure and we usually write $\mu = e^{\phi} \ltimes \alpha$. If $g(x) > 0$ is a regular positive function such that $g(x) \in L^1(\alpha)$ then the modulation $g \ltimes \alpha$ simply means taking the product with $g(x)$ and normalising with a constant, i.e.

$$g \ltimes \alpha = \frac{g \cdot \alpha}{\int g(x)\, d\alpha(x)}. \tag{14}$$

We can for example construct the Bernoulli measure $\eta(x;p)$ as the modulation $e^{\phi(x)} \ltimes \upsilon(x)$ where $\upsilon(x)$ denotes the *uniform* Bernoulli measure on X and $\phi(x)$ is the linear potential

$$\phi(x) = \sum_{i \in S} \log p(i)(x_i).$$

Modulation of a Bernoulli measure with a linear potential results in a new Bernoulli measure.

The following rule shows that composition of modulation behaves naturally, i.e.

$$e^{\psi} \ltimes (e^{\phi} \ltimes \alpha) = e^{\psi+\phi} \ltimes \alpha, \tag{15}$$

provided $e^{\phi} \ltimes \alpha$ is unique. Another rule of computation is that of distributivity over products of measures, i.e.

$$e^{\psi(x_A)+\phi(x_B)} \ltimes (\alpha(x_A) \otimes \beta(x_B)) = (e^{\psi} \ltimes \alpha)(x_A) \otimes (e^{\phi} \ltimes \beta)(x_B). \tag{16}$$

Both rule (15) and (16) are immediate when applied to the specification determined by (13). Since we assume that gibbs measures are uniquely specified by their specifications, we can state the rules above as equalities between gibbs measures.

2.2.1. *The random cluster model.* A *random graph model* $\alpha(\gamma)$ is a probability distribution $\alpha(\gamma) \in \mathcal{M}(\Gamma(G))$ on the space $\Gamma(G)$ of (spanning) subgraphs γ of a fixed ambient graph G. We can identify γ with the corresponding configuration $\gamma : E(G) \to \{0,1\}$ so that $\Gamma(G) \cong \{0,1\}^{E(G)}$. In our context of "long range models", we will use the complete graph $G = K(V)$ on a countable vertex set V as the ambient graph and we write $\gamma \in \Gamma(V)$. However, we will almost surely have finite degrees $D(F,\gamma)$.

The Bernoulli graph model $\eta(\gamma;p)$ is uniquely specified by its *edge probabilities* $p : G \to [0,1]$, so that $\gamma_{ij} = 1$ with probability $p(ij)$ independently at each edge $ij \in E(G)$. The finite degree condition holds whenever $\sum_{j:ij \in E(G)} p(ij) < \infty$ for all i. Note that the Bernoulli graph model is independent of the graph structure and is the same for all ambient graphs with the same set of edges.

The *random cluster model* $\mathsf{RC}_q(\gamma;p)$ (or FK-model, see [20]) is the random graph distribution $\mu = q^{\omega(\gamma)} \ltimes \eta(\gamma;p)$ on $\Gamma(G)$ that one obtains if one modulates the Bernoulli graph $\eta(\gamma;p)$ with $q^{\omega(\gamma)}$. Note that, $\mathsf{RC}_1(\gamma;p) = \eta(\gamma;p)$.

Since we focus on the Ising model, we will assume that $q = 2$ unless otherwise stated. We will only work in the sub-critical regimes where the uniqueness of the random cluster model $\mu = q^{\omega(\gamma)} \ltimes \eta(p)$ is well established [20].

It is well-known (see [20]) that the random cluster models satisfy a stochastic domination relation, so that

$$\mathsf{RC}_q(\gamma; p) \prec \mathsf{RC}_{q'}(\gamma'; p') \quad \text{when } p \leq p' \text{ and } q \geq q'.$$

In particular it holds that $\mathsf{RC}(\gamma; p) \prec \eta(\gamma; p)$. For a fixed vertex $\mathrm{o} \in V$, we use $C_{\mathrm{o}}(\gamma)$ to denote the cluster containing the vertex o. It is a fact, see [5], that if we condition on the cluster C_{o} then the distribution of the remaining graph $\gamma \setminus C_{\mathrm{o}}$ is the random cluster model with edge probabilities $p'(ij) = p(ij)1_{i,j \notin C_{\mathrm{o}}}$. It follows that the conditional distribution of $\gamma \setminus C_{\mathrm{o}}$ is dominated by the unconditional distribution of $\gamma \setminus C_{\mathrm{o}}$, so that

$$\mu(\gamma \setminus C_{\mathrm{o}} \mid C_{\mathrm{o}}) \prec \mu(\gamma). \tag{17}$$

The *random spin-cluster model* $\mathsf{RC}((x, \gamma); p)$ is the joint distribution of a random graph γ together with an Ising *spin* configuration, $x \in X = \{+1, -1\}^V$, on the vertex set. One can obtain the distribution

$$\mu(x, \gamma) = \mathsf{RC}((x, \gamma); p)$$

of (x, γ) by first considering the product distribution $\upsilon(x) \otimes \eta(\gamma)$ of the uniform distribution of $x \in X$ and the Bernoulli distribution $\eta(\gamma; p)$ and then conditioning on the event that x and γ are compatible in the sense that no edge in γ connects vertices of opposite spin. An alternate, perhaps more direct, way to derive the spin-cluster distribution $\mu(x, \gamma)$ is to first choose the random graph γ according to the random cluster model $\mu(\gamma)$ and then to choose a spin $x(C) \in \{+1, -1\}$ to each cluster $C \in \mathcal{C}(\gamma)$ uniformly at random.

For our purposes, one should note that the marginal distribution $\mu(x)$ of the spins $x \in X$ is the Gibbs measure corresponding to the potential

$$\Phi(x) = \sum_{ij \in E(G)} -\log(1 - p(ij)) x_i x_j. \tag{18}$$

The marginal distribution $\mu(\gamma)$ of γ is the random cluster model $\mathsf{RC}(\gamma; p)$. Percolation is the event that the random graph γ contains a cluster of infinite size. The almost sure existence of an infinite cluster coincide with the existence of multiple Gibbs measures for the spins $x \in X$.

2.2.2. *Cylinder probabilities.* Let F be a finite subset of V and consider the cylinder $[x]_F$ of spins. Let $B_F(x, \gamma) \in \{0, 1\}$ indicate the event that the graph γ is compatible with the cylinder: That is, that no *path* in γ connects $i, j \in F$ such that the spins x_i and x_j have opposite signs. Recall that $\omega_F(\gamma)$ denote the number of clusters in γ that intersects F. From the alternate

way to derive the spin-cluster distribution, we deduce that the probability of the cylinder is

$$\mu([x]_F) = \int 2^{-\omega_F(\gamma)}\, B_F(x,\gamma)\, d\mu(\gamma), \tag{19}$$

since the probability that the cluster-wise assignment of spins $\{x(C)\}$ give rise to the cylinder $[x]_F$ equals $2^{-\omega_F(\gamma)}$ provided the graph γ is compatible with $[x]_F$.

Note that $\omega_F(\gamma) \le |F|$ is a bounded function. From (14), (19) and the potential equality (5), we arrive at the following expression for the cylinder probability

$$\mu([x]_F) = \frac{1}{\int 2^{-\omega_F}\, d\mu} \cdot \int B_F(x,\gamma)\, d\mu^F(\gamma), \tag{20}$$

where

$$\mu^F(\gamma) = 2^{\omega(\gamma^F)} \ltimes \eta(\gamma^F; p).$$

denote the random cluster model μ *contracted at* F. Since $\Gamma(G) \cong \Gamma(G^F)$ as sets, we may choose to consider $\mu^F(\gamma)$ as a perturbed random cluster distribution for $\gamma \in \Gamma(G)$ or the random cluster distribution for the contraction $\gamma^F \in \Gamma(G^F)$.

2.2.3. *Decomposition of the random cluster model across a cut.* Consider the decomposition in (6) of a graph γ across a cut (V_+, V_-). Let $\mu = 2^{\omega(\gamma)} \ltimes \eta(\gamma)$ be the full "two-sided" random cluster model. Similary, let $\nu(\gamma_\pm) = 2^{\omega(\gamma_\pm)} \ltimes \eta(\gamma_\pm)$ be the "one-sided" random cluster models for the graphs $\gamma_\pm$ on vertex sets $V_\pm$. Assume that $F \Subset V_+$ is a fixed finite subset of V_+. Let also $\mu^F(\gamma)$ and $\nu^F(\gamma_+)$ be the contractions at F of $\mu(\gamma)$ and $\nu(\gamma_+)$, respectively. From (6), it is clear that the Bernoulli distribution $\eta(\gamma) = \eta(\gamma; p)$ factorises into three Bernoulli measures

$$\eta(\gamma) = \eta(\gamma_+) \otimes \eta(\gamma_0) \otimes \eta(\gamma_-). \tag{21}$$

We construct these Bernoulli measures by restricting the given edge probability.

A similar factorisation for the contracted random cluster measure μ^F uses (16). From (21) and (9), we obtain that

$$\begin{aligned}\mu^F(\gamma) &= 2^{\omega(\gamma_+^F)+\omega(\gamma_-)-|\gamma_0|+R_F(\gamma)} \ltimes (\eta(\gamma_+) \otimes \eta(\gamma_0) \otimes \eta(\gamma_-)) = \\ &= 2^{R_F} \ltimes \Big((2^{\omega(\gamma_+^F)} \ltimes \eta(\gamma_+)) \otimes (2^{-|\gamma_0|} \ltimes \eta(\gamma_0)) \otimes (2^{-\omega(\gamma_-)} \ltimes \eta(\gamma_-)) \Big)\end{aligned}$$

and, hence, we have the following factorisation

$$\mu^F(\gamma) = 2^{R_F(\gamma)} \ltimes \left(\nu^F(\gamma_+) \otimes \tilde{\eta}(\gamma_0) \otimes \nu(\gamma_-) \right). \tag{22}$$

where the measure $\tilde{\eta}$ is the Bernoulli measure

$$\tilde{\eta}(\gamma_0) = 2^{-|\gamma_0|} \ltimes \eta(\gamma_0) = \eta(\gamma_0; \tilde{p}) \quad \text{where } \tilde{p} = p/(2-p). \tag{23}$$

Clearly, $\tilde{\eta}(\gamma_0) \prec \eta(\gamma_0)$.

Specialising (22) with $F = \emptyset$, allow us to write the random cluster model $\mu(\gamma)$ as

$$\mu(\gamma) = 2^{R(\gamma)} \ltimes (\nu(\gamma_+) \otimes \tilde{\eta}(\gamma_0) \otimes \nu(\gamma_-)) \,. \tag{24}$$

We shall see that, for the Dyson model, $2^{R(\gamma)} \in L^1(\mu)$, which shows that the marginal distribution $\mu(\gamma_+)$ of γ_+ under the two-sided model is absolutely continuous with respect to the one-sided measure $\nu(\gamma_+)$.

2.3. **The Dyson model.**

2.3.1. *The one-sided and two-sided models.* Let $\bar{X} = \mathcal{A}^{\mathbb{Z}}$ with projection $\bar{X} \to X$ given by $\bar{x} \mapsto x = \bar{x}|_{\mathbb{N}}$. For the analysis of the long range one-dimensional Dyson model, we consider the "two-sided" random spin-cluster model

$$\mu(\bar{x}, \gamma) = \mathsf{RC}((\bar{x}, \gamma); p)$$

with vertex set $V = \mathbb{Z}$ where the edge probabilities are

$$p(ij) = 1 - e^{-J(ij)} \quad \text{where} \quad J(ij) = \frac{\beta}{|i-j|^\alpha}. \tag{25}$$

By (18), the marginal spin distribution $\mu(\bar{x})$ is the Gibbs measure

$$\bar{\Phi}(\bar{x}) = \sum_{i,j} \frac{\beta}{|i-j|^\alpha} \bar{x}_i \bar{x}_j = \sum_{k=-\infty}^{\infty} \bar{\phi}(T^k \bar{x}) \tag{26}$$

where $\bar{\phi}$ is the lift to $\bar{X}$ of one point potential ϕ from (2). By symmetry and uniqueness, $\mu(\bar{x}, \gamma)$ is translation invariant with respect the left shift T on $\mathbb{Z}$. In particular, so is the marginal distribution $\mu(x)$ of $x \in X$.

Taking the cut of $\mathbb{Z} = V_+ \uplus V_-$ where $V_+ = \mathbb{N} = \{0, 1, 2, \dots\}$ and $V_- = \{\dots, -2, -1\}$, we also consider the two "one-sided" spin-cluster models

$$\nu(x_\pm, \gamma_\pm) = \mathsf{RC}(x_\pm, \gamma_\pm; p_\pm).$$

By the vertex map $j \mapsto -1-j$, $j \in \mathbb{N}$, we have an isomorphism $\nu((x_+, \gamma_+)) \cong \nu((x_-, \gamma_-))$. For this one-sided model, the spin distribution $\nu(x)$ is the Gibbs measure corresponding to the one-sided potential $\Phi(x)$

$$\Phi(x) = \sum_{k=0}^{\infty} \phi(T^k x) \tag{27}$$

for $x \in X$, where we drop the subscripting with $\pm$ on the spin sequences. The Gibbs measure $\nu(x)$ for the potential $\Phi(x)$ in (27) is also the eigenmeasure for $\mathcal{L}_\phi^*$ since the definition of $\mathcal{L}_\phi^*$ gives that

$$\mathcal{L}^* \nu(x) = e^{\phi(x)} \cdot \nu(Tx)$$

and the right hand side is, up to the normalising constant $\frac{1}{\lambda}$, the Gibbs measure for $\Phi(x)$ due to the identity $\Phi(x) = \phi(x) + \Phi(Tx)$.

2.3.2. *Cluster size distribution.* It is well-known (see e.g. [1], [20] or [12]) that for all α, $1 < \alpha \leq 2$, and $q \geq 1$ there exists a critical parameter $\beta_c = \beta_c(\alpha, q)$, such that percolation does not occur with probability one for $0 \leq \beta < \beta_c$ (the "sub-critical" regime), while it occurs with probability one for $\beta_c < \beta < \infty$. The random cluster model is moreover unique except for possibly at $\beta = \beta_c$.

We claim that there is a $\beta_* > 0$ such that, for $0 < \beta < \beta_*$, the moment generating function of the cluster size has a positive radius of convergence. In other words: For some $t_0 = t_0(\alpha, \beta) > 0$, such that for $0 < t < t_0$

$$\mathsf{E}\left(e^{t \cdot |C_{\mathrm{o}}(\gamma)|}\right) = \sum_{k=0}^{\infty} \frac{\mathsf{E}(|C_{\mathrm{o}}|^k)}{k!} t^k < \infty, \tag{28}$$

where o is any fixed vertex. The property (28) follows if the cluster size $|C_{\mathrm{o}}|$ has exponentially bounded tails, i.e. if

$$\mathsf{P}(|C_{\mathrm{o}}| > n) \leq A \cdot e^{-t_0 n}/\sqrt{n} \tag{29}$$

for some A and $t_0 > 0$. We can easily see that (29) implies (28) using the identity

$$\mathsf{E}\left(e^{sX}\right) = \int_0^{\infty} s e^{sx} \mathsf{P}(X > x)\, dx.$$

Moreover, it is well-known that (29) holds (see Panagiotis [28] Theorem 1.2.1; Aizenman and Newman [2], Proposition 5.1) for $\beta < \beta_c^1(\alpha)$. By stochastic domination we have $\beta_c^1(\alpha) \leq \beta_c(\alpha)$ and we can thus infer that

$$\beta_c^1(\alpha) \leq \beta_* \leq \beta_c(\alpha) \tag{30}$$

as claimed in discussion following Theorem 1. From now on we assume that $\beta < \beta_*$ and thus that (28) holds.

A major part of our argument depends on the following lemma stating that the moment generating function (MGF) $\mathsf{E}(e^{sQ})$ of Q from (10) is finite for all s.

Lemma 2. *If $\nu(\gamma_-)$ satisfies* (28) *then*

$$\int \exp\left(sQ(\gamma_-, \gamma_0)\right)\, d\tilde{\eta}(\gamma_0)\, d\nu(\gamma_-) < \infty, \tag{31}$$

for every $s > 0$.

2.3.3. *Proof of Lemma 2.* The edge-indicators $\gamma_0(ij)$ distributed according to $\tilde{\eta}$ are independent with Bernoulli distribution $\mathrm{Be}(\tilde{p}(ij))$ where $\tilde{p}(ij) \leq p(ij) = 1 - e^{-J(ij)}$. Since $\mathrm{Be}(1 - e^{-J}) \prec \mathrm{Po}(J)$, we have

$$\tilde{\eta}(\gamma_0) \prec \eta(\gamma_0) \prec \bigotimes_{ij} \mathrm{Po}(J(ij)).$$

We assume an underlying probability space $(\Omega, \mathcal{F}, \mathsf{P})$, carrying the processes $(\gamma_-, \eta) \sim \nu_- \otimes \tilde{\eta}$ as in (31). In addition, we assume a discrete Poisson process $C \mapsto X(C)$, $C \subset V_-$, specified by

$$X(C) := \sum_{i \in C} \sum_{j \in V_+} X(ij) \sim \mathrm{Po}(\lambda(C)), \quad \lambda(C) = \sum_{i \in C} \sum_{j \in V_+} J(ij),$$

where $D(C) = \sum_{i \in C} \sum_j \gamma_0(ij) \leq X(C)$.

Let $Y(C) = (X(C) - 1)_+$. To prove Lemma 2, it is enough to show that

$$\mathsf{E}\left(\exp\left(s \cdot \sum_{C \in \mathcal{C}(\gamma_-)} Y(C)\right)\right) < \infty, \tag{32}$$

for $s > 0$.

Choose $m_0 \geq 2$ so that

$$t = \frac{e^s \beta}{\alpha'(m_0 - 1)^{\alpha'}} < t_0. \tag{33}$$

where $t_0 = t_0(\beta)$ is the radius of convergence from (28). Let $S = \{-1, -2, \ldots, -m_0\}$ and let $\mathcal{C}' = \{C \setminus S \mid C \in \mathcal{C}(\gamma)\}$. Note that, for every $C \subset V_-$, we have

$$Y(C) \leq X(S \cap C) + Y(C \setminus S).$$

Hence, it follows that

$$\sum_{C \in \mathcal{C}(\gamma_-)} Y(C) \leq X(S) + \sum_{C \in \mathcal{C}'} Y(C).$$

Since $\sum_{C \in \mathcal{C}'} Y(C)$ is independent of $X(S) \sim \mathrm{Po}(\lambda(S))$ by disjointness, it is enough to show that $\mathsf{E}\left(e^{s \sum_{C \in \mathcal{C}'} Y(C)}\right) < \infty$. This amounts to show that,

$$K_3 := \mathsf{E}\left(\prod_{C \in \mathcal{C}'} \Psi(C)\right) < \infty, \tag{34}$$

where

$$\begin{aligned} \Psi(C) &= \mathsf{E}\left(e^{s\,(X(C)-1)_+} \mid C\right) \\ &= \sum_{k=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^k}{k!} \cdot e^{s\,(k-1)_+} \\ &= e^{-\lambda} + \lambda e^{-\lambda} + e^{-\lambda - s} \cdot \sum_{k=2}^{\infty} \frac{(e^s \lambda)^k}{k!}, \quad \text{with } \lambda = \lambda(C). \end{aligned} \tag{35}$$

For $i \geq 2$, an elementary integral estimate of $\lambda(\{i\}) = \sum_{j\in\mathbb{N}} J(ij)$ gives that

$$\lambda(\{i\}) \leq \frac{\beta}{\alpha' \cdot (|i|-1)^{\alpha'}},$$

where $\alpha' = \alpha - 1$. Hence, for any $C \subset V_-$

$$\lambda(C) \leq \frac{\beta}{\alpha' \cdot (J(C)-1)^{\alpha'}} \cdot |C|, \tag{36}$$

where $J(C) = \min\{|i| : i \in C\}$ is the rightmost (first) element of C.

Since $e^{-\lambda} + \lambda e^{-\lambda} < 1$ and $e^{-\lambda - s} < 1$, the expression (35) and the bound (36) implies that

$$\Psi(C) \leq 1 + \sum_{k=2}^{\infty} \left(\frac{e^s \beta}{\alpha'(J(C)-1)^{\alpha'}} \right)^k \cdot \frac{|C|^k}{k!} \tag{37}$$

Since $J(C) \geq m_0$, for all $C \in \mathcal{C}'$, we obtain

$$\Psi(C) \leq 1 + w(J) \cdot \Theta(|C|) \tag{38}$$

where

$$w(J) = \frac{(m_0-1)^{2\alpha'}}{(J-1)^{2\alpha'}} \quad \text{and} \quad \Theta(N) = \sum_{k=2}^{\infty} t^k \cdot \frac{N^k}{k!} < \infty,$$

with $t < t_0$ as in (33).

Order the elements in $\mathcal{C}' = \{C_1, C_2, \dots\}$ so that

$$m_0 + 1 = J(C_1) < J(C_2) < \cdots .$$

Note that $J(C_k) = \min\{|i| : i \notin S \cup C_1 \cup \cdots \cup C_{k-1}\}$, i.e., we can determine $J(C_k)$ from the preceding clusters. By induction on (17), we see that

$$\mathsf{P}(|C_k| \mid C_1, \dots, C_{k-1}) \prec \mathsf{P}(|C_{\mathrm{o}}|)$$

and, hence, with $\Theta(N)$ as in (38), we have, for all k,

$$\mathsf{E}\left(\Theta(|C_k|) \mid C_1, C_2, \dots, C_{k-1}\right) \leq \mathsf{E}\left(\Theta(|C_{\mathrm{o}}|)\right) =: \Theta_0. \tag{39}$$

which, since $t < t_0$, is less than ∞ by (28).

To complete the proof, we take conditional expectations in (34) and obtain from (38) and (39) that

$$\begin{aligned} K_3 &\leq \mathsf{E}\left(\prod_{k=1}^{\infty} \mathsf{E}\left(1 + w(J(C_k)) \cdot \Theta(|C_k|) \mid C_1, C_2, \dots, C_{k-1}\right) \right) \\ &\leq \mathsf{E}\left(\prod_{k=1}^{\infty} \left(1 + w(J(C_k)) \cdot \Theta_0\right) \right) \\ &\leq \exp\left(\Theta_0 \cdot \mathrm{const} \cdot \sum_{k=m_0}^{\infty} \frac{1}{(m_0+k-1)^{2\alpha'}} \right) < \infty, \end{aligned}$$

since $J(C_k) \geq m_0 + k$ and $w(k)$ is decreasing in k and the sum is finite due to $2\alpha' > 1$.

□

2.4. **The proof of the theorem.** Recall that our aim is to show that the sequence of the local likelihood ratios

$$h_n(x) = \frac{\mu([x]_n)}{\nu([x]_n)}$$

in (4) is a Cauchy sequence with respect to the supremum norm. That is, we aim to show that

$$\|h_n(x) - h_m(x)\|_\infty \to 0 \quad \text{as } n, m \to \infty \tag{40}$$

which means that the limit $h(x)$ is a continuous function bounded away from 0 and ∞.

We will refer to previous relations concerning representations and cylinder probabilities such as (24), (22), (20), etc. which use notation for a more general setting. We can now specialise to the case where we consider cylinders at $F = [0, n-1] = \{0, 1, \ldots, n-1\}$ where $n \to \infty$. For notational simplicity, we use subscript n instead of F when referring to cylinders, cluster counts and measures, etc. i.e. $[x]_n$ stands for the cylinder $[x]_{[0,n-1]}$ and we write ω_n for $\omega_{[0,n-1]}$, ν^n for ν^F, R_n for R_F, etc.

From (22) and (20), we have in this notation

$$\mu([x]_n) = k_2 \cdot \int B_n(x, \gamma) \cdot 2^{R_n(\gamma)} \, d\left(\nu^n(\gamma_+) \otimes \tilde{\eta}(\gamma_0) \otimes \, d\nu(\gamma_-)\right)$$

and

$$\nu([x]_n) = k_1 \cdot \int B_n(x, \gamma_+) \, d\nu^n(\gamma_+),$$

where $k_2 = \int 2^{-\omega_n(\gamma)} \, d\mu(\gamma)$ and $k_1 = \int 2^{-\omega_n(\gamma_+)} \, d\nu(\gamma_+)$. Hence, by taking the ratio, we have

$$h_n(x) = K_n \cdot \frac{1}{L_n} \cdot I_n(x) \tag{41}$$

where K_n and L_n are

$$K_n = \frac{k_2}{k_1} = \frac{\int 2^{-\omega_n(\gamma)} \, d\mu(\gamma)}{\int 2^{-\omega_n(\gamma_+)} \, d\nu(\gamma_+)} \tag{42}$$

$$L_n = \int 2^{R_n(\gamma)} \, d(\nu^n(\gamma_+) \otimes \tilde{\eta}(\eta) \otimes \nu(\gamma_-)). \tag{43}$$

Only the integral $I_n(x)$ depends on $x \in X$ and we have

$$I_n(x) = \frac{\int B_n(x, \gamma) \cdot 2^{R_n(\gamma)} \cdot \, d(\nu^n(\gamma_+) \otimes \tilde{\eta}(\gamma_0) \otimes \nu(\gamma_-))}{\int B_n(x, \gamma_+) \, d\nu^n(\gamma_+)}. \tag{44}$$

Note that K_n and L_n does not depend on x.

Let

$$B'_n(x,\gamma) = \begin{cases} 1 & B_n(x,\gamma_+) = 0 \\ B_n(x,\gamma) & \text{otherwise.} \end{cases}$$

Thus $B'_n(x,\gamma)$ is zero only if $B_n(x,\gamma_+) = 1$ and their is some cluster C in $\mathcal{C}(\gamma_-)$ sends a pair of edges in γ_0 that joins two clusters in γ_+ that intersect $[1, n-1]$ at positions with opposite spins. We can now rewrite $I_n(x)$ as

$$I_n(x) = \int B'_n(x,\gamma) 2^{R_n(\gamma)} \, d(\hat{\nu}_n(\gamma_+) \otimes \tilde{\eta}(\gamma_0) \otimes \nu(\gamma_-)), \tag{45}$$

where $\hat{\nu}_n(\gamma_+)$ is the probability measure given by

$$\hat{\nu}_n = \frac{B_n(x,\gamma_+) \cdot d\nu^n(\gamma_+)}{\int B_n(x,\gamma_+) \, d\nu^n(\gamma_+)}. \tag{46}$$

In other words, it is the measure $\nu^n(\gamma_+)$ conditioned on γ_+ and $[x]_n$ being compatible.

We define the endpoint of the "last" irrelevant edge as

$$N = \max\{j \in V_+ \mid ij \in \gamma_0, \ i \in C \in \mathcal{C}(\gamma_-), \ D(C) \geq 2\}. \tag{47}$$

By Lemma 2 $\mathsf{P}(N < \infty) = 1$. Let $A(x, \gamma_-, \gamma_0)$ indicate the event that no cluster $C \in \mathcal{C}(\gamma_-)$ sends two edges in γ_0 to opposite spins of x. We have

$$B'_n(x,\gamma) = A(x, \gamma_-, \gamma_0) \quad \text{for all } n \geq N. \tag{48}$$

Moreover, by (12), we have for $n \geq N$

$$R_n(\gamma) = Q(\gamma_-, \gamma_0). \tag{49}$$

We note that the quantities B'_n and R_n are conditionally independent on the event $n \geq N$.

We can now start to establish the convergence of the quantities (42), (43) and (44). From (49) it is clear that the integrals L_n converge

$$L_n \to \int 2^{Q(\gamma_0,\gamma_-)} \, d(\tilde{\eta}(\gamma_0) \otimes \nu(\gamma_-)),$$

as $n \to \infty$. This is finite by Lemma 2.

It also follows from (48) and (49) that conditioning on (γ_-, γ_0) gives

$$g_n(x) := \mathsf{E}(I_n(x) \mid \gamma_-, \gamma_0) = A(x, g_-, \gamma_0) \cdot 2^{Q(\gamma_-,\gamma_0)} > 0$$

on $n \geq N$ and hence $g_n(x) - g_m(x)$ are eventually equal to 0. It follows from dominated convergence that

$$\|I_n(x) - I_m(x)\|_\infty \leq \|2^Q\|_{L^1} \cdot \mathsf{P}(N \geq \min(n,m)),$$

which goes to zero as $n, m \to \infty$. Thus, the functions $\{I_n(x)\}$ constitute a Cauchy sequence with respect to the supremum norm and the limit $I(x) = \lim I_n(x)$ is thus a continuous function and it is also clear that $I_n(x) > 0$ for all x.

In order to establish (40), we must also show that the limit of K_n exists as a value bounded away from zero and infinity. That is, we want to show that

$$\lim_{n\to\infty} \log K_n = \log K \tag{50}$$

exists as a finite value. By the representation (24) of μ, we can write

$$\begin{aligned}\log K_n &= \log \int 2^{-\omega_n(\gamma)}\, d\mu(\gamma) - \log \int 2^{-\omega_n(\gamma_+)}\, d\nu(\gamma_+) \\ &= \log \frac{\int 2^{\omega_n(\gamma_+)-\omega_n(\gamma)+R(\gamma)} \cdot 2^{-\omega_n(\gamma_+)}\, d(\nu(\gamma_+)\otimes\tilde{\eta}(\gamma_0)\otimes\nu(\gamma_-))}{\int 2^{-\omega_n(\gamma_+)}\, d\nu(\gamma_+)} \\ &= \log \int 2^{\omega_n(\gamma_+)-\omega_n(\gamma)+R(\gamma)}\, d(\hat{\nu}_n(\gamma_+)\otimes\tilde{\eta}(\gamma_0)\otimes\nu(\gamma_-))\end{aligned}$$

where $\hat{\nu}_n(\gamma_+)$ is the probability distribution $2^{-\omega_n(\gamma_+)} \ltimes \nu(\gamma_+)$.

But, each cluster C in γ_- can only contribute with at most $(D(C)-1)_+$ to the difference $\omega_n(\gamma_+)-\omega_n(\gamma)$, since each irrelevant edge from C can join at most two clusters intersecting $S=n$. It follows that

$$\omega_n(\gamma_+) - \omega_n(\gamma) \le Q(\gamma_-,\gamma_0) \tag{51}$$

Thus by (11), we have

$$\omega_n(\gamma_+) - \omega_n(\gamma) + R(\gamma) \le 2\cdot Q(\gamma_-,\gamma_0),$$

and, since $\int 2^{2Q}\, d\tilde{\eta}(\gamma_0)\, d\nu(\gamma_-) < \infty$ by Lemma 2, the dominated convergence theorem implies (50). □

References

[1] M. Aizenman, J. Chayes, L. Chayes and C. Newman, Discontinuity of the magnetization in the one-dimensional $1/|x-y|^2$ Ising and Potts models, *J. Statist. Phys.* **50** (1988), 1–40.

[2] M. Aizenman and C. Newman, Tree Graph Inequalities and Critical Behavior in Percolation Models, *J. Statist. Phys.* , **36** (1984), 107–143.

[3] H. Berbee, Chains with Infinite Connections: Uniqueness and Markov Representation, *Probab. Theory Related Fields* **76** (1987), 243–253.

[4] H. Berbee, Uniqueness of Gibbs measures and absorption probabilities, *Ann. Probab.* **17** (1989), no. 4, 1416–1431.

[5] J. van den Berg J., O. Häggstrom, and J. Kahn, Some conditional correlation inequalities for percolation and related processes *Random Structures and Algorithms*, **29** (4) (2006), 417–435.

[6] N. Berger, C. Hoffman and V. Sidoravicius, Nonuniqueness for specifications in $l^{2+\epsilon}$, *Ergodic Theory Dynam. Systems* **38** (2018), no. 4, 1342–1352.

[7] N. Berger, D. Conache, A. Johansson, and A. Öberg, Doeblin measures – uniqueness and mixing properties, preprint.

[8] R. Bissacot, E.O. Endo, A. C. D. van Enter, and A. Le Ny, Entropic Repulsion and Lack of the g-measure Property for Dyson Models, *Comm. Math. Phys.* **363** (2018), 767–788.

[9] M. Campanio and A.C.D. van Enter, Weak versus strong uniqueness of Gibbs measures: a regular short-range example, Letter to the Editor, *J. Phys. A: Math. Gen.* **28** (1995), L45–L47.

[10] L. Cioletti and A. Lopes, Ruelle Operator for Continuous Potentials and DLR-Gibbs Measures, *Discr. Cont. Dyn. Systems* **40** (2020), no. 8, 4625–4652.

[11] W. Doeblin and R. Fortet, Sur des chaâines à liaisons complètes, *Bull. Soc. Math. France* **65** (1937), 132–148.

[12] H. Duminil-Copin, C. Garban, and V. Tassion, Long-range models in 1D revisited, preprint.

[13] F.J. Dyson, Non-existence of spontaneous magnetisation in a one-dimensional Ising ferromagnet, *Commun. Math. Phys.* **12** (1969), no. 3, 212–215.

[14] E.O. Endo, A.C.D. van Enter, and A. Le Ny, The Roles of Random Boundary Conditions in Spin Systems, in *Progress in Probability* **77**, 2020.

[15] A.C.D. van Enter and E.A. Verbitskiy, On the Variational Principle for Generalized Gibbs Measures, *Markov Processes and Related Fields* **10** (2004), no. 3, 411-434.

[16] R. Fernandez and G. Maillard, Chains with Complete Connections: General Theory, Uniqueness, Loss of Memory and Mixing Properties, *J. Statist. Phys.* **118** (2005), 555–588.

[17] R. Fernandez and G. Maillard, Chains with complete connections and one-dimensional Gibbs measures, *Electron. J. Probab.* **9** (2004), no. 6, 145–176.

[18] J. Fröhlich and T. Spencer, The phase transition in the one-dimensional Ising Model with $1/r^2$ interaction energy, *Comm. Math. Phys.* **4** (1982), no. 1, 87–101.

[19] C. Gallesco, S. Gallo, and D.Y. Takahashi, Dynamic uniqueness for stochastic chains with unbounded memory, *Stochastic Process. Appl.* **128** (2018), 689–706.

[20] G. R. Grimmett, *The Random Cluster Model*, Springer 2006.

[21] P. Hulse, An example of non-unique g-measures, *Ergodic Theory Dynam. Systems* **26** (2006), no. 2, 439–445.

[22] J. Jacod and A.N. Shiryaev, *Limit Theorems for Stochastic Processes*, 2nd ed., Grundlehren der mathematischen Wissenschaften **288**, Springer-Verlag 2003.

[23] A. Johansson and A. Öberg, Square summability of variations of g-functions and uniqueness of g-measures, *Math. Res. Lett.* **10** (2003), no. 5–6, 587–601.

[24] A. Johansson, A. Öberg and M. Pollicott, Countable state shifts and uniqueness of g-measures, *Amer. J. Math.* **129** (2007), no. 6, 1501–1511.

[25] A. Johansson, A. Öberg and M. Pollicott, Unique Bernoulli g-measures, *JEMS* **14** (2012), 1599–1615.

[26] A. Johansson, A. Öberg and M. Pollicott, Phase transitions in long-range Ising models and an optimal condition for factors of g-measures, *Ergodic Theory Dynam. Systems* **39** (2019), no. 5, 1317–1330.

[27] M. Keane, Strongly mixing g-measures, *Invent. Math.* **16** (1972), 309–324.

[28] C. Panagiotos, *Interface theory and Percolation*, PhD Thesis, University of Warwick 2020.

[29] Ya.G. Sinai, Gibbs measures in ergodic theory, *Russian Mathematical Surveys* **27** (4) (1972), 21–69.

[30] P. Walters, Ruelle's operator theorem and g-measures, *Trans. Amer. Math. Soc.* **214** (1975), 375–387.

[31] P. Walters, Convergence of the Ruelle Operator, *Trans. Amer. Math. Soc.* **353** (2000), no. 1, 327–347.

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