

An elementary proof that the Rauzy gasket is fractal

Mark Pollicott & Benedict Sewell*

July 12, 2022

Abstract

We present an elementary proof that the Rauzy gasket has Hausdorff dimension strictly smaller than two.

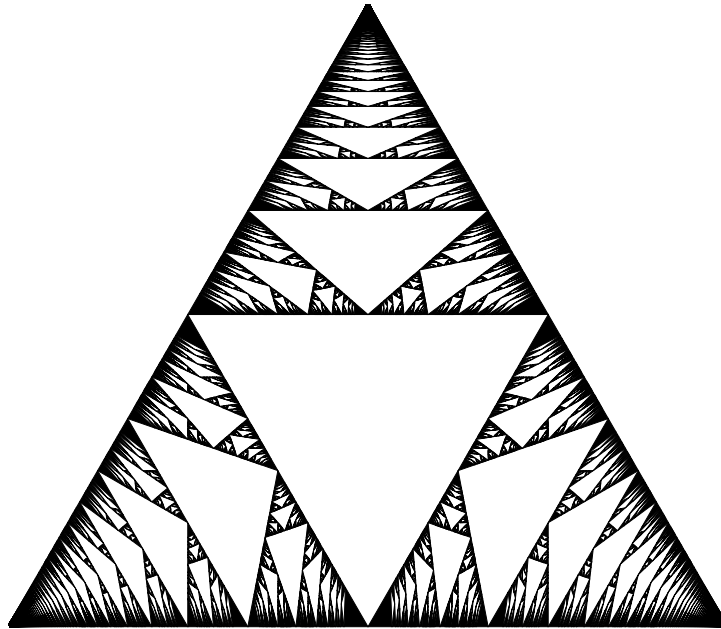


Figure 1: The Rauzy gasket, \mathcal{G}_3 .

1 Introduction

The Rauzy gasket is set which lies within the two-dimensional simplex. It is an important subset of parameter space in numerous dynamical or topological problems (see the introductions of [4] and [5]), having been discovered independently at least three times in different contexts.

*The first author is partly supported by ERC-Advanced Grant 833802-Resonances and EPSRC grant EP/T001674/1, and the second by the Alfréd Rényi Young Researcher Fund.

It was a conjecture of Arnoux [2] that the Hausdorff dimension of this set is smaller than two¹, and this was confirmed by [1]. In this short note we give a completely elementary proof of this fact.

Our method of proof is fairly flexible, and we illustrate this by applying it to a family of higher-dimensional examples \mathcal{G}_d ($d \geq 3$) which generalise the Rauzy gasket. That is, the purpose of this note is to give an elementary proof of the following.

Theorem 1.1. $\dim_H(\mathcal{G}_d) < d - 1$ for any $d \geq 3$.

We note that for the $d = 3$ case, this result has been strengthened with explicit (albeit loose) upper bounds [8, 10], and complemented with a lower bound [9].

In section 2 below, we define \mathcal{G}_d . In section 3 we provide a covering lemma that allows us to reduce to considerations of volumes of certain simplices, and in section 4 we give an explicit formula for these volumes. In section 5, we use the formula to prove some iterative inequalities, which combine with the renewal theorem to get our main technical result. We then verify the assumptions of this technical result in section 6, to complete the proof of Theorem 1.1.

2 Definition of \mathcal{G}_d

We begin with the general definition of a Rauzy gasket in the standard d -simplex, first defined in [2]. These were shown to have non-trivial Hausdorff dimension in [8] using estimates of [3].

Definition 2.1. Fix $d \geq 3$ as the dimension of the ambient space, and let $\Delta \subset \mathbb{R}^d$ be the standard $d - 1$ simplex:

$$\Delta = \{x \in [0, 1]^d : \|x\| = 1\},$$

where $\|x\| = \|x\|_{l^1} = \sum_{j=1}^d v_j$ denotes the usual l^1 norm. For $j \in \{1, 2, \dots, d\}$, define the matrix $M_j \in \{0, 1\}^{d \times d}$ by

$$(M_j)_{i,k} = \begin{cases} 1 & \text{if } i = j \text{ or } i = k, \\ 0 & \text{otherwise,} \end{cases}$$

i.e., M_j has ones on the main diagonal and the j th row, and zeros elsewhere.

We define \mathcal{G}_{d-1} as the attractor of the projectivised maps $\{T_j\}_{j=1}^d$,

$$T_j : \Delta \rightarrow \Delta, \quad T_j(x) = \frac{M_j \cdot x}{\|M_j \cdot x\|},$$

that is, \mathcal{G}_{d-1} is the unique non-empty compact subset of Δ such that

$$\mathcal{G}_{d-1} = \bigcup_{j=1}^d T_j(\mathcal{G}_{d-1})$$

(see [6] for equivalent definitions).

¹It is also a special case of a conjecture of Novikov, that this dimension is strictly between one and two.

Example 2.2 ($d = 3$). The original Rauzy gasket, depicted in Figure 1, corresponds to $d = 3$. The three matrices are

$$M_1 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

and the first map is

$$T_1(x, y, z) = \left(\frac{1}{2-x}, \frac{y}{2-x}, \frac{z}{2-x} \right).$$

We now introduce some notation to be used throughout. We shall also write $|\underline{i}| = n$ as shorthand for $\underline{i} \in \{1, 2, \dots, d\}^n$.

Notation 2.3. Throughout we write, for $\underline{i} = (i_1, \dots, i_n) \in \{1, 2, \dots, d\}^n$,

- $M_{\underline{i}} := M_{i_1} M_{i_2} \cdots M_{i_n}$,
- $T_{\underline{i}} := T_{i_1} \circ T_{i_2} \circ \cdots \circ T_{i_n}$, and
- $\Delta_{\underline{i}} := T_{\underline{i}}(\Delta)$.

3 Covering lemma

Our first step in the proof of Theorem 1.1 is the following lemma.

Lemma 3.1. Given $\delta \in (0, 1)$, $\dim_H(\mathcal{G}_d) \leq d + \delta - 2$, whenever

$$X_n := \sum_{|\underline{i}|=n} \text{vol}_{d-1}(\Delta_{\underline{i}})^\delta \rightarrow 0$$

as $n \rightarrow \infty$, where $\text{vol}_{d-1}(\Delta_{\underline{i}})$ denotes the $(d-1)$ -dimensional volume of $\Delta_{\underline{i}}$.

Note that, where appropriate, we write $f \ll g$ to mean that there exists a constant $C > 0$, depending only on d and δ , such that $f \leq Cg$.

Proof. From the definition of Hausdorff dimension [6], to show $\dim_H(\mathcal{G}_d) \leq d + \delta - 2$ it suffices to exhibit a family of open covers $\{\mathcal{C}_n\}_{n=1}^\infty$ of \mathcal{G} such that

$$\sum_{S \in \mathcal{C}_n} \text{diam}(S)^{d+\delta-2} \rightarrow 0$$

as $n \rightarrow \infty$. To define these covers, it follows from the definition of \mathcal{G}_d , and the fact that $\mathcal{G}_d \subset \Delta$, that

$$\mathcal{G}_d \subset \bigcup_{|\underline{i}|=n} \Delta_{\underline{i}}.$$

Thereby, providing a cover of each $\Delta_{\underline{i}}$ by open balls and taking the union gives an open cover of \mathcal{G}_n . For each $|\underline{i}| = n$, the construction is as follows.

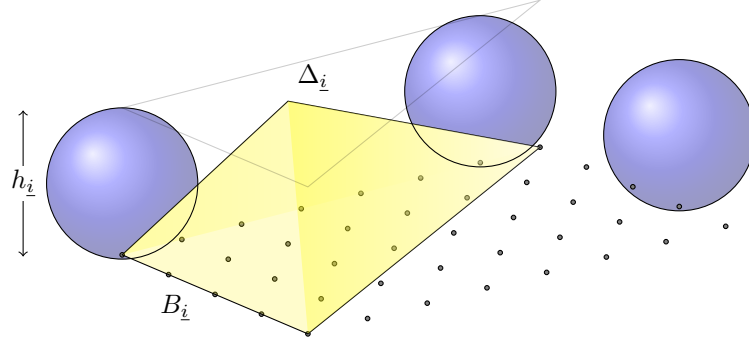


Figure 2: A collection of balls of diameter $h_{\underline{i}}$ covering a copy of $B_{\underline{i}}$. The bases of the balls are denoted by dots. Scaling each ball by a factor of $\sqrt{2}$ about its centre gives a cover of $\Delta_{\underline{i}}$.

Since $T_{\underline{i}}$ is the projectivisation of an injective linear map, $\Delta_{\underline{i}}$ is a $(d-1)$ -simplex. Choose a maximal-volume face $B_{\underline{i}}$ of $\Delta_{\underline{i}}$ as its base, so that $h_{\underline{i}}$, its height measured from $B_{\underline{i}}$, is the smallest such height. A simple induction shows that $B_{\underline{i}}$ is contained in a $(d-2)$ -dimensional hypercuboid, whose side lengths are at least $h_{\underline{i}}$ and whose volume is at most $(d-2)! \text{vol}_{d-2}(B_{\underline{i}})$. It follows that this cuboid and hence $B_{\underline{i}}$ can be covered with $n_{\underline{i}}$ open balls of radius $h_{\underline{i}}$, where

$$n_{\underline{i}} \ll \frac{\text{vol}_{d-2}(B_{\underline{i}})}{h_{\underline{i}}^{d-2}} \ll \frac{\text{vol}_{d-2}(B_{\underline{i}})^{d-1}}{\text{vol}_{d-1}(\Delta_{\underline{i}})^{d-2}}.$$

Treating $\Delta_{\underline{i}} \subset B_{\underline{i}} \times [0, h_{\underline{i}}]$, taking such a cover of $B_{\underline{i}} \times \{h_{\underline{i}}/2\}$ and enlarging the balls by a factor of $\sqrt{2}$ gives a cover of $\Delta_{\underline{i}}$. Doing this construction simultaneously for all such $|\underline{i}| = n$ defines the cover \mathcal{C}_n . Then, since

$$\sum_{S \in \mathcal{C}_n} \text{diam}(S)^{d+\delta-1} \ll \sum_{|\underline{i}|=n} \frac{\text{vol}_{d-2}(B_{\underline{i}})^{d-1}}{\text{vol}_{d-1}(\Delta_{\underline{i}})^{d-2}} \cdot \left(\frac{\text{vol}_{d-1}(\Delta_{\underline{i}})}{\text{vol}_{d-2}(B_{\underline{i}})} \right)^{d+\delta-2} = \sum_{|\underline{i}|=n} \text{vol}_{d-1}(\Delta_{\underline{i}})^\delta \text{vol}_{d-2}(B_{\underline{i}})^{1-\delta}$$

and $\text{vol}_{d-2}(B_{\underline{i}}) \ll 1$, our assumption gives the required convergence to zero as $n \rightarrow \infty$. \square

4 Volume formula

We now show that the volume of $\Delta_{\underline{i}}$ can be expressed simply in terms of the entries of $M_{\underline{i}}$.

Lemma 4.1. *For any tuple \underline{i} ,*

$$\frac{\text{vol}_{d-1}(\Delta_{\underline{i}})}{\text{vol}_{d-1}(\Delta)} = \nu(M_{\underline{i}}),$$

where, for any $N \in \mathbb{R}^{d \times d}$,

$$\nu(N) := \prod_{j=1}^d \left(\sum_{k=1}^d N_{k,j} \right)^{-1} = \prod_{j=1}^d \|N \cdot e_j\|^{-1},$$

and where $e_j = (0, \dots, 0, 1, 0, \dots, 0)$ is the j th standard basis vector in \mathbb{R}^d , i.e., vertex of Δ .

Proof. We define $\Delta_{\underline{i}}^* := \{\lambda v : \lambda \in [0, 1], x \in \Delta_{\underline{i}}\}$, i.e., the d -simplex with vertices consisting of those of $\Delta_{\underline{i}}$ plus the origin, and define Δ^* analogously. Integrating appropriately gives

$$\frac{\text{vol}_{d-1}(\Delta_{\underline{i}})}{\text{vol}_{d-1}(\Delta)} = \frac{\text{vol}_d(\Delta_{\underline{i}}^*)}{\text{vol}_d(\Delta^*)}. \quad (1)$$

The two d -simplices are related by a linear action: $\Delta_{\underline{i}}^* = M_{\underline{i}}^* \cdot \Delta^*$, where

$$M_{\underline{i}}^* := (T_{\underline{i}}(e_1) | T_{\underline{i}}(e_2) | \cdots | T_{\underline{i}}(e_d)) = \left(\frac{M_{\underline{i}} \cdot e_1}{\|M_{\underline{i}} \cdot e_1\|} \mid \frac{M_{\underline{i}} \cdot e_2}{\|M_{\underline{i}} \cdot e_2\|} \mid \cdots \mid \frac{M_{\underline{i}} \cdot e_d}{\|M_{\underline{i}} \cdot e_d\|} \right).$$

That is, $M_{\underline{i}}^*$ is $M_{\underline{i}}$ with each j th column multiplied by a factor of $\|M_{\underline{i}} \cdot e_j\|^{-1}$. Therefore, using that $\det(M_j) = 1$ for each $j = 1, \dots, d$, the right hand side of (1) is none other than

$$\det(M_{\underline{i}}^*) = \nu(M_{\underline{i}}) \det(M_{\underline{i}}) = \nu(M_{\underline{i}}),$$

as required. \square

5 Renewal theorem

We first give a convenient partition of $\{1, \dots, d\}^n$ and decomposition of X_n . Note that these are incomplete, and this gives rise to the remainder term r_n in Lemma 5.3.

Definition 5.1. For each $n > k \geq 1$, let

$$A_{n,k} = \{\underline{i} = (i_1, i_2, \dots, i_n) \in \{1, 2, \dots, d\}^n \mid i_1 = \cdots = i_k \neq i_{k+1}\}$$

and

$$X_{n,k} = \sum_{\underline{i} \in A_{n,k}} \text{vol}_{d-1}(\Delta_{\underline{i}})^\delta.$$

The following lemma allows us to focus on the convergence of $X_{n,1}$ in place of X_n .

Lemma 5.2. $X_n \ll X_{n+2,1}$.

Proof. By a change of variables, we simply have

$$X_{n+2,1} \geq \sum_{|\underline{i}|=n} \text{vol}_{d-1}(T_1 T_2(\Delta_{\underline{i}}))^\delta \geq \min_{x \in \Delta} (\text{Jac}_x T_1 T_2)^\delta \sum_{|\underline{i}|=n} \text{vol}_{d-1}(\Delta_{\underline{i}})^\delta =: C X_n.$$

This constant C is positive as T_1 and T_2 are injective. \square

The next important lemma is our main tool to guarantee that $X_{n,1}$ converges to zero.

Lemma 5.3. For all $n > k \geq 1$,

$$X_{n+1,k+1} \leq b_k X_{n,k}, \quad \text{and} \quad X_{n+1,1} \leq \sum_{j=1}^{n-1} a_k X_{n,j} + r_n,$$

where

$$b_k = \max_{v \in R_k \cap \Delta_1} \|M_1 \cdot v\|^{-d\delta}, \quad a_k = \max_{v \in R_k \cap \Delta_1} \sum_{j=2}^d \|M_j \cdot v\|^{-d\delta}$$

and $r_n \ll n^{(1-d)\delta}$, and where

$$R_k := \text{cl} \left(\bigcup_{j=1}^d T_j^k(\Delta) \setminus T_j^{k+1}(\Delta) \right), \quad \text{i.e.,} \quad R_k \cap \Delta_1 = \text{cl} \left(T_1^k(\Delta) \setminus T_1^{k+1}(\Delta) \right).$$

Proof. First note that, by the definition of $A_{n,k}$, if $\underline{i} = (i_1, \dots, i_n) \in A_{n,k}$ then

- $(i_1; \underline{i}) := (i_1, i_1, i_2, \dots, i_n) \in A_{n+1,k+1}$; and
- $(j; \underline{i}) := (j, i_1, i_2, \dots, i_n) \in A_{n+1,1}$ for any $j \neq i_1$.

A similar statement applies for $\underline{i} \in \{1\}^n \cup \{2\}^n \cup \dots \cup \{d\}^n$.

From this, $\underline{i}' \in A_{n+1,k+1}$ if and only if there exists a unique $\underline{i} \in A_{n,k}$ such that $\underline{i}' = (i_1; \underline{i})$. Consequently,

$$X_{n+1,k+1} = \sum_{\underline{i}' \in A_{n+1,k+1}} \text{vol}_{d-1}(\Delta_{\underline{i}'}^\delta) = \sum_{\underline{i} \in A_{n,k}} \text{vol}_{d-1}(T_{i_1}(\Delta_{\underline{i}}))^\delta. \quad (2)$$

Using the formula in Lemma 4.1 and that

$$\underline{i} \in A_{n,k} \implies \Delta_{\underline{i}} \in R_k := \text{cl} \left(\bigcup_{j=1}^d T_j^k(\Delta) \setminus T_j^{k+1}(\Delta) \right)$$

(i.e., $T_{i_1}(e_j) \in R_k \cap \Delta_{i_1}$ for each j), we have

$$\begin{aligned} \frac{\text{vol}_{d-1}(T_{i_1}(\Delta_{\underline{i}}))^\delta}{\text{vol}_{d-1}(\Delta_{\underline{i}'}^\delta)} &= \prod_{j=1}^d \frac{\|M_{i_1} M_{\underline{i}} \cdot e_j\|^{-\delta}}{\|M_{\underline{i}} \cdot e_j\|^{-\delta}} = \prod_{j=1}^d \left\| M_{i_1} \cdot \frac{M_{\underline{i}} \cdot e_j}{\|M_{\underline{i}} \cdot e_j\|} \right\|^{-\delta} \\ &= \prod_{j=1}^d \|M_{i_1} \cdot T_{\underline{i}}(e_j)\|^{-\delta} \leq \prod_{j=1}^d \max_{v \in R_k \cap \Delta_{i_1}} \|M_{i_1} \cdot v\|^{-\delta} \\ &= \prod_{j=1}^d \max_{v \in R_k \cap \Delta_1} \|M_1 \cdot v\|^{-\delta} = \max_{v \in R_k \cap \Delta_1} \|M_1 \cdot v\|^{-d\delta} \\ &=: b_k, \end{aligned} \quad (3)$$

using symmetry and linearity. Applying this estimate to (2) gives the required inequality:

$$X_{n+1,k+1} \leq \sum_{i \in A_{n,k}} b_k \operatorname{vol}_{d-1}(\Delta_{\underline{i}})^\delta = b_k X_{n,k}.$$

The proof of the second inequality in the lemma is slightly more nuanced. From our first consideration, we have

$$X_{n+1,1} = \sum_{|\underline{i}|=n} \sum_{\substack{1 \leq \omega \leq d: \\ \omega \neq i_1}} \operatorname{vol}_{d-1}(T_\omega(\Delta_{\underline{i}}))^\delta,$$

and we bound the internal sum in two cases:

Case 1: $\underline{i} \in A_{n,k}$ for some k . This case is similar to the proof of (3), but this time we also apply the AM-GM inequality:

$$\begin{aligned} \frac{\sum_{\omega \neq i_1} \operatorname{vol}_{d-1}(T_\omega(\Delta_{\underline{i}}))^\delta}{\operatorname{vol}_{d-1}(\Delta_{\underline{i}})^\delta} &= \sum_{\omega \neq i_1} \prod_{j=1}^d \frac{\|M_\omega M_{\underline{i}} \cdot e_j\|^{-\delta}}{\|M_{\underline{i}} \cdot e_j\|^{-\delta}} = \sum_{\omega \neq i_1} \prod_{j=1}^d \|M_\omega \cdot T_{\underline{i}}(e_j)\|^{-\delta} \\ &\leq \frac{1}{d} \sum_{\omega \neq i_1} \sum_{j=1}^d \|M_\omega \cdot T_{\underline{i}}(e_j)\|^{-d\delta} = \frac{1}{d} \sum_{j=1}^d \sum_{\omega \neq i_1} \|M_\omega \cdot T_{\underline{i}}(e_j)\|^{-d\delta} \\ &\leq \max_{v \in R_k \cap \Delta_{i_1}} \sum_{\omega \neq i_1} \|M_\omega \cdot v\|^{-d\delta} = \max_{v \in R_k \cap \Delta_{i_1}} \sum_{j=2}^d \|M_j \cdot v\|^{-d\delta} \\ &=: a_k. \end{aligned} \tag{4}$$

Summing over $\underline{i} \in A_{n,k}$ hence gives

$$\sum_{i \in A_{n,k}} \sum_{\omega \neq i_1} \operatorname{vol}_{d-1}(T_\omega(\Delta_{\underline{i}}))^\delta \leq \sum_{i \in A_{n,k}} a_k \operatorname{vol}_{d-1}(\Delta_{\underline{i}})^\delta = a_k X_{n,k}.$$

Case 2: $\underline{i} \in \{1\}^n \cup \{2\}^n \cup \dots \cup \{d\}^n$. This is an explicit calculation, using that all $\binom{d}{2}$ of the summands for this case are equal:

$$r_n := \sum_{j=1}^d \sum_{\omega \neq j} \operatorname{vol}_{d-1}(T_\omega T_j^n(\Delta))^\delta \ll \operatorname{vol}_{d-1}(T_1 T_2^n(\Delta))^\delta = (2^{1-d} (2n+1)^{-1} (n+1)^{2-d})^\delta \ll n^{\delta(1-d)},$$

using the formula in Lemma 4.1 and the explicit form of $M_1 M_2^n$. This case completes the proof of the second inequality and hence of the lemma. \square

The only missing piece, before we prove the main technical result of this paper, is to furnish a_k and b_k with values. The proof is simple, but for convenience we defer it to the appendix.

Lemma 5.4. *For each $k \in \mathbb{N}$,*

$$b_k = \left(\frac{k+2}{k+3} \right)^{d\delta} \quad \text{and} \quad a_k = \left(\frac{k+1}{2k+1} \right)^{d\delta} + 2^{-d\delta} (d-2).$$

We now use the proof of a particular case of the renewal theorem [7, p330] to conclude the following result from the previous.

Theorem 5.5. *If $\delta > (d-1)^{-1}$ and $\sum_{k=1}^{\infty} a_k \prod_{j=1}^{k-1} b_j < 1$, i.e.,*

$$3^{d\delta} \sum_{k=1}^{\infty} \left(\frac{k+1}{2k+1} \right)^{d\delta} (k+2)^{-d\delta} + 2^{-d\delta} (d-2)(k+2)^{-d\delta} < 1,$$

then $X_{n,1} \rightarrow 0$ as $n \rightarrow \infty$. Consequently, $\dim_H(\mathcal{G}_d) \leq d-2+\delta$

Proof. Applying the first inequality of Lemma 4.3 to the summands of the second gives

$$X_{n+1,1} \leq \sum_{k=1}^{n-1} a_k \prod_{j=1}^{k-1} b_j X_{n+1-k,1} + r_n.$$

Writing $\lambda_k = a_k \prod_{j=1}^{k-1} b_j$ for succinctness, we then have

$$\begin{aligned} \sum_{n=1}^N X_{n+1,1} &\leq \sum_{n=1}^N \sum_{k=1}^{n-1} \lambda_k X_{n+1-k,1} + \sum_{n=1}^N r_n = \sum_{k=1}^{N-1} \lambda_k \sum_{n=k+1}^N X_{n+1-k,1} + \sum_{n=1}^N r_n \\ &\leq \sum_{k=1}^{N-1} \lambda_k \sum_{j=1}^{N+1} X_{j,1} + \sum_{n=1}^N r_n, \end{aligned}$$

i.e.,

$$\sum_{n=1}^N X_{n+1,1} \leq \frac{x_1 \sum_{k=1}^{N-1} \lambda_k + \sum_{n=1}^N r_n}{1 - \sum_{k=1}^N \lambda_k}.$$

The right hand side is bounded in N by our assumptions, and the result follows. \square

6 Proof of Theorem 1.1

The remainder of the proof of Theorem 1.1 is to show that Theorem 5.5 holds for $\delta = 1$ and any $d \geq 3$, since by continuity, it will then apply for any $\delta < 1$ sufficiently close to 1. More explicitly, we wish to show that

$$\sum_{k=1}^{\infty} \left(\frac{3(k+1)}{(2k+1)(k+2)} \right)^d + (d-2) \left(\frac{3}{2(k+2)} \right)^d = \sum_{k=1}^{\infty} \left(\frac{1}{2k+1} + \frac{1}{k+2} \right)^d + (d-2) \left(\frac{3}{2(k+2)} \right)^d$$

is strictly less than 1. This uses elementary calculus. Since each term on the right hand side is decreasing in d for $d \geq 3$, it suffices to show just the $d = 3$ case, i.e., that

$$\frac{27}{8} \sum_{k=1}^{\infty} \left(\frac{2(k+1)}{(2k+1)(k+2)} \right)^3 + (k+2)^{-3} < 1. \quad (5)$$

This follows by simply bounding the tail of the sum by an integral. For any $n \in \mathbb{N}$, the left hand side of (5) is at most

$$\begin{aligned} & \frac{27}{8} \sum_{k=1}^n \left(\frac{2(k+1)}{(2k+1)(k+2)} \right)^3 + (k+2)^{-3} \quad + \quad \frac{27}{8} \int_{n-1}^{\infty} 8(2x+1)^{-3} + (x+2)^{-3} \, dx \\ &= \frac{27}{8} \sum_{k=1}^n \left(\frac{2(k+1)}{(2k+1)(k+2)} \right)^3 + (k+2)^{-3} \quad + \quad \frac{27}{4(2n-1)^2} + \frac{27}{16(n+1)^2}, \end{aligned}$$

and for $n = 3$, this expression equals

$$\left(\frac{2}{3}\right)^3 + \left(\frac{1}{2}\right)^3 + \left(\frac{9}{20}\right)^3 + \left(\frac{3}{8}\right)^3 + \left(\frac{12}{35}\right)^3 + \left(\frac{3}{10}\right)^3 + \frac{27}{4 \times 5^2} + \frac{27}{16 \times 4^2} = \frac{574898507}{592704000} < 1.$$

Hence (5) holds, completing the proof of the theorem.

A Proof of Lemma 5.4

Noting that $\|M_j \cdot v\| = (2 - v_j)$ and

$$R_k \cap \Delta_1 := \left\{ v \in \Delta \mid \frac{k}{k+1} \leq v_1 \leq \frac{k+1}{k+2} \right\},$$

we have

$$b_k = \max \left\{ (2 - v_1)^{-d\delta} \mid \frac{k}{k+1} \leq v_1 \leq \frac{k+1}{k+2} \right\} = \left(2 - \frac{k+1}{k+2} \right)^{-d\delta} = \left(\frac{k+2}{k+3} \right)^{d\delta}.$$

Regarding the value of a_k , we write $a_k = \max_{R_k \cap \Delta_1} f$, where

$$f(v) := \sum_{j=2}^d (2 - v_j)^{-d\delta}.$$

f is convex (since its summands are convex), so the maximum value it takes on $T_1^j(\Delta) \supset R_k \cap \Delta_1$ is obtained at one of its d vertices. By symmetry, this maximum is either $f(e_1)$ or $f(\frac{k}{k+1}e_1 + \frac{1}{k+1}e_j)$, and we find it is the latter:

$$f(e_1) = 2^{-d\delta}(d-1) \leq \left(\frac{k+1}{2k+1} \right)^{d\delta} + 2^{-d\delta}(d-2) = f\left(\frac{k}{k+1}e_1 + \frac{1}{k+1}e_j \right),$$

since $\frac{k}{k+1}e_1 + \frac{1}{k+1}e_2$ lies in $R_k \cap \Delta_1$, a_k takes the claimed value.

References

- [1] A. Avila, P. Hubert and A. Skripchenko. On the Hausdorff dimension of the Rauzy gasket. *Bull. Soc. Math. France* **144** (2016), no. 3, 539–568. doi:10.24033/bsmf.2722
- [2] P. Arnoux and Š. Starosta. The Rauzy gasket. *Further developments in fractals and related fields*, Trends Math., Birkhäuser/Springer: New York, 2013, 1–23.
- [3] A. Baragar. The exponent for the Markoff-Hurwitz equations *Pacific J. Math.* **182** (1998), no. 1, 1–21. doi:10.2140/pjm.1998.182.1
- [4] R. DeLeo and I. A. Dynnikov. Geometry of plane sections of the infinite regular skew polyhedron $\{4, 6 \mid 4\}$. *Geom. Dedicata* **138** (2009), 51–67. doi:10.1007/s10711-008-9298-1
- [5] I. A. Dynnikov, P. Hubert, A. Skripchenko. *Dynamical systems around the Rauzy gasket and their ergodic properties* (2020). arXiv:2011.15043
- [6] K. Falconer. *Fractal geometry: Mathematical foundations and applications*. Third edition. John Wiley & Sons, Ltd.: Chichester 2014.
- [7] W. Feller. *An introduction to probability theory and its applications*. Vol. I. Third edition. John Wiley & Sons, Ltd.: London 1968.
- [8] C. Fougeron. *Dynamical properties of simplicial systems and continued fraction algorithms* (2020). arXiv:2001.01367
- [9] R. Gutiérrez-Romo and C. Matheus. Lower bounds on the dimension of the Rauzy gasket. *Bull. Soc. Math. France* **148** (2020), no. 2, 321–327. doi:10.24033/bsmf.2807
- [10] M. Pollicott and B. Sewell. An upper bound on the dimension of the Rauzy gasket arXiv:2110.07264

M. POLLICOTT, MATHEMATICS DEPARTMENT, UNIVERSITY OF WARWICK, COVENTRY, CV4 7AL, UK.

B. SEWELL, ALFRÉD RÉNYI INSTITUTE, 13–15 REÁLTONODA UTCA, BUDAPEST, 1053, HUNGARY.