Topics in Fractal Geometry

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Chapter 1

Introduction

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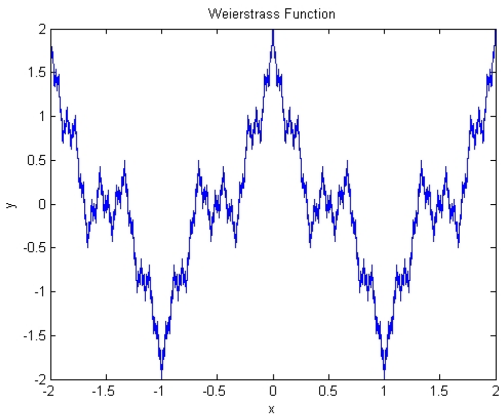
1.0.1 In the beginning

There is no single generally accepted definition of a *fractal* set. They generally take the form of complicated subsets on Euclidean space.

On 18 July, 1872 the famous german mathematician Karl Weierstrass presented a paper at the Royal Prussian Acadamy of science in which he gave an example of a continuous function $f : \mathbb{R} \to \mathbb{R}$ on the real line which was nowhere differentiable.



This was part of the programme of Weierstrass to put real analysis onto a more rigorous footing. Prior to his work, it was commonly (and incorrectly) assumed that continuous functions were automatically differentiable. In fact, in 1806 Amphere had published a paper in the Journal de l'Ecole Polytechnique claiming (erroneously) that continuous functions were almost everywhere differentiable. [1] The confusion that lead to his erroneous proof mainly arose from the lack of clarity in the defintions. Although it was widely known that the proof was flawed, the conclusion was still widely accepted.

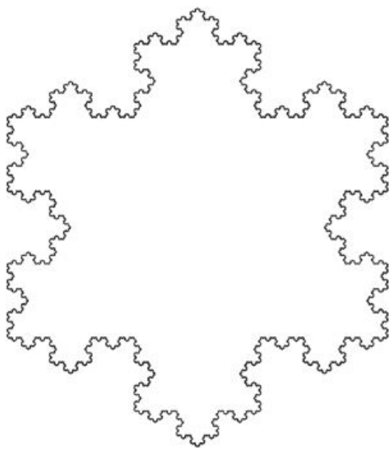
The Weierstrass function is defined using an infinite series

$$f(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x)$$

where $0 < a < 1$ and $b \in 2\mathbb{N} - 1$ satisfy $ab > 1 + \frac{3\pi}{2}$. The graph of this function might be viewed as the first example of a *fractal*.

In 1883, Cantor (who had attended lectures of Weirstrass) gave examples of what are now usually called Cantor sets in the real line.

On the other hand, von Koch was unsatisfied with Weierstrass' analytic approach and in 1906 proposed a more geometric constrution based on interating scaled down versions of the original picture to get a *von Koch snowflake*.



In a similar spirit to the constructions of both Cantor and von Koch, the polish number theorist Sierpinski constructed in 1915-16 his triangle and Gasket in the plane.

In 1918, Hausdorff developed the definition of the dimension of fractals. One of the possible ways to define fractals is to say that their Hausdorff

[1] Amphere's father was guilloteed after the french revolution and he himself made significant contributions to physics which are commerated by the use of his name as a unit of electrical current.

dimension is strictly bigger than their topological definition. [2] The idea of Hausdorff dimension was very effective in understanding many problems, and it was used extensively by Besicovich in the 1930s. It was only in 2018 that its proven that the graph of the Wierestrass function has dimension $2 + \log_b a$. [3] A simpler, but less subtle notion of dimension called Box dimension was introduced by Bouligand in 1928 (based on earlier ideas of Minkowski).

In the same year that Hausdorff proposed his definition, two french mathmematicians Julia and Fatou independently initiated the study of what are now called *Julia sets* in complex dynamics and where are important examples of fractal sets. Julia published a 199-page paper in 1918 entitled *Mémoire sur l'iteration des fonctions rationelle* describing the Julia set. With this paper, Julia won the Grand Prix of the Académie des Sciences and became extremely famous in mathematical circles throughout the 1920s. However, this work fell into obscurity for about fifty years. In contrast, Fatou, who producted similar retults using different methods, did not achieve the same level of fame as Julia. [4]

In the 1960's Benoit Mandlebrot popularised the study of fractal structures accross the sciences. His name is now used to complex dynamics to call a set in the parameter space of families of rational maps. [5]

A recurrent theme is to describe the *size* of these fractal sets. This reflects their complexity.

1.0.2 The notion of dimension

For d a natural number there is a perfectly reasonable intuitive definition of dimension: *A space is d-dimensional if locally it looks like a patch of* $\mathbb{R}^d$. (Of course, "looks like" requires some interpretation. For the moment we shall loosely interpret as "diffeomorphic to"). This immediately allows us to say: the dimension of a point is zero; the dimension of a line is 1; the

[2] He was forced to give up his post as a professor at the University of Bonn in 1935 and his work could only be published outside of Germany. Eventually, facing the prospect of being sent to a concentration camp, he, along with his wife and sister-in-law, committed suicide in January 1942.

[3] This was proved by someone who used to work at Warwick.

[4] He submitted an announcement of his results to Comptes Rendus. Unfortunately, Julia, ever protective of his work, sent letters to Comptes Rendus asking them to investigate whose results had priority. The publication duly launched an investigation and included a note on Julia's findings in the same issue as the Fatou's announcement. This apparently discouraged Fatou enough to keep him from entering for the Grand Prix. However, the Académie des Sciences gave him some recognition and awarded him a prize for his paper on the topic.

[5] Mandelbrot's uncle, Szolem Mandelbrojt, was a pure mathematician in Paris, who took an interest in the young Mandelbrot and tried to steer him towards mathematics. In fact, in 1945, Mandelbrojt showed his nephew the works of Fatou and Julia, though the young Mandelbrot initially did not take much of an interest.

dimension of a plane is 2; the dimension of $\mathbb{R}^d$ is d. Moreover, we want the dimension of a circle to be 1; the dimension of a surface to be 2, etc. The difficulty comes with more complicated sets "fractals" for which we might want some notion of dimension which can be any real number.

There are several different notions of dimension for more general sets, some more easy to compute and others more convenient in applications. We shall concentrate on *Hausdorff dimension*. Hausdorff introduced his defintion of dimension in 1919 and this was used to study such famous objects such as Koch's snowfalke curve. In fact, his definition was actually based on earlier ideas of Carathéodory. Further contributions and applications, particularly to number theory, were made by Besicovitch.

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1.0.3 In search of a good definition

To begin at the very beginning: How can we best define the dimension of a closed bounded set X in $\mathbb{R}^n$, say? Ideally, we might want a definition so that:

(i) When X is a manifold then the value of the dimension is an integer which coincides with the usual notion of dimension;

(ii) For more general sets X we can have "fractional" dimensional; and

(iii) Points, and countable unions of points, have zero dimension.

Perhaps the earliest attempt to define the dimension was the following:

First Definition. We can define the *Topological dimension* $\dim_T(X)$ by induction. We say that X has zero dimension if for every point $x \in X$ every sufficiently small ball about x has boundary not intersecting X. We say that X has dimension d if for every point $x \in X$ every sufficiently small ball about x has boundary intersecting X in a set of dimension $d - 1$.

This definition satisfies out first requirement, in that it co-incides with the usual notion of dimensions for manifolds. Unfortunately, the topological dimension is always a whole number. (For example, the topological dimension of the Cantor set $\mathcal{C}$ is zero). In particular, this definition fails the second requirement. Thus, let us try another definition.

Second Definition. Given $\epsilon > 0$, let $N(\epsilon)$ be the smallest number of ϵ-balls needed to cover X. We can define the *Box dimension* to be

$$\dim_B(X) = \limsup_{\epsilon \to 0} \frac{\log N(\epsilon)}{\log(1/\epsilon)}$$

Again this co-incides with the usual notion of dimensions for manifolds. Furthermore, the box dimension can be fractional (e.g., the dimension of the Cantor set X is $\log 2/\log 3$). We have used the limit supremum to avoid problems with convergence. Strictly speaking, this is usually called the upper box dimension and the box dimension is usually said to exist when the limit exists (and is thus equal to the limsup). However, we have the following:

Lemma 1. *There exist countable sets such that condition (iii) fails for the box dimension.*

As a particular, example we can consider the countable set

$$X = \left\{ \frac{1}{n} : n \geq 1 \right\} \cup \{0\}$$

Then the box dimension is equal to 1/2. We will give a proof in the next section.

Figure 1.1: Covering the coastline of Britain by boxes

Example 1 (The coastline of countries)**.** *Of course, to begin with there is no reason that either the Box dimension or the Hausdorff dimension of a coastline would actually be well defined. However, instead of taking a limit as ϵ tends to zero one could just take ϵ to be "sufficiently small" and see*

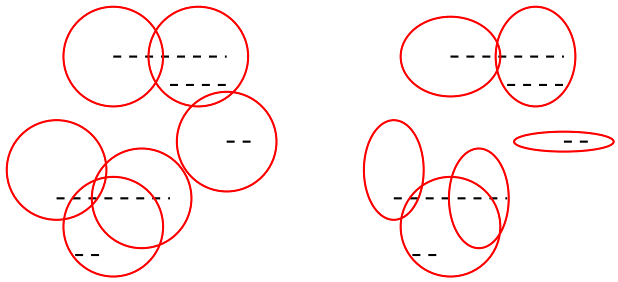


Figure 1.2: (i) A cover by balls of diameter ϵ; (ii) A cover by open sets of diameter ϵ

what sort of values one can get. Empirically, we can attempt to estimate what the Box dimension d would be, if it was well defined.

More precisely, we can count how many balls are needed to cover the coastline on a range of different scales (e.g., radius 100 *miles,* 10 *miles,* 1 *mile). This leads to interesting (if not particularly rigorous) results, as was observed by Lewis Fry Richardson. For example:*

Germany, d = 1.12;
Great Britain, d = 1.24; and
Portugal, d = 1.12.

Finally, let us try a third definition,

Third Definition. We can define the *Hausdorff dimension* (or Hausdorff-Besicovitch dimension) as follows.

Given X we can consider a cover $U = \{U_i\}_i$ for X by open sets. For $\delta > 0$ we can define $H^\delta_\epsilon(X) = \inf_U\{\sum_i \mathrm{diam}(U_i)^\delta\}$ where the infimum is taken over all open covers $U = \{U_i\}$ such that $\mathrm{diam}(U_i) \leq \epsilon$. We define $H^\delta(X) = \lim_{\epsilon\to 0} H^\delta_\epsilon(X)$ and, finally,

$$\dim_H(X) = \inf\{\delta : H^\delta(X) = 0\}.$$

As for the previous two definitions this coincides with the usual notion of dimensions for manifolds. Furthermore, the Hausdorff dimension can be fractional (e.g., the dimension of the Cantor set X is again $\log 2/\log 3$). Finally, for any countable set X property (iii) holds:

Lemma 2. *For any countable set X we have that $dim_H(X) = 0$.*

We will give a proof of this fact in chapter 4.

At first sight, the definition of Hausdorff dimension seems quite elaborate. However, its many useful properties soon become apparent. Conveniently, in many of the examples we will consider later $\dim_H(X) = \dim_B(X)$. In fact, one inequality is true in all cases:

Lemma 3. *The definitions are related by $dim_H(X) \leq dim_B(X)$.*

We will give her proof of this results in chapter 4.

After this rather rapid gallop through the definitions, we will now settle down to a more gentle canter through the definitions.

1.0.4 Books: A few of my favorite things

There are an number of excellent mathematical treatments of Hausdorff dimension and its properties. Amongst my particular favorites are *Fractal Geometry* by K.J.Falconer and *Geometry of sets and measures in Euclidean spaces* by P. Matilla. In the context of Dynamical Systems and Dimension Theory an excellent book is *Dimension Theory in Dynamical Systems: Contemporary Views and Applications* by Y. Pesin.

Chapter 2

A zoo of examples of fractal sets

We want to begin my considering a selection of examples of candidates to be called fractal sets.

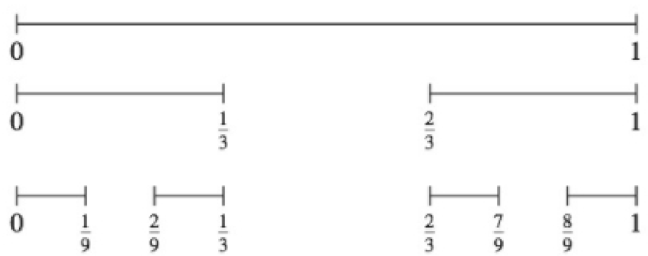
2.1 Cantor sets

The simplest examples of fractal sets are already well known to most people, namely Cantor sets. The most familiar example of a Cantor set is the middle third Cantor set.

2.1.1 Middle $\frac{1}{3}$-Cantor set

One can delete from the unit interval $[0, 1] = \{x \in \mathbb{R} : 0 \leq x \leq 1\}$ a countable sequence of open intervals to give the standard Cantor set. More precisely, one first deletes the central interval $(\frac{1}{3}, \frac{2}{3})$ of length $\frac{1}{3}$ leaving behind the union of two closed intervals $[0, \frac{1}{3}]$ and $[\frac{2}{3}, 1]$ of length $\frac{1}{3}$.

The next step is to delete from each closed interval the middle third intervals $(\frac{1}{9}, \frac{2}{9})$ and $(\frac{7}{9}, \frac{8}{9})$ leaves four closed intevals$[\frac{1}{9}, \frac{2}{9}]$, $[0, \frac{1}{9}]$, $[\frac{2}{9}, \frac{1}{3}]$, $[\frac{2}{3}, \frac{7}{9}]$ and $[\frac{8}{9}, 1]$. When this is iterated n-times we have 2^n intervals of length $\frac{1}{3^n}$. Eventually one arrives at a closed set C, namely the middle third Cantor set.



It is easy to see that we can also write this in the form

$$C = \left\{ \sum_{n=1}^{\infty} \frac{2i_n}{3^n} : i_1, i_2, i_3, \cdots \in \{0,1\} \right\}$$

i.e., those numbers where the digit 1 doesn't occur in the base 3 expansion. In particular, there is a bijection $\pi : \Sigma \to C$ from the space of sequences

$$\Sigma = \{0,1\}^{\mathbb{N}} := \{(i_1, i_2, i_3, \cdots) : i_1, i_2, i_3, \cdots \in \{0,1\}\}$$

to the Cantor set C defined by

$$\pi : (i_1, i_2, i_3, \cdots) \mapsto 2 \sum_{n=1}^{\infty} \frac{i_n}{3^n}.$$

Exercise 1. *Show that these two defintions of the middle third Cantor set actually coincide.*

2.1.2 Middle λ-Cantor sets

A simple variant on this construction is where we choose $0 < \lambda < 1$ and we delete the middle λ interval at each stage, rather than the middle third interval. More precisely, we first delete the interval $(\frac{1-\lambda}{2}, \frac{1+\lambda}{2})$ leaving the intervals $[0, \frac{1-\lambda}{2}]$ and $[\frac{1+\lambda}{2}, 1]$, each of which has length $\frac{1-\lambda}{2}$. We then delete from each of these intervals their middle intervals, of length $\frac{\lambda(1-\lambda)}{2}$ leaving 4 intervals of length $\left(\frac{(1-\lambda)}{2}\right)^2$. Continuing in this way at the nth step we have 2^n closed intervals of the length $\left(\frac{(1-\lambda)}{2}\right)^n$ Continuing iteratively we end up with a closed set C_λ.

Example 2 ($\lambda = \frac{1}{3}$)**.** *In the particular case that $\lambda = \frac{1}{3}$ this reduces to the previous construction and $C = C_\lambda$.*

We can also write

$$C_\lambda = \left\{ \left(\frac{1+\lambda}{2}\right) \sum_{n=1}^{\infty} i_n \left(\frac{1-\lambda}{2}\right)^n : i_1, i_2, i_3, \cdots \in \{0,1\} \right\}$$

Exercise 2. *Show that these two defintions of the Cantor set C_λ actually coincide.*

Remark 1. *To show that the Cantor set actually exists (as a non-empty set) we formally need to invoke a little metric space theory. More precisely, assume that we have a nested sequence of compact sets (in this case unions of closed intervals)*

$$C_1 \supset C_2 \supset C_3 \supset \cdots .$$

then we claim that $C = \cap_n C_n \neq \emptyset$. *Here is a simple argument. Choose* $c_n \in C_n \subset C_1$ *and by compactness of* C_1 *choose a convergent subsequence* $c_{n_k} \to x \in C_1$, *say. Moreover, for any* j *we see that* $(c_n)_{n \geq j} \subset C_j$ *and since* C_j *is closed we see* $x \in C_j$. *Thus* $x \in \cap_j C_j$, *which is therefore nonempty.*

A similar argument can be applied to some of the other constructions (e.g., Sierpinski gasket, Bedford-McMullen carpets).

2.1.3 Cantor sets of zero Lebesgue measure

Formally, Cantor sets are totally disconnected perfect closed sets, and all such sets are homeomorphic. However we are more interested in their metric structure.

Without appealing to too much measure theory, we can say what it means for a set to have zero Lebesgue measure.

Definition 1. *We say that* $X \subset [0,1]$ *has zero Lebesgue measure if for any* $\epsilon > 0$ *we can choose a finite (or countable) set of subintervals* $I_1, I_2, \cdots, I_n \subset [0,1]$ *such that*

$$X \subset \cup_{i=1}^N I_i \text{ and } \sum_{i=1}^N \lambda(I_i) < \epsilon,$$

where $\lambda(\cdot)$ *is the measure (i.e., length) of the interval.* [1]

It is also easy to see countable unions of zero measure sets have zero measure.

Exercise 3. *Show that if a countable family* X_i *each have zero measure then so does its union* $\cup_i U_i$.

It is easy to see that the middle third Cantor set has zero measure. At the nth stage in the cosntruction we have $N := 2^n$ intervals I_i each of length $1/3^n$. In particular, using these intervals we see that

$$\sum_{i=1}^N \ell(I_i) = \left(\frac{2}{3}\right)^n < \epsilon$$

provided that $n > \frac{\log \epsilon}{\log \frac{2}{3}} > 0$.

In the case that $\lambda < 1$ we similarly see that at the nth level of the construction we have that the Cantor set C_λ is covered by 2^n intervals I_i of size $\left(\frac{1-\lambda}{2}\right)^n$. In particular, using these intervals we see that

$$\sum_{i=1}^N \ell(I_i) = (1-\lambda)^n < \epsilon$$

[1] In $\mathbb{R}^d$ there is an analogous defintions with cubes replacing intervals.

provided that $n > \frac{\log \epsilon}{\log(1-\lambda)} > 0$.

In summary, these Cantor sets are all homeomorphic and have zero Lebesge measure. Later we will introduce defintions of dimension to destinguish their size which will help to distinguish them.

2.1.4 Hölder bijections

Let $0 < \lambda, \nu < 1$. There is a natural bijection between the Cantor sets $\pi = \pi_{\lambda,\nu} : C_\lambda \to C_\nu$ given by

$$\pi : \left(\frac{1+\lambda}{2}\right) \sum_{n=1}^{\infty} i_n \left(\frac{1-\lambda}{2}\right)^n \mapsto \left(\frac{1+\nu}{2}\right) \sum_{n=1}^{\infty} i_n \left(\frac{1-\nu}{2}\right)^n.$$

This is a homeomorphism. However we can show a stronger result, after recalling the following definition.

Definition 2. *We say that π is Lipschitz if there exists $C > 0$ and*

$$|\pi(x) - \pi(y)| \leq C|x - y|.$$

Given $\gamma > 0$ we say that π is γ-Hölder if there exists $C > 0$ and

$$|\pi(x) - \pi(y)| \leq C|x - y|^\gamma.$$

In particular, if $\gamma = 1$ then a 1-Holder function is Lipschitz.

Proposition 1. *The map π is γ-Holder when $\gamma = \frac{\log\left(\frac{1+\lambda}{2}\right)}{\log\left(\frac{1+\nu}{2}\right)}$.*

Proof. Given $x, y \in C_\lambda$ we can write

$$x = \left(\frac{1+\lambda}{2}\right) \sum_{n=1}^{\infty} i_n \left(\frac{1-\lambda}{2}\right)^n \text{ and } y = \left(\frac{1+\lambda}{2}\right) \sum_{n=1}^{\infty} j_n \left(\frac{1-\lambda}{2}\right)^n.$$

For $x \neq y$ we can let

$$N = \min\{n : i_n \neq j_n\}$$

then $i_N \neq j_N$. In particular, there exists $c > 0$ such that

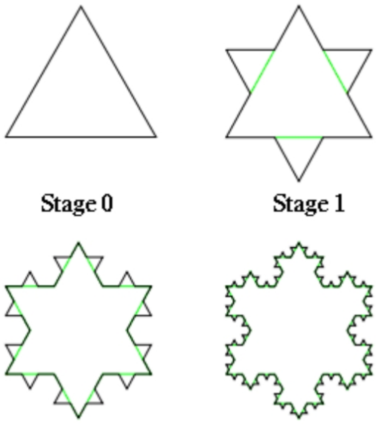
$$|x - y| \geq c \left(\frac{1+\lambda}{2}\right)^N.$$

Similarly, there exists $d > 0$ such that

$$|\pi(x) - \pi(y)| \geq d \left(\frac{1+\nu}{2}\right)^N.$$

It is then easy to see the result. □

Exercise 4. *Complete the details of the proof.*



2.2 von Koch curve and snowflakes

The von Koch curve is defined by an iterative process.

Starting from from an equilateral triangle the middle third segment of each side is replaced by the other two sides of an equilateral triangle, i.e., replacing the each middle third of each side by the other two sides of an equilateral triangle pointing outward of side length 1/3 the size of the original edge lengths entered on the edge.

We can continue this process repeatedly and it ”converges”” to the von Koch curve.

2.3 Convergence of sets

One has to ask what convergence means in these contexts. This introduces us to the notion of the Hausdorff metric on sets. Given a compact non-empty set [2] $X \subset \mathbb{R}^d$ and $\epsilon > 0$ we define an ϵ-neighbourhood

$$B(X, \epsilon) = \{y \in \mathbb{R}^d : \exists x \in X \text{ with } d_{\mathbb{R}^d}(y, x) < \epsilon\},$$

where $d_{\mathbb{R}^d}(y, x) = \sqrt{\sum_{i=1}^d (x_i - y_i)^2}$, for $x = (x_1, \cdots, x_d)$ and $y = (y_1, \cdots, y_d)$, is the usual Euclidean metric.

Definition 3. *Given two non-empty compact sets $X, Y \subset \mathbb{R}^d$ we define the Hausdorff distance of two compact sets X and Y in $\mathbb{R}^d$ by*

$$d(X, Y) = \inf\{\epsilon > 0 : X \subset B(Y, \epsilon) \text{ and } Y \subset B(X, \epsilon)\}.$$

Remark 2. *We can see that we require the sets bounded since if we choose $X, Y \subset \mathbb{R}$ to be $X = \{0\}$ and $Y = \mathbb{R}$ then $d(X, Y) = +\infty$. We also see that*

[2]Recall that in Euclidean space a closed bounded sets correspond to the compact sets

we want to consider only closed sets since for $X = [0,1]$ and $Y = (0,1)$ we have $d(X,Y) = 0$ but $X \neq Y$.

Lemma 4. *The Hausdorff metric d is a metric on the set of compact subsets of $\mathbb{R}^d$.*

Proof. To see this is a metric we need to estblish three properties. Firstly, if $X = Y$ then we see from the definitions that $X \subset B(X,\epsilon)$ for any $\epsilon > 0$ and so deduce that $d(X,X) = 0$. Conversely, if $d(X,Y) = 0$ then we see that for any $x \in X$ and each $n \geq 1$, there exists $y_n \in Y$ wih $d_{\mathbb{R}^d}(x, y_n) < \frac{1}{n}$. Thus $y_n \to x$ and we deduce that $x \in Y$ since Y is closed. Thus $X \subset Y$. Similarly, $Y \subset X$ by symmetry.

Secondly, we observe from the symmetry in the definitions that $d(X,Y) = d(Y,X)$, i.e., the metric is symmetric.

Finally, to prove the triangle inequality let $X, Y, Z \subset \mathbb{R}^d$ be compact subsets. Choose $\epsilon, \delta > 0$ such

$$X \subset B(Y,\epsilon), Y \subset B(X,\epsilon), Y \subset B(Z,\delta), Z \subset B(Y,\delta) \tag{1}$$

For $x \in X \subset B(Y,\epsilon)$ there exists $y \in Y$ with $d_{\mathbb{R}^d}(x,y) < \epsilon$. Moreover, since $y \in Y \subset B(Z,\delta)$ there exists $z \in Z$ with $d_{\mathbb{R}^d}(y,z) < \delta$. By the Euclidean triangle inequality

$$d_{\mathbb{R}^d}(x,z) \leq d_{\mathbb{R}^d}(x,y) + d_{\mathbb{R}^d}(y,z) < \epsilon + \delta$$

and thus $X \subset B(Z, \epsilon + \delta)$. Similarly, we can show that $Z \subset B(X, \epsilon + \delta)$ and deduce that $d(X,Z) \leq \epsilon + \delta$. Taking the infimum over both ϵ and δ satisfying (1) gives the triangle inequality for the Hausdorff metric:

$$d(X,Z) \leq d(X,Y) + d(Y,Z).$$

This complete the proof.

□

An equivalent definition of the Hausdorff metric is

$$d(X,Y) = \max\left\{\sup_{x\in X}\inf_{y\in Y} d_{\mathbb{R}^d}(x,y), \sup_{y\in Y}\inf_{x\in X} d_{\mathbb{R}^d}(x,y)\right\}$$

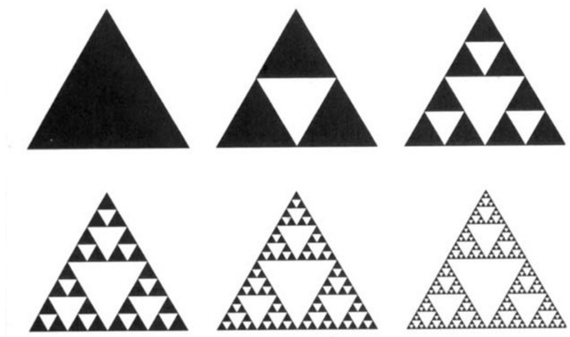
Exercise. Show that the two defintions of $d(X,Y)$ are equivalent.

Example 3. *Let $X_n = \{\frac{i}{n} : i = 0,1,2,\cdots,n\}$ and $X = [0,1]$. We see that $d(X_n, X) = \frac{1}{2n}$ and so $X_n \to X$ in the Hausdorff metric.*

2.4 Sierpinski trianges and carpets

2.4.1 Sierpinski triangle

The Sierpinski triangle is rather like a two dimensional version of the middle third Cantor set where we iteratively deleted open intervals. This time around we start from an equilateral triangle and delete the open inscribed (inverted) middle triangle. This leaves three triangles each of which is half the size of the original triangle. We continute to delete the sclaed down open middle triangles and continue iteratively.



Remark 3. *To see that the limit exists we observe that we have a decreasing sequence $T_1 \supset T_2 \supset T_3 \supset$ where T_n is a union of 3^n triangles of size $\frac{1}{2^n}$. The Sierpinski triangle is given by $T = \cap_{n=1}^{\infty} T_n$. Compactness shows that T is non-empty by analogy with the case of Cantor sets.*

2.4.2 Bedford-McMullen carpets

A similar construction, with more variations available, is the following.

The Bedford-McMullen carpet is a closed subset of the unit square constructed by analogy with the Sierpinski triangle. The construction appeared independently in the Warwick doctoral thesis of Tim Bedford and the first paper of Curt McMullen. [3]

Let $n, m \geq 2$ and let $S \subset \{0, \cdots, n-1\} \times \{0, \cdots, m-1\}$. Divide the unit square $[0,1]^2$ into subrectangles of size $\frac{1}{m} \times \frac{1}{n}$ and keep only the squares

$$\left[\frac{i}{n}, \frac{i+1}{n}\right] \times \left[\frac{j}{n}, \frac{j+1}{n}\right]$$

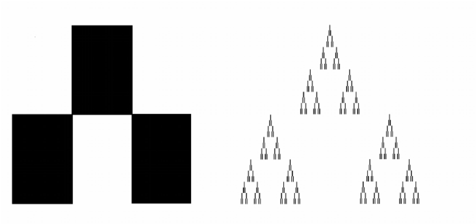
where $(i,j) \in S$. We then iterate the procedure. The resulting set is the

[3]McMullen told the lecturer about his work in the tea room at IHES, who then wrote to Bedford informing him

carpet. It can also be written as

$$X = \left\{ \left(\sum_{k=1}^{\infty} \frac{x_k}{n^k}, \sum_{k=1}^{\infty} \frac{y_k}{m^k} \right) : (x_k, y_k) \in S, \forall k \geq 1 \right\}$$

Example 4. *We can consider the special case $n = 3$ and $m = 2$ and $\mathcal{S} = \{(0,1),(1,1),(2,0)\}$.*



Consider the following three affine maps of $\mathbb{R}^2$:

$$T_i : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} c_i \\ d_i \end{pmatrix}, \quad i = 1, 2, 3,$$

where

$$\begin{pmatrix} c_1 \\ d_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} c_2 \\ d_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{2} \end{pmatrix}, \begin{pmatrix} c_3 \\ d_3 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} \\ 0 \end{pmatrix}.$$

=1.5in bedford.eps

The first two steps in the Bedford-McMullen example The limit set takes the form

$$\Lambda = \left\{ \left(\sum_{n=1}^{\infty} \frac{i_n}{3^n}, \sum_{n=1}^{\infty} \frac{j_n}{2^n} \right) : (i_n, j_n) \in \{(0,0),(1,1),(2,0)\} \right\},$$

and is closely related to what is called Hironaka's curve.

Example 5 (Sieprinski Carpet)**.** *We can consider the special case $n = 3$ and $m = 3$ and $\mathcal{S} = \{0,1,2\} \times \{0,1,2\} - \{(1,1)\}$.*

Exercise 5. *Show that providing $|\mathcal{S}| < nm$ the carpet has zero Lebesgue measure.*

Exercise 6. *When is the final set connected?*

2.5 Less linear examples

All of the previous example have been very linear in their construction. However, we can now describe a couple of examples where the construction of the fractal set is a little different.

2.5.1 Circle packings

The curvilinear analogue of the Sierpinski Gasket is the so called Apollonian Gasket.

Beginning with the unit circle we consider three mutually tangent inscribed circles. We successively inscribe into each triangle between three circles another circle. The closure of the union of these circles is a circle packing.

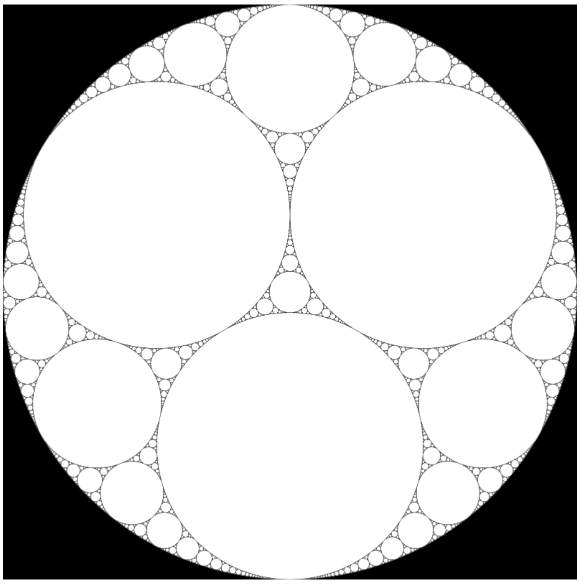


Figure 2.1: An apollonian circle packing made up of infinitely many inscribed circles

Remark 4. *The following estimate on the size of the circles was only proven ten years ago: There exists $C > 0$ and $\delta > 1$ such that*

$$\lim_{\epsilon \to 0} \frac{\text{The number of circles of radius} \geq \epsilon}{\epsilon^{\delta}} = C.$$

2.5.2 Quasi-circles

Consider circls $C_1, \cdots, C_n$ be a finite set of circles in $\mathbb{C}$. Assume that the circles have disjoint interiors and C_i touches C_{i+1} at a single point (and C_n

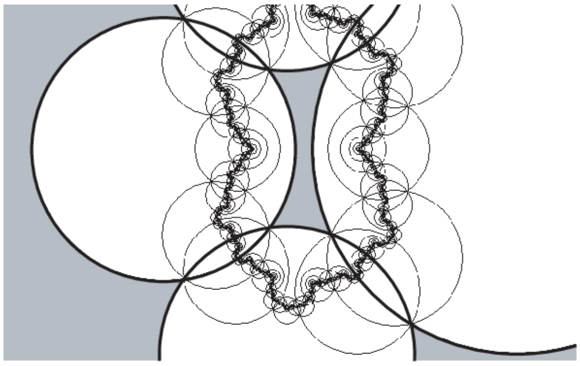


Figure 2.2: A quasi circle preserved by reflection in the linked circles

touches C_1). footnoteAlternatively we can assume that C_i intersects C_{i+1}, and C_n intersects C_1, at exactly 2 points and at these points of intersection the two circles meet at right angles. We can define inversions in the circle $C_i = \{z \in \mathbb{C} : |z - z_i| = r_i\}$ to be the map $T_i : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ defined by

$$T_i(z) = \frac{(z - z_i)r_i^2}{|z - z_i|^2} + z_i, \quad \text{for } (i = 1, \cdots, n).$$

The smallest closed set $X \subset \mathbb{C}$ such that $T_i(X) = X$, for $i = 1, \cdots, n$ is either:

1. Another circle in $\mathbb{C}$; or
2. A "fractal" non-rectifiable curve (i.e., *not* the Lipschitz image of a circle)

2.5.3 Julia sets

Let $c \in \mathbb{C}$. The Julia set of a polynomial $p(z) = z^2 + c$ is a closed subset $J \subset \mathbb{C}$. The simplest definition is in terms of fixed points of iterates of p. More precisely, for each n we can consider fixed points

$$z = p^n(z) \text{ where } p^n = p \circ \cdots \circ p.$$

We call such a periodic point repelling if $|(p^n)'(z)| > 1$. The Julia set

$$J = \overline{\{z : z \text{ is a reprelling periodic point}\}}$$

is the closure of the repelling periodic points. This is merely one of several equivalent definitions.

When $c = 0$ this is merely a circle. But for values $c \neq 0$ the Julia sets have a fractal structure. We can further subdivide the parameter sets into those for which the associated Julia set is connected or (totally) disconnected. The collection of points c with the former property for the Mandelbrot set. This forms a compact set in $\mathbb{C}$ whose boundary is again fractal in appearance.

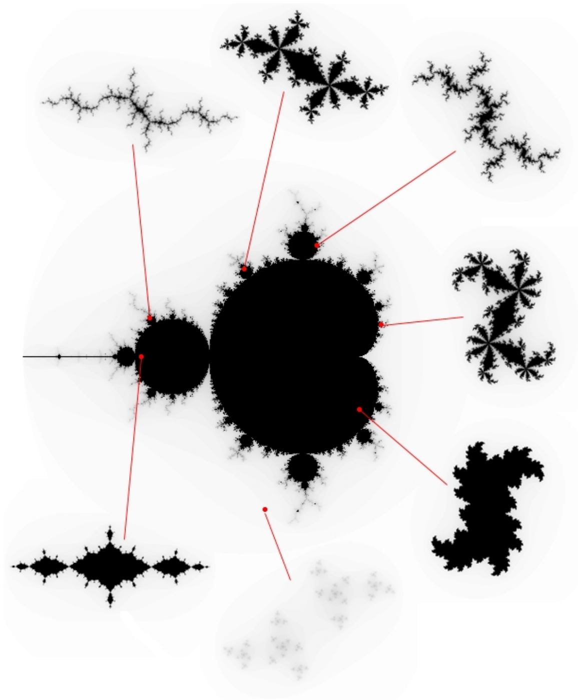


Figure 2.3: The Mandlebrot set is in the parameter space, and different choices of c give rise to different Julia sets

2.6 Digit frequencies

We complete our wanderings through examples of "fractal sets" by considering a classical result which will have echos later. Here the sets in question will actually be dense sets in the unit interval characterised by properties of their decimal (and other bases) expansions.

2.6.1 Normal numbers and Borel's Theorem

Given any real number $0 < x < 1$ we can consider its decimal expansion

$$x = 0.a_1a_2a_3\ldots \qquad \text{where } a_1, a_2, a_3, \in \{0, 1, 2, \cdots, 9\}.$$

This will be unique, except in a countable set of values. More generally, for any natural number $b \geq 2$ we can consider its expansion in base b:

$$x = \sum_{n=1}^{\infty} \frac{a_n}{b^n} \text{ where } a_1, a_2, a_3, \in \{0, 1, 2, \cdots, b-1\}.$$

Again, this expansion is unique except for a countable set of values.

We say that x is normal to base b if the digits in the base b expansion all occur wih equal frequency $\frac{1}{b}$, i.e., for all $j \in \{0, 1, \cdots, b-1\}$ we have

$$\lim_{N \to +\infty} \frac{1}{N} \mathrm{Card}\{1 \leq n \leq N : a_n = j\} = \frac{1}{b}.$$

A normal number x is one which is normal to every base $b \geq 2$.

A very significant result of E. Borel is the following.

Theorem 1 (Borel, 1909)**.** [4] *The set of points $0 < x < 1$ which are not normal have zero Lebesgue measure.*

In the case $b = 2$. Let us deal with the case $b = 2$, the Back iothers being similar. To reformulate the result in a more cBack ionvenient form to prove, we define a sequence of funcions $\chi_n : [0, 1] \to \{-1, 1\}$ defined by

$$\chi_n(x) = \begin{cases} 1 & \text{if } a_n = 0 \\ -1 & \text{if } a_n = 1 \end{cases}$$

where

$$x = \sum_{k=1}^{\infty} \frac{a_n}{2^n}.$$

Thus for x to be normal in base b is equivalent to

$$\lim_{N \to +\infty} \frac{1}{N} \sum_{n=1}^{N} \chi_n(x) = 0.$$

5

We can avoid using measure theory by using a simple direct construction. We begin with the following simple result on series

Claim 1. Let $c_n \geq 0$ with $\sum_n c_n < +\infty$. Then there is a sequence $0 \leq b_n \to +\infty$ such that $\sum_n a_n b_n < +\infty$.

Proof of Claim 1. Since the tail of a convergent sequence tends to zero, we can choose $0 = k_0 < k_1 < k_2 < \cdots$ such that $k_1 = 0$ and

$$\sum_{n=k_j}^{\infty} c_n < 2^{-j} \text{ for } j = 2, 3, 4, \cdots.$$

[4] The veracity of Borel's Theorem is now beyond question. However, to paraphrase Doob, Borel's original derivation contains an "unmendably faulty" error. Borel himself was aware of the gap in his proof, and asked for a complete argument. His plea was answered a year later by Faber and also later by Hausdorff, using what is now called the Borel-Cantelli lemma. The more modern proof uses the Birkhoff Ergodic Theorem from 1931.

[5] These functions χ_n are sometimes called Rademacher functions

For each $n \in \mathbb{N}$ we define $b_n = j$ where $k_j < n \leq k_{j+1}$. In particular,

$$\begin{aligned}\sum_{n=1}^{\infty} c_n b_n &= \sum_{n=1}^{k_2} c_n + \sum_{j=2}^{\infty}\left(\sum_{n=k_j+1}^{k_{j+1}} c_n b_n\right)\\ &\leq \sum_{n=1}^{k_2} c_n + \sum_{j=2}^{\infty}\left(j \sum_{n=k_j+1}^{k_{j+1}} c_n\right)\\ &\leq \sum_{n=1}^{k_2} c_n + \sum_{j=2}^{\infty}\left(j \sum_{n=k_j+1}^{\infty} c_n\right)\\ &\leq \sum_{n=1}^{k_2} c_n + \sum_{j=2}^{\infty} j2^{-j} < +\infty\end{aligned}$$

We can now proceed with the proof of Borel's Theorem as follows. Let us write

$$\phi_n(x) = \frac{1}{n}\sum_{k=1}^{n} \chi_k(x)$$

then we want to show that $\phi_n(x) \to 0$ as $n \to +\infty$, s except on a set of zero measure. *Excercise* Show that the functions $\phi_n(x)$ take only finitely many values and are constant on the dyadic intervals $[i/2^n, (i+1)/2^n]$.

Claim 2. $\sum_{n=1}^{\infty} \int_0^1 |\phi_n(x)|^4 dx < +\infty$

Proof of claim 2. We can expand

$$\begin{aligned}\int_0^1 |\phi_n(x)|^4 dx &= \frac{1}{n^4}\int_0^1 \left(\sum_{k=1}^{n} \chi_k(x)\right)^4 dx\\ &= \frac{1}{n^4}\sum_{k=1}^{n}\int \chi_k^4(x)dx + \frac{1}{n^4}\sum_{k_1 \neq k_2}\int \chi_{k_1}(x)^2\chi_{k_2}(x)^2 dx\\ &+ \frac{1}{n^4}\sum_{k_1\neq k_2}\int \chi_{k_1}(x)\chi_{k_2}(x)^3 dx + \frac{1}{n^4}\sum_{k_1,k_2,k_3 \text{distinct}}\int \chi_{k_1}(x)\chi_{k_2}(x)\chi_{k_3}(x)^2 dx\\ &+ \frac{1}{n^4}\sum_{k_1,k_2,k_3,k_4\text{distinct}}\int \chi_{k_1}(x)\chi_{k_2}(x)\chi_{k_3}(x)\chi_{k_3}(x)dx\end{aligned} \tag{1}$$

We can simply this by noting that

1. Since $\chi_k^2(x) = 1$,

   (a) $\int \chi_k(x)^4 dx = 1$ and

   (b) $\int \chi_{k_1}(x)^2\chi_{k_2}(x)^2dx = 1$ for k_1, k_2 distinct.

2. Since $\int \chi_{k_1}(x)\chi_{k2}(x)dx = 0$ for k_1, k_2 distinct, and thus

   (a) we have $\int \chi_{k_1}(x)^3\chi_{k_2}(x) = 0$ (using) $\chi_k(x)^3 = \chi(x)$) and

   (b) $\int \chi_{k_1}(x)^2\chi_{k2}(x)\chi_3(x)dx = \int \chi_{k2}(x)\chi_3(x)dx = 0$ for k_1, k_2, k_3 distinct (using $\chi_{k_1}(x)^2 = 1$).

3. $\int \chi_{k_1}(x)\chi_{k2}(x)\chi_{k3}(x)\chi_{k3}(x)dx = 0$ for k_1, k_2, k_3, k_4 distinct.

Exercise. Verify these equalities.

In particular, we see that

$$\frac{1}{n^4}\sum_{k=1}^{n}\int \chi_k^4(x) = \frac{1}{n^3}dx \text{ and } \frac{1}{n^4}\sum_{k_1\neq k_2}\int \chi_{k_1}(x)^2\chi_{k_1}(x)^2dx = \frac{n-1}{n^3}$$

and all of the other terms in (1) vanish.

Exercise. Verify these estimates

In particular, we see that $\sum_{n=1}^{\infty}\int_0^1 |\phi_n(x)|^4dx < +\infty$ as claimed.

The proof of Borel's theorem will follow immediately from Claim 2 and Claim 3 below.

Claim 3.[A first brush with the strong law of large numbers] If $\sum_n \int_0^1 |\phi_n(x)|^4dx < +\infty$ then $\lim_{n\to+\infty}\phi_n(x) = 0$ for almost all x (i.e., except on a set of zero measure).

Proof of Claim 3. We want to show that the set $Z \subset [0,1]$ of those x for which $|\phi_n(x)|$ *doesn't* converge to zero, has measure zero.

Let $c_n = \int |\phi_n(x)|^4dx$ and then since by claim 2 $\sum_n c_n < +\infty$ we can apply claim 1 to find a sequence $b_n \to +\infty$. If $x \in Z$ then clearly

$$|\phi_n(x)|^4 > \frac{1}{b_n} \text{ for infinitely many } n. \tag{1}$$

Let us denote

$$A_n = \left\{y \in [0,1] : |\phi_n(y)|^4 > \frac{1}{b_n}\right\} \text{ for } n \geq 1.$$

then by (1) we see that $x \in \cup_{k=n}^{\infty}A_k$ for all $n \geq 1$, i.e., $Z \subset \cup_{k=n}^{\infty}A_k$.

Moreover, since $|\phi_n(y)|^4$ takes only finitely many values we see that we can write $A_n = \cup_{i=1}^{N_n}J_i^{(n)}$ where $J_1^n, J_2^n, \cdots J_{N_n}^n$ are disjoint intervals. In particulae, $\cup_{k=n}^{\infty}A_k$ is covered by a countable union of intervals

$$\mathcal{J}_n = \{J_i^{(k)} : 1 \leq i \leq N_k, k \geq 1\}$$

whose union must also cover Z, i.e., $Z \subset \cup_{J \in \mathcal{J}_n} J$. Furthermore, we can then write the total lengths of these intervals as

$$\lambda(A_n) = \sum_{i=1}^{N_n} \lambda(J_i^{(n)}), n \geq 1.$$

By definition we see that for each $y \in A_n$ we have $|\phi_n(y)|^4 b_n \geq 1$ and thus

$$\sum_n \lambda(A_n) \leq \sum_n b_n \int_0^1 |\phi_n(x)|^4 dx < +\infty.$$

Since this series converges, its tail must tend to zero, i.e., $\lim_{n \to +\infty} \sum_{k=n}^{\infty} \lambda(A_n) = 0$. In particular, for any $\epsilon > 0$ we can write

$$\lambda\left(\cup_{J \in \mathcal{J}_n} J\right) \leq \sum_{k=n}^{\infty} \lambda(A_n) < \epsilon$$

for n sufficiently large. This complete the proof.

R □

Remark 5 (More measure theory makes for light work)**.** *With more measure theory we can shorten the proof. In claim 2 we had that*

$$\sum_{n=1}^{\infty} \int_0^1 \left(\frac{1}{n} \sum_{k=1}^{n} \chi_k(x) \right)^4 dx < +\infty.$$

In this case we can interchange the summation and integral to deduce

$$\int_0^1 \sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^{n} \chi_k(x) \right)^4 dx < +\infty.$$

We can then deduce that

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^{n} \chi_k(x) \right)^4 dx < +\infty$$

for almost all x. From this we can deduce that

$$\lim_{n \to +\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_k(x) = 0$$

as required.

We will next turn to the problem of describing the size of these zero measure sets

Chapter 3

Box Dimension

There are two particularly popular notions of quantifying the size of fractal sets which we will consider, namely Box and Hausdorff dimension. Both measure how efficiently a set X can be covered by balls.

Box dimension requires covering the set X by balls of the same size. This makes it particularly easy to compute, but it lacks many desirable properties. On the other hand, in the definition of Hausdorff dimension we will allow coverings by sets of different sizes. This gives a better behaved notion of dimension, but (as we shall see) is usually much more difficult to compute.

We first introdcuce Box dimension and its properties.

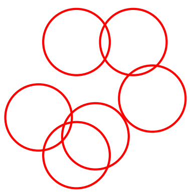
3.1 Definitions

3.1.1 The definition of box dimension

We begin with the definition of Box dimension (or Minkowski dimension as it is sometimes called). We first need to introduce a very simple notion.

Definition 4. *Suppose $X \subset \mathbb{R}^d$ is a bounded set. Let $\epsilon > 0$. Let $N(X, \epsilon)$ be the minimal number of ϵ-balls needed to cover X, i.e.,*

$$N(X, \epsilon) = \inf\{n\colon \exists x_1, \cdots, x_n \in X \text{ such that } X \subset \cup_{i=1}^n B(x_i, \epsilon)\}$$



Since X is bounded it is easy to see that $N(X, \epsilon)$ is finite. Similarly, it follows immediately from the defintions that if $\epsilon < \epsilon'$ then $N(X, \epsilon) \geq N(X, \epsilon')$, since given any (minimal) cover by ϵ-balls also corresponds to a (possibly non-minimal) cover by ϵ'-balls with the same centres.

The Box dimension measures the way in which the numbers $N(X, \epsilon)$ depend on ϵ as $\epsilon \to 0$. It doesn't always exist, but even when it doesn't there are the more general notions of upper and lower box dimension, which we define below.

Definition 5. *We define the upper Box dimension (or Minkowski dimension) of X as* [1]

$$\overline{\dim}_B = \limsup_{\epsilon \to 0} -\frac{\log N(X, \epsilon)}{\log \epsilon}$$

and the lower box (or Minkowski dimension) of X as [2]

$$\underline{\dim}_B(X) = \liminf_{\epsilon \to 0} -\frac{\log N(X, \epsilon)}{\log \epsilon}.$$

If the two values agree, then the limit

$$\dim_B(X) = \lim_{\epsilon \to 0} -\frac{\log N(X, \epsilon)}{\log \epsilon}$$

exists and is simply called the box dimension (or Minkowski dimension) of X.

As we shall see later, there are examples with $\underline{\dim}_B(X) < \overline{\dim}_B(X)$, so the box dimension isn't always defined.

1. It doesn't matter if we assume the centres of the balls x_i are chosen in $\mathbb{R}^d$ rather than X. This would change the value of $N(X, \epsilon)$ but not the different values of the dimension(s).

2. Intuitively when the box dimension exists it means that for any $\delta > 0$ the number of balls of size ϵ needed to cover X grows as

$$\epsilon^{-(\dim B(X) - \delta} \leq N(X, \epsilon) \leq \epsilon^{-(\dim B(X) + \delta}$$

for $\epsilon > 0$ sufficiently small.

3. It is clear from the definition that $\underline{\dim}(X) \geq 0$ since $\frac{\log N(X,\epsilon)}{\log \epsilon} \geq 0$.

Often it is more convenient ()both practically and conceptually to take the lim inf and lim sup through a subsequence, in which the following simple lemma is helpful.

Lemma 5 (Trivial, but useful lemma)**.** *Let $\epsilon_n \to 0$ be a monotone decreasing sequence such that*

1. $\lim_{n \to +\infty} \frac{\log \epsilon_{n+1}}{\log \epsilon_n} = 1$*; and*

---

[1] By $\limsup_{\epsilon \to 0} a_\epsilon = a$ we mean that $\forall \delta > 0$, $\exists \epsilon_0 > 0$ such that $|\sup_{0 < \epsilon < \epsilon_0} a_\epsilon - a| < \delta$

[2] By $\liminf_{\epsilon \to 0} a_\epsilon = a$ we mean that $\forall \delta > 0$, $\exists \epsilon_0 > 0$ such that $|\inf_{0 < \epsilon < \epsilon_0} a_\epsilon - a| < \delta$

2. $\lim_{n\to+\infty} \frac{\log N(X,\epsilon_{n+1})}{\log N(X,\epsilon_n)} = 1$*;*

Then

$$\overline{\dim}_B(X) = -\limsup_{n\to+\infty} \frac{\log N(X,\epsilon_n)}{\log \epsilon_n} \text{ and } \underline{\dim}_B(X) = -\liminf_{n\to+\infty} \frac{\log N(X,\epsilon_n)}{\log \epsilon_n}$$

Proof. For any $\epsilon > 0$ we can choose $\epsilon_{n+1} \le \epsilon < \epsilon_n$ and we know that $N(X,\epsilon_n) \le N(X,\epsilon) \le N(X,\epsilon_{n+1})$. Therefore

$$\frac{\log N(\epsilon_n)}{\log(\frac{1}{\epsilon_{n+1}})} \le \frac{\log N(\epsilon)}{\log(\frac{1}{\epsilon})} \le \frac{\log N(\epsilon_{n+1})}{\log(\frac{1}{\epsilon_n})}$$

Letting $n \to +\infty$ and using 1 and 2 gives result. □

Remark 6 (The coastline of countries)**.** *Of course, there is no reason that either the Box dimension of a coastline would actually be well defined. However, instead of taking a limit as ϵ tends to zero one could just take ϵ to be "sufficiently small" and see what sort of values one can get. Empirically, we can attempt to estimate what the Box dimension d would be, if it was well defined. More precisely, we can count how many balls are needed to cover the coastline on a range of different scales (e.g., radius* 100 *miles,* 10 *miles,* 1 *mile). This leads to interesting (if not particularly rigorous) results, as was observed by Lewis Fry Richardson. For example:*

Germany, dimension $\asymp 1.12$;
Great Britain, dimension $\asymp 1.24$; *and*
Portugal, dimension $\asymp 1.12$.

The first mathematical example we will consider is trivial, but the conclusion is reassuring.

Example 6 (Single points)**.** *Let $X = \{x\}$ be a single point. Then for any $\delta > 0$ we have that $N(X,\delta) = 1$ since we can cover x by the single ball $B(x,\epsilon)$. Thus we can deduce that $\overline{\dim}_B(X) = 0$ and thus $\dim_B(X) = 0$.*

The next example is almost as simple, and equally reassuring.

Example 7 (Unit interval)**.** *Suppose $X = [0,1]$. Then we observe that for any $\epsilon > 0$ we have that*

$$[1/(2\epsilon)] \le N(X,\epsilon) \le [1/\epsilon] + 1$$

where $[\cdot]$ is the integer part. The upper bound is apparent since we can consider the points $x_i = \epsilon i$, for $0 \le i \le [1/\epsilon]$ and then we have an open cover with $[1/\epsilon]+1$ balls $B(x_i,\epsilon)$. On the other hand, to get the lower bound we observe that since any ball $B(x,\epsilon)$ is an interval of length 2ϵ in the real line we need at least $[1/(2\epsilon)]$ such intervals to cover the unit interval.

In particular, using these bounds on $N(X,\epsilon)$ we see that

$$\overline{\dim}_B(X) \leq \limsup -\frac{\log([1/\epsilon]+1)}{\log \epsilon} = 1$$

and

$$\underline{\dim}_B(X) \geq \liminf -\frac{\log([1/(2\epsilon)])}{\log \epsilon} = 1$$

In particular, we deduce that the box dimension exists and $\dim_B(X) = 1$.

It may seem a little strange to call this box dimension rather than, say, ball dimension, since we uses covers by balls rather than boxes. However, we can go some way to explaining this in the next subsection.

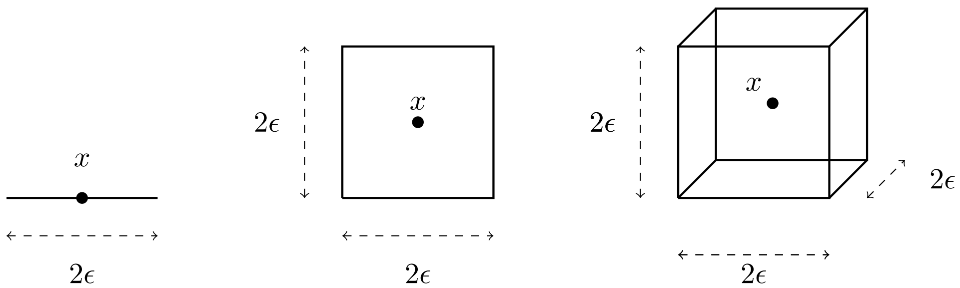
3.1.2 Variants on the definition of box dimension

It isn't actually very important to use balls in the covering for X. For example, could easily replace balls of size ϵ by "squares" or boxes $S(x,\epsilon)$ of size ϵ instead. This perhaps helps to explain the name "box dimension".

Definition 6. *Let $x = (\xi_1, \cdots, \xi_d)$ then we denote*

$$S(x,\epsilon) = (\xi_1 - \epsilon, \xi_1 + \epsilon) \times (\xi_2 - \epsilon, \xi_2 + \epsilon) \times \cdots \times (\xi_d - \epsilon, \xi_d + \epsilon)$$

i.e., a cube with edge sides length 2ϵ.



In one dimension these definitions of squares and balls coincide, but for $d \geq 2$ it can be more convenient to use one definition rather than the other.

Definition 7. *Let $\epsilon > 0$. By analogy with the cover by ϵ-balls, let $N_S(X,\epsilon)$ be the minimal number of ϵ-boxes needed to cover X, i.e.,*

$$N_S(X,\epsilon) = \min\{n\text{: } \exists x_1, \cdots, x_n \in X \text{ with } X \subset \cup_{i=1}^n S(x_i,\epsilon)\}$$

The next lemma shows that it doesn't really matter if we use balls or cubes to define the (upper and lower) Box dimension(s).

Lemma 6. *We can rewrite the upper box dimension as*

$$\overline{\dim}_B(X) = \limsup_{\epsilon \to 0} -\frac{\log N_S(X, \epsilon)}{\log \epsilon}$$

and the lower Box dimension as

$$\underline{\dim}_B(X) = \liminf_{\epsilon \to 0} -\frac{\log N_S(X, \epsilon)}{\log \epsilon}$$

Proof. Before we begin, we first note that a ball $B(x, \epsilon)$ of radius ϵ will fit snuggly inside a box $S(x, \epsilon)$ of side length 2ϵ. In particular, a (minimal) cover of X by ϵ-boxes gives rise to a cover by ϵ-balls with the same centres, from which we deduce that

$$N(x, \epsilon) \geq N_S(x, \epsilon). \tag{3}$$

On the other hand, an $\epsilon/\sqrt{d}$ -box $B(x, \epsilon/\sqrt{d})$ of side length $\epsilon/\sqrt{d}$ will sit inside a ball $B(x, \epsilon)$ of radius ϵ. In particular, we see that

$$N_S(x, \epsilon/\sqrt{d}) \geq N(x, \epsilon) \tag{4}$$

The result easily follows from (3) and (4) from the definitions since for the upper box dimension we have

$$\begin{aligned}
\overline{\dim}_B(X) &= \limsup_{\epsilon \to 0} -\frac{\log N(X, \epsilon)}{\log \epsilon} \\
&\leq \limsup_{\epsilon \to 0} -\frac{\log N_S(X, \epsilon/\sqrt{d})}{\log \epsilon} \\
&\leq \limsup_{\epsilon \to 0} -\frac{\log N(X, \epsilon/\sqrt{d})}{\log \epsilon} \\
&= \limsup_{\epsilon \to 0} -\frac{\log N(X, \epsilon/\sqrt{d})}{\log \epsilon/\sqrt{d}} = \overline{\dim}_B(X)
\end{aligned}$$

(since $\lim_{\epsilon \to 0} \frac{\log \epsilon}{\log(\epsilon/\sqrt{d})}) = 1$) and similarly for the lower Box dimension we have

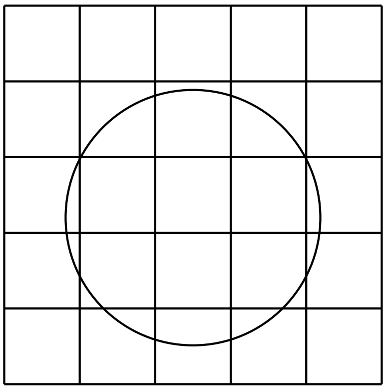
$$\begin{aligned}
\underline{\dim}_B(X) &= \liminf_{\epsilon \to 0} -\frac{\log N(X, \epsilon)}{\log \epsilon} \\
&\leq \liminf_{\epsilon \to 0} -\frac{\log N_S(X, \epsilon/\sqrt{d})}{\log \epsilon} \\
&\leq \liminf_{\epsilon \to 0} -\frac{\log N(X, \epsilon/\sqrt{d})}{\log \epsilon} = \underline{\dim}_B(X)
\end{aligned}$$

□

Figure 3.1: Boxes stacked like a cover by three dimensional cubes

Having shown that we can replace the balls by squares we can next see that we could restrict the squares to be those in a grid.

Example 8 ($d = 2$)**.** *If $d = 2$ then we could imagine X as a set drawn on graph paper of width ϵ. We could then count the number of squares in the graph paper which intersect X.*



Example 9 ($d = 3$)**.** *For $d = 3$ then this cover might resemble boxes stacked in a warehouse.*

In general, for any $d \geq 1$ and then a given $\epsilon > 0$ we can denote the family of ϵ-boxes associated with the standard ϵ-grid by

$$\left\{ \prod_{j=1}^{d} [m_j\epsilon, (m_j + 1)\epsilon] \, : \, m = (m_1, \cdots, m_d) \in \mathbb{Z}^d \right\}.$$

We then let

$$N_G(X, \epsilon) = \mathrm{Card} \left\{ m = (m_1, \cdots, m_d) \in \mathbb{Z}^d \, : \, \prod_{j=1}^{d} [m_j\epsilon, (m_j + 1)\epsilon] \cap X \neq \emptyset \right\}$$

denote the number of such ϵ-boxes intersecting the set X.

The next lemma shows that even with these particular choices of boxes we still recover the upper and lower box dimensions.

Lemma 7. *We can rewrite the upper box dimension as*

$$\overline{\dim}_B(X) = \limsup_{\epsilon \to 0} \frac{\log N_G(X, \epsilon)}{-\log \epsilon}$$

and the lower box dimension as

$$\underline{\dim}_B(X) = \limsup_{\epsilon \to 0} \frac{\log N_G(X, \epsilon)}{-\log \epsilon}$$

Proof. Let $\epsilon > 0$. By definition X intersects $N_G(X, \epsilon)$ of the ϵ-grid boxes described above. Consider the corresponding $(\epsilon\sqrt{d})$-balls $B('m + v, \epsilon\sqrt{d})$ centred at the *middle* points of

$$m' = \left(m_1 + \frac{1}{2}, m_2 + \frac{1}{2}, \cdots, m_d + \frac{1}{2}\right)$$

of boxes associated with the lattice points $m = (m_1, \cdots, m_d) \in \mathbb{Z}^d$ the grid corresponding to boxes intersecting X we see that since each box is contained in the corresponding ball, i.e.,

$$\prod_{i=1}^{d} [m_i\epsilon, (m_i + 1)\epsilon] \subset B(m', \epsilon\sqrt{d})$$

this gives a cover for X by $N_G(X, \epsilon)$ $\epsilon\sqrt{d}$- balls. Thus since $N(X, \delta\sqrt{d})$ is the cardinality of the smallest such cover we deduce that

$$N_G(X, \epsilon) \geq N(X, \epsilon\sqrt{d}). \tag{5}$$

On the other hand, we begin with the simple geometric observation:

Claim (Geometric observation). Let $\delta > 0$. Given $U \subset \mathbb{R}^d$ with

$$\mathrm{diam}(U) = \sup_{x,y \in U} |x - y| \leq \delta$$

then U is contained in a union of 3^d standard δ-grid cubes (consisting of any cube intersecting U and the $3^d - 1$ neighbouring cubes). This reminiscent of a Rubik cube.

Let $\epsilon > 0$. Assume that we have a (minimal) cover by $N(X, \epsilon)$ ϵ-balls. Since ϵ-ball has diameter 2ϵ and so we can apply the geometric observation with $\delta = 2\epsilon$ and where U is an ϵ-ball. In particular, every ϵ-ball in the cover can itself be covered by 3^n of these 2ϵ-cubes. In particular, X intersect

Figure 3.2: For $d = 3$ there are $3^3 - 1 = 26$ cubes neighbouring the original cube

$3^d N(X, 2\epsilon)$ of the 2ϵ-cubes from the standard grid with spacing 2ϵ, and thus it can only intersect a smaller number of such cubes, i.e.,. $3^d N(X, \epsilon) \geq N_G(X, 2\epsilon)$ or, on replacing ϵ by $\epsilon/2$,

$$3^d N(X, \epsilon/2) \geq N_G(X, \epsilon) \tag{6}$$

The result easily follows from (5) and (6) and the definitions, since for the upper box dimension we have

$$\begin{aligned}\overline{\dim}_B(X) &= \limsup_{\epsilon\to 0} -\frac{\log N(X, \epsilon\sqrt{d})}{\log \epsilon}\\ &\leq \limsup_{\epsilon\to 0} -\frac{\log N_G(X, \epsilon)}{\log \epsilon}\\ &\leq \limsup_{\epsilon\to 0} -\frac{\log(3^d N(X, \epsilon/2))}{\log \epsilon} = \overline{\dim}_B(X)\end{aligned}$$

since $\frac{d \log 3}{\log \epsilon} \to 0$ and $\frac{d \log 3}{\log(\epsilon/2)} \to 1$ as $\epsilon \to 0$. In particular, we have qualities throughout. Similarly, for the lower Box dimension we have

$$\begin{aligned}\underline{\dim}_B(X) &= \liminf_{\epsilon\to 0} -\frac{\log N(X, \epsilon\sqrt{d})}{\log \epsilon}\\ &\leq \liminf_{\epsilon\to 0} -\frac{\log N_G(X, \epsilon/\sqrt{d})}{\log \epsilon}\\ &\leq \liminf_{\epsilon\to 0} -\frac{\log(3^d N(X, \epsilon/2))}{\log \epsilon} = \underline{\dim}_B(X)\end{aligned}$$

again giving equalities throughout.

$\square$

In additional to the conceptual advantage, one of the advantage sof using a grid is that it simplifies the computation of the dimension in specific examples. Rather than having to consider all possible covers and choosing the most efficient we can restrict to covers using the grid boxes.

We illustrate this with the simple example of a cube (generalizing the previous example of the unit interval).

Example 10 (d-dimensional cube)**.** *Suppose $X = [0,1]^d$. Then by the previous construction*

$$[1/\epsilon]^d \leq N_G(X, \epsilon) \leq ([1/\epsilon] + 1)^d$$

In particular we see that

$$-d\frac{\log[1/\epsilon]}{\log \epsilon} \leq -\frac{\log N_G(X, \epsilon)}{\log \epsilon} \leq -d\frac{\log([]1/\epsilon] + 1)}{\log \epsilon}$$

Taking the limsup (and liminf) of the upper and lower bounds as $\epsilon \to 0$ gives

$$\overline{\dim}_B(X) = -\limsup_{\epsilon \to 0} \frac{\log N_G(X, \epsilon)}{\log \epsilon} = d \text{ and } \underline{\dim}_B(X) = -\liminf_{\epsilon \to 0} \frac{\log N_G(X, \epsilon)}{\log \epsilon} = d$$

Therefore we have that $\dim_B(X) = d$.

3.2 Examples of Cantor sets

A marginally more interesting example is the middle third Cantor set.

Example 11 (Middle third Cantor)**.** *Let $X = C_{1/3}$ be the usual middle third Cantor set. By virtue of its construction it is covered by the intervals left at ant stage in its iterative construction. More precisely, for each $n \geq 1$ we can cover the Cantor set by the union of 2^n intervals of the form*

$$C_{1/3}^{(n)} = \bigcup_{i_1=0}^{1} \cdots \bigcup_{i_n=0}^{1} \left[\frac{2i_1}{3} + \frac{2i_2}{3^2} + \cdots + \frac{2i_n}{3^n}, \frac{2i_1}{3} + \frac{2i_2}{3^2} + \cdots + \frac{2i_n}{3^n} + \frac{1}{3^n} \right] \tag{7}$$

where each interval has length $\frac{1}{3^n}$. Given $0 < \epsilon < 1$ we can choose n such that $\frac{1}{3^{n+1}} \leq 2\epsilon \leq \frac{1}{3^n}$. Since the 2^{n+1} intervals above at the nth level are each of length $\frac{1}{3^{n+1}}$ they can be increased in size to give intervals which give a cover by 2^{n+1} ϵ-balls (i.e., 2ϵ intervals). We thus deduce that

$$N(C, \epsilon) \leq 2^{n+1}.$$

In particular, we see from the definitions

$$\overline{\dim}_B(C) = -\limsup_{\epsilon \to 0} \frac{\log N(C, \epsilon)}{\log \epsilon} \leq -\limsup_{n \to +\infty} \frac{\log 2^{n+1}}{\log(3^n/2)} = \frac{\log 2}{\log 3}.$$

Conversely, given $\epsilon > 0$ consider a (minimal) cover of C by $N(C,\epsilon)$ ϵ-balls (i.e., intervals of length 2ϵ). We can choose n such that $\frac{1}{3^{n+1}} \le 2\epsilon \le \frac{1}{3^n}$. Since at the nth level the individual intervals in the construction of the Cantor set have separation at least $1/3^n$, we see that any ball in the minimal cover (of length $2\epsilon < \frac{1}{3^n}$) can intersect at most one of these 2^n intervals. In particular, we see that

$$N(C,\epsilon) \ge 2^n$$

Thus

$$\underline{\dim}_B(C) = -\limsup_{\epsilon \to 0} \frac{\log N(C,\epsilon)}{\log \epsilon} \ge -\limsup_{n \to +\infty} \frac{\log 2^n}{\log(3^{n+1}/2)} = \frac{\log 2}{\log 3}.$$

Therefore, comparing these inequalites and recalling $\underline{\dim}_B(C) \le \overline{\dim}_B(C)$, the box dimension exists and we have

$$\dim_B(C) = \frac{\log 2}{\log 3}.$$

A variation on this argument leads to the following generalization to the middle λ-Cantor set.

Example 12 (Middle λ-Cantor set)**.** *Let $0 < \lambda < 1$ and consider the λ-Cantor set C_λ then a similar argument gives that the Minkowski dimension is*

$$\dim_B(C_\lambda) = \frac{\log 2}{\log\left(\frac{1-\lambda}{2}\right)}.$$

Exercise 7. *Check the above formula.*

3.3 The limitations of Box Dimension

The next example starts to show the limitations of the box dimension. We might expect that a countable family of points has zero dimension, but this is not the case.

Example 13. *Consider the countable set*

$$X = \left\{ \frac{1}{k} : k \in \mathbb{N} \right\}.$$

Observe that the distance between the consecutive points $\frac{1}{k}$ and $\frac{1}{k+1}$ is

$$\left| \frac{1}{k+1} - \frac{1}{k} \right| = \frac{1}{k(k+1)} \le \frac{1}{k^2}.$$

Given $\epsilon > 0$ we can choose n so that $1/(n+1)^2 < 2\epsilon \leq 1/n^2$ then at least n distinct ϵ-balls are needed to cover the points $\{1, 1/2, \cdots, 1/n\}$. Thus we deduce that $N(X,\epsilon) \geq n$. Moreover the remaining points

$$\left\{\frac{1}{k} : k \geq n+1\right\} \subset \left[0, \frac{1}{n+1}\right]$$

in X can be covered by just n more intervals of length 2ϵ, or equivalently ϵ-balls. Thus we see that $N(X,\epsilon) \leq 2n$. We can now observe that

$$\underline{\dim}_B(X) = -\liminf_{\epsilon\to 0} \frac{\log N(X,\epsilon)}{\log \epsilon} \geq \limsup_{n\to 0} \frac{\log n}{\log(2(n+1)^2)} = \frac{1}{2}.$$

and

$$\underline{\dim}_B(X) = -\liminf_{\epsilon\to 0} \frac{\log N(X,\epsilon)}{\log \epsilon} \leq \limsup_{n\to 0} \frac{\log(2n)}{\log(2n^2)} = \frac{1}{2}.$$

In particular, since $\underline{\dim}_B(X) \leq \overline{\dim}_B(X)$ we see that the inequalities in the last two expressions are all equalities and deduce that $\underline{\dim}_B(X) = \dim_B(X) = \frac{1}{2}$ and thus $\dim_B(X) = \frac{1}{2}$.

In particular, in this example X is a *countable* set with $\dim_B(X) > 0$ which is a less desirable property for a dimension.

As an exercise we can consider a variant on this set.

Exercise 8. . *Fix $s > 0$ and consider the set*

$$X = \left\{\frac{1}{n^s} : n \in \mathbb{N}\right\}.$$

What is the Box dimension $\dim_B(X)$ of X? Justify your answer.

3.4 Basic properties of Box dimension

In this section we collect together some simple properties of Box dimension, and attempt to illustrate them with the examples we have been studied.

Proposition 2 (Lipschitz images)**.** *Let $X, Y \subset \mathbb{R}^d$ be bounded subsets. It $\pi : X \to Y$ is Lipschitz and surjective then $\overline{\dim}_B Y \leq \overline{\dim}_B X$ and $\underline{\dim}_B Y \leq \underline{\dim}_B X$*

Proof. Recall that for π to be Lipschitz there exists $C > 0$ such that $\|\pi(x) - \pi(y)\| \leq C\|x - y\|$ for all $x, y \in X$.

Assume we have a minimal ϵ-cover $\{B(x_i,\epsilon)\}_{i=1}^N$ for X by $N = N(X,\epsilon)$ ϵ-balls. Since for $y \in B(x_i,\epsilon)$ we have that $\|y - x_i\| < \epsilon$, the Lipschitz property implies that $\|\pi(y) - \pi(x_i)i\| < C\epsilon$ and thus the images of the balls will satisfy $\pi(B(x_i,\epsilon)) \subset B(\pi(x_i), C\epsilon)$, for $i = 1, \cdots, N$. In particular, by surjectivity of π the balls $\{B(\pi(x_i), C\epsilon)\}_{i=1}^N$ form a $C\epsilon$-cover for Y. In particular, $N(Y,\epsilon) \leq N(X, C\epsilon)$. The result then follows from the definitions, i.e.,

$$\underline{\dim}_B(Y) = -\liminf_{\epsilon\to 0} \frac{\log N(Y,\epsilon)}{\log \epsilon} \leq -\liminf_{\epsilon\to 0} \frac{\log N(X,C\epsilon)}{\log \epsilon} = \underline{\dim}_B(X)$$

and

$$\overline{\dim}_B(Y) = -\limsup_{\epsilon\to 0} \frac{\log N(Y,\epsilon)}{\log \epsilon} \leq -\limsup_{\epsilon\to 0} \frac{\log N(X,C\epsilon)}{\log \epsilon} = \overline{\dim}_B(X)$$

since $\lim_{\epsilon\to 0} \frac{\log C}{\log \epsilon}$.

□

A simple example of this is where π is a projection.

Example 14. *In particular, let $\pi : \mathbb{R}^2 \to \mathbb{R}$ given by $\pi(x,y) = x$. Then we let $\pi(X) = \{\pi(x,y) : (x,y) \in X\}$.*

Since $\pi : \mathbb{R}^2 \to \mathbb{R}$ is a Lipschitz map we have that $\overline{\dim}_B Y \leq \overline{\dim}_B X$ and $\underline{\dim}_B Y \leq \underline{\dim}_B X$.

An interesting application is the following.

Example 15 (Apollonian Circle Packings)**.** *The Apollonian circle packing described earlier depends on the initial choice of sizes of circles. However, the dimension of the limit set does not.*

The reason for this is because given two sets of three initial tangent circles inscribed inside the unit circle there is a Möbius map which maps one set of circles to the other. Furthermore, since Möbius maps take circles to circles we deduce that the entire circle packings are mapped to each other. Finally, a Möbius map is clearly Lipschitz so they share the same box dimension.

We also have a simple, but useful, corollary.

Corollary 1. *If we homothetically scale a set by a. map $\pi : \mathbb{R}^d \to \mathbb{R}^d$ defined by*

$$\pi(x_1, \cdots, x_d) = (\lambda x_1, \cdots, \lambda x_n). + (z_1, \cdots, z_d)$$

where $\lambda \in \mathbb{R}^+$ and $(z_1, \cdots, z_d) \in \mathbb{R}^d$, we have that $\dim_B(X) = \dim_B(\pi(X))$

Proof. We first observe that $\|\pi(x) - \pi(y)\| = \lambda\|x - y\|$, for $x, y \in \mathbb{R}^d$. In particular, π is Lipschitz. Similarly, π^{-1} is Lipschitz. By the previous results, since $\pi^{-1} \circ \pi(x) = x$ we have

$$\overline{\dim}_B(X) = \overline{\dim}_B(\pi^{-1}\pi X) \leq \overline{\dim}_B(\pi(X)) \leq \overline{\dim}_B(X)$$

and thus $\overline{\dim}_B(X) = \overline{\dim}_B(\pi(X))$. Similarly, $\underline{\dim}_B(X) = \underline{\dim}_B(\pi(X))$.

□

A slight generalization of the result for Lipschitz maps is the following.

Proposition 3. *Assume that $\pi : X \to Y$ is α-Hölder continuous and surjective. Then* $\overline{\dim}_B Y \leq \alpha \overline{\dim}_B X$ *and* $\underline{\dim}_B Y \leq \alpha \underline{\dim}_B X$

Exercise. Prove the result above

By way of a reality check on these inequalities we can consider what happens for Cantor sets.

Example 16. *Given a middle λ-Cantor set C_λ we have observed that there is a surjective Hölder continuous map $\pi : C_\lambda \to C_{1/3}$. The formulae for the dimensions give us bounds on the possible Hölder exponents of any such maps.*

Another type of basic property is the following.

Proposition 4 (Inclusion)**.** *Let $X \subset Y \subset \mathbb{R}^d$ be bounded sets then*

$$\underline{\dim}_B(X) \leq \underline{\dim}_B(Y) \text{ and } \overline{\dim}_B(X) \leq \overline{\dim}_B(Y)$$

Proof. Let $\epsilon > 0$. Let $\{B(x_i, \epsilon)\}_{i=1}^N$ be an ϵ-cover for Y of minimal cardinality $N = N(Y, \epsilon)$. Since $X \subset Y \subset \cup_{j=1}^N B(x_i, \epsilon)$ we can deduce that $N(X, \epsilon) \leq N(Y, \epsilon)$. We see from the definitions

$$\underline{\dim}_B(X) = -\liminf_{\epsilon \to 0} \frac{\log N(X, \epsilon)}{\log \epsilon} \leq -\liminf_{\epsilon \to 0} \frac{\log N(Y, \epsilon)}{\log \epsilon} = \underline{\dim}_B(Y).$$

□

Yet another type of result is the following

Proposition 5 (Topological results)**.** *Given $X \subset \mathbb{R}^d$ be a bounded subset.*

1. *Let $\overline{X}$ denote the closure of X then* $\underline{\dim}_B(X) = \underline{\dim}_B(\overline{X})$ *and* $\overline{\dim}_B(X) = \overline{\dim}_B(\overline{X})$.
2. *If $X \subset \mathbb{R}^d$ has nonempty interior then* $\dim_B(X) = d$.

Proof. For part 1, since $X \subset \overline{X} =: Y$ we can apply the inclusion property to get $\underline{\dim}_B(X) \leq \underline{\dim}_B(\overline{X})$ and $\overline{\dim}_B(X) \leq \overline{\dim}_B(\overline{X})$.

Given $\epsilon > 0$, we can choose a (minimal) cover $\{B(x_i, \epsilon)\}_{i=1}^N$ for X, where $N = N(X, \epsilon)$. We therefore see that $\{B(x_i, 2\epsilon)\}_{i=1}^N$ for $\overline{X}$, Thus $N(\overline{X}, 2\epsilon) \leq N(X, \epsilon)$. Thus

$$\overline{\dim}_B(X) = -\limsup_{\epsilon \to 0} \frac{\log N(\overline{X}, 2\epsilon)}{\log \epsilon} \leq -\liminf_{\epsilon \to 0} \frac{\log N(X, \epsilon)}{\log \epsilon} = \overline{\dim}_B(X)$$

and

$$\underline{\dim}_B(X) = -\limsup_{\epsilon \to 0} \frac{\log N(X, 2\epsilon)}{\log \epsilon} \leq -\limsup_{\epsilon \to 0} \frac{\log N(Y X \epsilon)}{\log \epsilon} = \overline{\dim}_B(Y).$$

For part 2, we can choose such a π with $\pi([0,1]^d)$ contained in the interior of X. In particular, $\underline{\dim}_B(U) \geq \underline{\dim}_B(\pi([0,1]^d)) = \underline{\dim}_B([0,1]^d) = d$ by the previous corollary. We can easily see that $\overline{\dim}_B(U) \leq d$ and thus $underline\dim_B(U) = overline\dim_B(U) = d$. Therefore $\dim_B(U) = d$, as required. $\square$

We now come to simple result.

Lemma 8 (Finite domination)**.** *Given bounded sets $X, Y \subset \mathbb{R}^d$ then*

$$\overline{\dim}_B(X \cup Y) = \max\{\overline{\dim}_B(X), \overline{\dim}_B(Y)\}.$$

Proof. Let $\epsilon > 0$¿. Let $\{B(x_i,\epsilon)\}_{i=1}^N$ be a (minimal) ϵ-cover of minimal cover with $N = N(X,\epsilon)$ and let $\{B(y_j,\epsilon)\}_{j=1}^M$ be a (minimal) ϵ-cover of minimal cover with $M = N(Y,\epsilon)$. We then have an ϵ-cover for $X \cup Y$ of the form $\{B(x_i,\epsilon)\}_{i=1}^N \cup \{B(y_j,\epsilon)\}_{j=1}^M$ and cardinality $N + M = N(X,\epsilon) + N(Y,\epsilon)$. In particular,

$$N(X \cup Y, \epsilon) \leq N(X,\epsilon) + N(Y,\epsilon) \leq 2\max\{N(X,\epsilon), N(Y,\epsilon)\}.$$

In particular,

$$\begin{aligned}
\overline{\dim}_B(X \cup Y) &\leq -\limsup_{\epsilon\to 0} \frac{\log N(Y \cup X, \epsilon)}{\log \epsilon} \\
&\leq -\limsup_{\epsilon\to 0} \frac{\log\left(2\max\{N(X,\epsilon), N(Y,\epsilon)\}\right)}{\log \epsilon} \\
&\leq \max\left\{-\limsup_{\epsilon\to 0} \frac{\log(2N(X,\epsilon))}{\log \epsilon}, -\limsup_{\epsilon\to 0} \frac{\log(2N(Y,\epsilon))}{\log \epsilon}\right\} \\
&\leq \max\{\overline{\dim}_B(X), \overline{\dim}_B(Y)\}
\end{aligned} \tag{1}$$

by the nature of the lim sup.

On the other hand, since $X, Y \subset X \cup Y$ we have that

$$\overline{\dim}_B(X) \leq \overline{\dim}_B(X \cup Y) \text{ and } \overline{\dim}_B(Y) \leq \overline{\dim}_B(X \cup Y)$$

and thus

$$\max\{\overline{\dim}_B(X), \overline{\dim}_B(Y)\} \leq \overline{\dim}_B(X \cup Y). \tag{2}$$

Comparing (1) and (2) proves the result.

$\square$

Remark 7. *The corresponding result doesn't hold for $\underline{\dim}_B$. There exist examples with*

$$\max\{\underline{\dim}_B(X), \underline{\dim}_B(Y)\} < \overline{\dim}_B(X \cup Y).$$

Remark 8. *In particular, as we observed in the introduction example ??? illustrates that box dimensions are not countably dominated. In particular, the set X a countable union of points x_i, each of which has zero box dimension, but is itself of non-zero box dimension (i.e., $\frac{1}{2} = \overline{\dim}_B(\cup_i x_i) > \sup_i\{\overline{\dim}_B(x_i)\} = 0$).*

We also have the following.

Lemma 9 (Products)**.** *For bounded sets $X \subset \mathbb{R}^d$ and $Y \subset \mathbb{R}^l$ we have for $X \times Y \subset \mathbb{R}^{d+l}$ that*

$$\overline{\dim}_B(X \times Y) \leq \overline{\dim}_B(X) + \overline{\dim}_B(Y)$$

and

$$\underline{\dim}_B(X \times Y) \geq \underline{\dim}_B(X) + \underline{\dim}_B(Y)$$

Proof. Let $\epsilon > 0$. It is convenient to use the definition using ϵ-grid boxes. In particular, given that the ϵ-grid for $\mathbb{R}^{d+l}$ is a product of the ϵ-grid for $\mathbb{R}^d$ and the ϵ-grid for $\mathbb{R}^l$, we then see that

$$N_G(X \times Y, \epsilon) = N_G(X, \epsilon) N_G(Y, \epsilon).$$

Thus we have that

$$\begin{aligned} \overline{\dim}_B(X \times Y) &\leq -\limsup_{\epsilon \to 0} \frac{\log(N_G(X \times Y, \epsilon))}{\log \epsilon} \\ &\leq -\limsup_{\epsilon \to 0} \frac{\log(N_G(X, \epsilon))}{\log \epsilon} - \limsup_{\epsilon \to 0} \frac{\log(N_G(Y, \epsilon))}{\log \epsilon} \\ &= \overline{\dim}_B(X) + \overline{\dim}_B(Y) \end{aligned}$$

(where for functions a_ϵ and b_ϵ we have that $\limsup_{\epsilon \to 0} a_\epsilon + b_\epsilon \leq \limsup_{\epsilon \to 0} a_\epsilon + \limsup_{\epsilon \to 0} b_\epsilon$).

The corresponding result for $\underline{\dim}_B$ is similar, i.e.,

$$\begin{aligned} \underline{\dim}_B(X \times Y) &\geq -\liminf_{\epsilon \to 0} \frac{\log(N_G(X \times Y, \epsilon))}{\log \epsilon} \\ &\geq -\liminf_{\epsilon \to 0} \frac{\log(N_G(X, \epsilon))}{\log \epsilon} - \liminf_{\epsilon \to 0} \frac{\log(N_G(Y, \epsilon))}{\log \epsilon} \\ &= \underline{\dim}_B(X) + \underline{\dim}_B(Y) \end{aligned}$$

(where for functions a_ϵ and b_ϵ we have that $\liminf_{\epsilon \to 0} a_\epsilon + b_\epsilon \geq \liminf_{\epsilon \to 0} a_\epsilon + \liminf_{\epsilon \to 0} b_\epsilon$.

□

This has an immediate corollary

Corollary 2. *For bounded sets $X \subset \mathbb{R}^d$ and $Y \subset \mathbb{R}^l$ where* $\dim_B(X)$ *and* $\dim_B(Y)$ *exist*

$$\dim_B(X \times Y) = \dim_B(X) + \dim_B(Y)$$

We can illustrate this with a simple example.

Example 17. *Consider the Cantor set obtained by replacing the unit square in the plane by four equal sub-squares of side length $0 < \lambda < \frac{1}{2}$. This is then iterated to get a two dimension Cantor set X by analogy with the middle third Cantor set.*

If the 4 boxes are horizontally and vertically then the corresponding set is $X = C_\lambda \times C_\lambda$ where $\nu = \frac{1-\lambda}{2}$. The product theorem tells us that

$$\dim_B(X) = -2.\dim_B(X) = 2\left(\frac{\log 2}{\log \alpha}\right)$$

3.5 More Examples

Let us consider a few more of the examples we encountered in the last chapter.

3.5.1 von Koch snowflake

The von Koch curve X is a standard fractal construction. Starting from the interval $X_0 = [0, 1]$ we associate to each piecewise linear curve X_n in the plane (which is a union of 4^n segments of length 3^{-n}) a new one X_{n+1}. This is done by replacing the middle third of each line segment by the other two sides of an equilateral triangle bases there. Alternatively, one can start from an equilateral triangle and apply this iterative procedure to each of the sides one gets a "snowflake curve".

Lemma 10. *For the von Koch curve both the Box dimension and the Hausdorff dimension are $\frac{\log 4}{\log 3}$.*

Proof. We start with a star shape X_1 which is the union of 12 straight line segments. When $\epsilon_n = \frac{1}{3^n}$, the set X_n is the union of 3×4^n intervals of length $\epsilon_n = 3^{-n}$. We can cover X_n by balls of size ϵ_n by associating to each edge a ball of radius $\frac{\epsilon_n}{2}$ centred at the midpoints of the side. It is easy to see that this is also a cover for X. Therefore, we deduce that $N(\epsilon_n) \leq 3 \times 4^n$.

Moreover, it is easy(-ish) to see that any ball of diameter ϵ_n intersecting X can intersect at most two intervals from X_n, and thus $N(\epsilon_n) \geq 3 \times 4^{n-1}$. For any $\epsilon > 0$ we can choose $\epsilon_{n+1} \leq \epsilon < \epsilon_n$ and we know that $N(\epsilon_n) \leq N(\epsilon) \leq N(\epsilon_{n+1})$. Then

$$\frac{n-1}{(n+1)}\frac{\log 4}{\log 3} \leq \frac{\log N(\epsilon_n)}{\log(\frac{1}{\epsilon_{n+1}})} \leq \frac{\log N(\epsilon)}{\log(\frac{1}{\epsilon})} \leq \frac{\log N(\epsilon_{n+1})}{\log(\frac{1}{\epsilon_n})} \leq \frac{(n+1)}{n}\frac{\log 4}{\log 3}.$$

Letting $n \to +\infty$ shows that $\dim_B(X) = \lim_{\epsilon\to 0} \frac{\log N(\epsilon)}{\log(\frac{1}{\epsilon})} = \frac{\log 4}{\log 3}$. We postpone the proof that $\dim_B(X) = \dim_H(X)$ until later, when we shall show a more general result. $\square$

3.5.2 Sierpinski triangle

Let X be the Sierpinski triangle.

Lemma 11. *For the Sierpinski carpet the Box dimension is equal to* $\dim_B(X) = \frac{\log 3}{\log 2}$

Proof. For any $n \geq 1$, we can consider the grid of size $\epsilon_n = \frac{1}{2^n}$. In particular, at the nth step of the construction we have a union X_n of 3^n triangles of size $\frac{1}{2^n}$. We can then write $X = \cap_{n=0}^{\infty} X_n$. However, each triangle in X_n corresponds to one half of a ϵ_n-grid box. From this we see that $N_G(X, \epsilon_n) = N_G(X_n, \epsilon_n) = 3^n$. In particular, for all $n \geq 1$ we have that

$$\frac{\log N_G(X \epsilon_n)}{\log(\frac{1}{\epsilon_n})} = \frac{\log 3}{\log 2}$$

Letting $n \to +\infty$, gives that $\dim_B(X) = \frac{\log 3}{\log 2}$. $\square$

3.5.3 Bedford-McMullen carpets

The Bedford-McMullen carpets have a satisfyingly complicated formula for the value of the box dimension.

Recall that the construction is based on choosing rectangles in a $m \times l$ grid of the unit square. More precisely, given

$$S \subset \{0, 1, \ldots, m-1\} \times \{0, 1, \ldots, l-1\}$$

we can associate an affine "Sierpinski carpet":

$$\Lambda = \left\{ \left(\sum_{n=1}^{\infty} \frac{i_n}{l^n}, \sum_{n=1}^{\infty} \frac{j_n}{m^n} \right) : (i_n, j_n) \in S \right\}$$

Let

$$a = \mathrm{Card}(\mathcal{S}) \leq lm$$

be the total number of rsuch ectangles. Assume for simplicity that $l \geq m \geq 2$ and that every row contains a rectangle.

Lemma 12. *If $a = Card(S)$ then*

$$dim_B(\Lambda) = 1 + \frac{\log \frac{a}{m}}{\log l}$$

Proof. At the nth level of the construction (denoted X_n) we have a^n rectangles of size $l^{-n} \times m^{-n}$. More precisely, each rectangle at the n th level of the construction corresponds to a finite string

$$(i_1, j_1), \cdots (i_n j_n) \in \mathcal{S}.$$

Each such rectangle has shorter side l^{-n} and longer side height m^{-n}. Moreover, we can cover each rectangle by approximately $(l/m)^n$ squares of size l^{-n} and, because no rows are empty, this many are all needed. Thus we get an upper bound on the number of l^{-n}-squares needed to cover X_n we get a bounds of the $N(X, l^{-j}) \approx a^j (l/m)^j$. Thus

$$\begin{aligned}
\dim_B(\Lambda) &= \lim_{\epsilon \to 0} -\frac{\log N(\epsilon)}{\log \epsilon} \\
&= \lim_{j \to +\infty} \frac{\log(a(l/m)^j)}{\log l^j} \\
&= \frac{\log a}{\log l} + 1 - \frac{\log m}{\log l} \\
&= 1 + \frac{\log \frac{a}{m}}{\log l}
\end{aligned}$$

as required. □

Example 18. *The Box dimension for the Hironaka curve* Λ *corresponding to* $S = \{(0,0), (1,1), (0,2)\}$ *can be explicitly computed. More precisely, in this case*

$$dim_B(\Lambda) = 1 + \log_3 \left(\frac{3}{2}\right) = 1.36907\ldots.$$

3.5.4 Examples where the lower box dimension is strictly less than the upper box dimension

We now return to a comment we made earlier, to the effect that the upper box dimension and lower box dimension may not agree. We now describe a simple construction.

In fact, we modify the easy middle third interval construction by removing at the nth stage either:

1. the middle third interval (leaving two intervals); or
2. both the left and right third intervals (leaving the middle interval).

Of course at the nth level of the construction the intervals have the same length 3^{-n}. However, the number of intervals required varies depending on which choice above is made.

Let us denote $m_k = 10^{k^2}$. At the nth step of the construction:

1. we take out the middle third interval (leaving two third intervals) if $m_{2k} \leq n < m_{2k+1}$, for some $k \geq 1$; and

2. we leave the middle third interval (taking out the left and right third intervals) if $m_{2k+1} \leq n < m_{2k+2}$ for some $k \geq 1$.

The length of the intervals at the nth stage of the construction will be $\frac{1}{3^n}$. The number of intervals at the nth level of the construction can be denoted by $N(n)$ We can now make some simple observations.

1. We see from the construction that the number of intervals at level m_{2k+2} is the same as the number of intervals at the level m_{2k+1}, i.e.,

$$N\left(m_{2k+2}\right) = N\left(m_{2k+1}\right)$$

2. On the other hand the number of intervals at the level m_{2k+1} is $2^{m_{2k+1}-m_{2k}}$ times the number of intervals at the level m_{2k}, i.e.,

$$N(m_{2k+1}) = 2^{m_{2k+1}-m_{2k}} N(n_{2k})$$

3. Furthermore, we observe that

$$\frac{m_{k+1} - m_k}{m_k} = \frac{10^{(k+1)^2} - 10^{k^2}}{10^{(k+1)^2}} = 1 - 10^{-(2k+1)} \to 0 \text{ as } k \to +\infty.$$

In particular,

Let X denote the associated Cantor set. In particular, following the same sort of reasoning as for the middle third Cantor set we can deduce that

$$\overline{\dim}_B(X) = \limsup_{k \to +\infty} \frac{\log N(m_{2k+1})}{\log 3^{m_{2k+1}}} = \frac{\log 2}{\log 3}$$

and

$$\underline{\dim}_B(X) = \limsup_{k \to +\infty} \frac{\log N(m_{2k})}{\log 3^{m_{2k}}} = 0$$

This shows the required difference.

3.6 Translates of Cantor sets

We finish with a simple application to adding Cantor sets. For context we consider the sum of the middle third Cantor set C with itself defined by

$$C + C := \{x + y : x, y \in C\}$$

The first observation is trivial.

Lemma 13. $C + C = [0, 2]$

Proof. We can expand $x, y \in C$ as

$$x = \sum_{n=1}^{\infty} \frac{a_n}{3^n}, y = \sum_{n=1}^{\infty} \frac{b_n}{3^n}, \text{ where } a_n, b_n \in \{0, 2\}.$$

Therefore, we can write

$$x + y = \sum_{n=1}^{\infty} \frac{a_n + b_n}{3^n} \text{ where } a_n + b_n \in \{0, 2, 4\}$$

In particular, we can achieve every expansion $2 \sum_{n=1}^{\infty} \frac{d_n}{3^n}$ where $d_n \in \{0, 1, 2\}$ from which the result immediately follows

□

Let C be the middle third Cantor set. Given $-1 \leq t \leq 1$ we denote

$$(C + t) \cap C = \{y \in C : \exists x \in C \text{ with } y = x + t\}$$

Theorem 2. *For all but a zero measure set of* $x \in [-1, 1]$ *we have that*

$$\dim_B((C + t) \cap C) = \frac{1}{3} \frac{\log 2}{\log 3}$$

Proof. To begin, we observe that any $t \in [-1, 1]$ can be written in the form

$$t = \sum_{n=1}^{\infty} \frac{c_n}{3^n}$$

where $c_n \in \{-2, 0, 2\}$. Moreover, this expansion is unique, except for finitely many cases where there are finitely many non-zero c_n. On the other hand, any $y \in C$ can be written uniquely in the form

$$y = \sum_{n=1}^{\infty} \frac{c_n}{3^n}$$

where $c_n \in \{0, 2\}$. If we also assume $y \in (C + t)$ then

$$\sum_{n=1}^{\infty} \frac{c_n + t_n}{3^n} \in C$$

which implies that $c_n + t_n \in \{0, 2\}$. This imposes the following conditions:

1. If $t_n = 2$ then $y_n = 2$; and
2. If $t_n = -2$ then $y_0 = 0$

Thus for $y \in (C + t) \cap C$ the coefficient y_n is completely determined unless $t_n = 0$, when it can take either of the two values. Since previously showed almost all t are normal (base 3) we know that $t_n = 0$ with frequency 1/3. In particular, we deduce that $N(C \cap (C + t)) = 2^{n/3(1+o(1)), \frac{1}{3^n}}$. From this we see that $\dim_B((C + t) \cap C) = \frac{1}{3} \frac{\log 2}{\log 3}$ □

Chapter 4

Hausdorff Dimension

We now want to move onto a more sophisticated version of dimension, called *Hausdorff dimension.* [1] For some sets the two notions agree, for others they disagree and then the value of the Hausdorff dimension gives more information.

4.1 Definition of Hausdorff Dimension

Given a set X and $\delta > 0$ and $\epsilon > 0$ we define

$$H_\epsilon^\delta(X) = \inf \left\{ \sum_{i=1}^{N} \operatorname{diam}(U_i)^\delta \,:\, X \subset \cup_{i=1}^n U_i, \operatorname{diam}(U_i) \leq \epsilon \right\}$$

where the infimum is taken over all finite open covers $\{U_i\}$ the diameter $\operatorname{diam}(U_i)$ of each set being atiso at most $\epsilon > 0$. Observe from the basic defintion that $\epsilon \mapsto H_\epsilon^\delta(X)$ is monotone decreasing. The δ-dimensional Hausdorff measure of X comes by taking the limit as ϵ tend to 0, i.e.,

$$H^\delta(X) = \lim_{\epsilon \to 0} H_\epsilon^\delta(X).$$

Lemma 14. *If $H^\alpha(X) < +\infty$ then $H^\beta(X) = 0$ for any $\beta > \alpha$. Similarly, if $H^\alpha(X) > 0$ then $H^\beta(X) = +\infty$ for any $\beta < \alpha$*

Proof. For the first part, it follows from the definition of $H^\alpha(X)$ that

$$H_\epsilon^\beta(X) \leq \epsilon^{\beta-\alpha} H_\epsilon^\alpha(X)$$

Letting $\epsilon \to 0$ gives the required result.

The other inequality follows similarly.

□

[1]It is named after Felix Hausdorff, a distinguished mathematician and great intellectual. Unfortunately, who came to am unfortunate end when his wife, sister-in-law and he committed suicide in Bonn in 1942, rather than being deported to a concentration camp.

Thus if we think of $H^\alpha_\epsilon(X)$ as a funtion of α it must take the values 0 or $+\infty$, except possibly an one particular value. This change occurs at the Hausdorff Dimension of the set.

Definition 8. *The Hausdorff dimension of X is the value*

$$\dim_H(X) = \inf\{\alpha : H^\alpha(X) = 0\} = \sup\{\beta : H^\beta(X) = +\}$$

The definition is clearly more complicated than in the case of Box dimension.

4.2 Hausdorff dimension is bounded above by Box dimension

The following relationship to Box dimension is clear.

Lemma 15. *The Hausdorff dimension of a set X is bounded by the lower box dimension, i.e.,* $\dim_H(X) \le \underline{\dim}_B(X)$.

Proof. Let $\delta > 0$. Let $\{B(x_i, \delta)\}$ be a (minimal) δ-cover for X by δ-balls of smallest cardinality, i.e., $N = N(X, \delta)$. Then since this constitutes a cover for X be sets of diameter less than 2δ we can write

$$H_t^{2\delta}(X) \le N(X, \delta)(2\delta)^t \tag{1}$$

For any $t > \underline{\dim}_B(X)$ it follows from the definition of $\underline{\dim}_B(X)$ that there is a sequence $\delta_k \to 0$ such that

$$-\frac{\log N(X, \delta_k)}{\log \delta_k} < t.$$

In particular, we have that

$$N(X, \delta_k) < \delta_k^{-t}. \tag{2}$$

For k sufficiently large, we can choose $\delta_k < \delta$ and then bound

$$H^t_\delta(X) \le H^t_{\delta_k}(X) \le N(X, \delta_k)(2\delta_k)^t < 2^t$$

using (1) and (2). Thus

$$H^t(X) \le \lim_{\delta \to 0} H^t_\delta(X) \le 2^t < +\infty.$$

In particular, $\dim_H(X) < t$. Since $t > \underline{\dim}_B(X)$ can be chosen arbitrarily we deduce that $\dim_H(X) \le \underline{\dim}_B(X)$. $\square$

4.3 Simple examples

As in the case of box dimension, we can test out the definition on some very simple examples.

Example 19 (Single point). *Let $X - \{x_0\}$ be a single point then since $\dim_H(X) \leq \underline{\dim}_B(X)$ we see that*

$$\dim_H(X) = \underline{\dim}_B(X) = 0$$

.

To see that there can be a strict inequality in $\dim_H(X) \leq \underline{\dim}_B(X)$ we revisit a familiar example.

Example 20. *Consider the countable set*

$$X = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}.$$

We have already see that $\dim_B(X) = \frac{1}{2}$. However, we claim that $\dim_H(X) = 0$. To see this, observe that for any $\epsilon > 0$ we can cover the each point $\frac{1}{n} \in X$ by an open set $U_n = B(\frac{1}{n}, \epsilon 2^{-n})$ of decreasing size. Thus for any $\delta > 0$ we can under this bound

$$H^\delta(X) \leq H^\delta_\epsilon(X) \leq \epsilon^\delta \sum_{k=1}^{\infty} 2^{-\delta k} = \frac{\epsilon^\delta}{1 - 2^{-\delta}} < +\infty.$$

Then by definition

$$\dim_H(X) = \inf\{\alpha : H^\alpha(X) = 0\} = 0$$

as claimed.

We can also consider the familiar example of a Cantor set

Example 21 (Middle third Cantor set). *Let X be the middle third Cantor set. We already know that*

$$\dim_H(X) \leq \dim_B(X) \leq \frac{\log 2}{\log 3}$$

We claim this is actually an equality and out method of showing this is a blueprint for a general method used for many different examples.

We want to associate to each of the 2^n intervals $I_1^{(n)}, I_2^{(n)}, \cdots, I_{2^n}^{(n)}$ occurring in the nth levels the same weight (or mass or measure) $\frac{1}{2^n}$, i.e., we can write $\mu(I_k^{(n)}) = \frac{1}{2^n}$ and observe that these are consistent in that for $I_{2k+1}^{(n)}, I_{2k+2}^{(n)} \subset I_k^{(n=1)}$ we have that

$$\mu(I_k^{(n)}) + \mu(I_k^{(n)}) = \frac{1}{2^n} + \frac{1}{2^n} = \frac{1}{2^{n-1}} = \mu(I_k^{(n-1)}).$$

Given $\delta > 0$ *we can consider a cover* $\{U_j\}$ *for* X *with* $\max_j \mathrm{diam}(U_j) < \delta$ *and then let* $V_j = U_j \cap X$ *be the restriction to* X*. For each* j *we can choose* $r_j \in \mathbb{N}$ *such that*

$$\frac{1}{3^{r_j+1}} \leq \mathrm{diam}(V_j) \leq \frac{1}{3^{r_j}}.$$

Since the distance between the 2^{r_j} *level* r_j *intervals is at least* $\frac{1}{3^{r_j}}$*, the set* V_j *can intersect at most one of the level* r_j *intervals and thus* $mu(V_j) \leq 2^{-r_j}$*. If* $\alpha = \frac{\log 2}{\log 3}$ *then we can write*

$$\mu(V_j) \leq 2^{-r_j} = 2.2^{-(r_j+1)} = 2.(3^{-(r_j+1)})^\alpha \leq 2(\mathrm{diam}(V_j))^\alpha.$$

Thus

$$1 = \mu(X) \leq \sum_j \mu(V_j) \leq 2\sum_j (\mathrm{diam}(V_j))^\alpha.$$

Therefore, for any δ*-cover we have*

$$\sum_j (\mathrm{diam}(U_j))^\alpha \geq \frac{1}{2}$$

and taking the infimum over all δ*-covers gives* $H_\alpha^\delta(X) \geq \frac{1}{2}$*. Moreover, letting* $\delta \to 0$ *we still have* $H^\alpha(X) \geq \frac{1}{2}$*. Since* $\dim_H(X) = \inf\{\beta : H^\beta(X) = 0\}$ *we see that* $\dim_H(X) \geq \alpha$*. Comparing this with the reverse bound which came from the box dimension gives the result.*

The same basic argument can be used in similar examples, such as the following.

Example 22. *Let us return to example ??. Since we have that*

$$\dim_H(X) \leq \dim_B(X) = -\frac{\log 4}{\log \lambda} =: d$$

It remains to get a lower bound by putting an appropriate measure on the set. To this end we can consider the probability measure on the set X *for which each of the* 4^n *squares of the* n*th level has same mass* $\frac{1}{4^n}$*. We claim that there exists* $C > 0$ *such that*

$$\mu(B(x,r)) \leq Cr^d \tag{1}$$

for all balls. To prove this claim, assume $B(x,r)$ *intersects* X *and choose* n *so that* $\lambda^{n+1} \leq r < \lambda^n$*. Then* $B(x,r)$ *intersects at most* 4 *of the* n*th generation squares and so*

$$\mu(B(x,r) \cap X) \leq 4\frac{1}{4^n} \leq 4\lambda^{nd} \leq 4\lambda^{-d}r^d$$

and the result follows with $C = 4\lambda^{-d}$*. Using this, we see that given any overing of* X *by ball* $\{B(x_i, r_i)\}$ *we have*

$$1 \leq \sum_i \mu(B(x_i, r_i)) \leq C \sum_i r_i^d$$

which shows that $H^d(X) \leq 1$*. Therefore* $\dim_H(X) \leq d$

This argument proves a basic version of the Mass Distribution Principle: upper bounds for Hausdorff dimension come from the Minkowski dimension, but lower bounds come from finding a suitable measure supported on the set X*. We will formulate this more generally below.*

4.4 Lower bounds on Hausdorff Dimension

It is convenient to formulate the method above as a general principle.

4.4.1 Mass distribution theorem

We now consider one of the basic techniques for Hausdorff dimension. The usual way to get a lower bound on the Hausdorff dimension is to use probability measures.

For the moment we only need to associate values $\mu(A) \in \mathbb{R}^+$ where A is either an element of the refined partition or an open set or an intersection of such sets. Moreover, we only need the natural properties:

1. If $A \subset A'$ then $\mu(A) \subset \mu(A')$;
2. If $(A_i)_i$ then $\mu(\cup_i A_i) \leq \sum_i \mu(A_i)$

A measure μ on X is called a probability measure is $\mu(X) = 1$.

We can consider measures μ as associating to appropriate sets Y a *mass* or postive weight.

Moreover, if we partition such a set Y into smaller suitable subsets $Y = Y_1 \cup \cdots \cup Y_k$ then the mass from Y need to be distributed between $Y_1, \cdots, Y_k$.

In the previous example, the suitable sets were the 4^n squares at the nth level.

Lemma 16 (Mass distribution principle)**.** *Let* $\alpha > 0$*. Assume that the compact set* $X \subset \mathbb{R}^d$ *supports a probability measure* μ *and there exists* $C > 0$ *such that for every* $x \in X$ *we have a uniform bound*

$$\mu(B(x, r)) \leq Cr^\alpha \text{ for all } r > 0.$$

Then $\dim_H(X) > \alpha$*.*

Proof. Given $\epsilon > 0$, let $\{U_i\}$ be a ϵ-cover for X. For each open set U_i can choose $r_i > \mathrm{diam}(U_i)$ and balls $B(x_i, r_i) \supset U_i$ containing the set. By assumption we have

$$\mu(U_i) \leq \mu(B(x_i, r_i)) \leq C r_i^\alpha \leq C(\mathrm{diam}(U_i))^\alpha.$$

In particular, we see that

$$\sum_i (\mathrm{diam}(U_i))^\alpha \geq \sum_i \frac{\mu(U_i)}{C} \geq \frac{\mu(X)}{C} = \frac{1}{C}.$$

Since this lower bound is independent of ϵ we see that

$$H^\alpha(X) \geq \frac{1}{C} > 0.$$

Since $\dim_H(X) \geq \inf\{t > 0 : H^t(X) = 0\}$ this implies that $\dim_H(X) \geq \alpha$. □

Recall that we saw that a set X with non-empty interior has $\dim_B(X) = d$. As an easy application we have the following.

Application. Assume that X has non-empty interior. Then $\dim_H(X) = d$

We already know that $\dim_H(X) \leq \dim_B(X) \leq \dim_B([0,1]^d)$. Let us choose a small box $Y \subset X$. But since we are assuming X has positive Lebesgue measure we can consider the normalization of the lebesgue measure $\mu(B) = \lambda(B \cap Y)/\lambda(Y)$ where λ denotes Lebesgue measure. We can then apply the Mass Distribution Principle with $\alpha = d$. [2]

We can revisit an example we already saw.

Example 23 (Middle third Cantor set revisited)**.** *We already saw for the middle third Cantor set X we have*

$$\dim_H(X) = \frac{\log 2}{\log 3}.$$

We already saw μ which gave equal measure 2^{-n} to each of the 2^n intervals in the nth level of the construction, which is the measure in the Mass Distribution Principle. If we choose n with $\frac{1}{3^n} \leq 2r \leq \frac{1}{3^{n-1}}$ then any $x \in X$ the ball $B(x,r)$ will contain at least one of the intervals at the nth level and thus

$$\mu(B(x,r)) \geq \frac{1}{2^n} = \left(\frac{1}{3^n}\right)^\alpha \geq \left(\frac{2}{3}\right)^\alpha r^\alpha$$

[2] Assume that X has positive Lebesgue measure then the same conclusion holds. A little more technical detail is needed here. If a set has positive measure then the density points also have positive measure. For a density point x we have that for some $C > 0$ we have $\lambda(B(x,r) \cap X) \geq C\lambda r^2$ for $r > 0$ sufficiently small

Example 24 (Koch curve)**.** *We can start from the* 12 *straightline segments at the zero stage, of equal length* 1*, say. In the construction, at each step we replace each segments by four segments of one third the previous length. Thus at the nth level one has* 12×4^n *segments of length* $\frac{1}{3^n}$*.*

It is therefore easy to see that $N(X, \frac{1}{3^n}) \leq 12 \times 4^n$ *and deduce that*

$$\overline{\dim}_B(X) \leq \limsup_{n \to +\infty} -\frac{N(X, \frac{1}{3^n})}{\log 3} = \frac{\log 4}{\log 3}.$$

The Mass Distribution Principle can be applied where each of the segments at the nth stage each has measure $\frac{1}{12.4^n}$*. In particular, there exists* $c > 0$ *such that for any* $r > 0$ *we have*

$$\mu(B(x,r)) \geq cr^\alpha$$

where $\alpha = \frac{\log 4}{\log 3}$*.*

We will return to these examples in the next chapter, where we will describe a general method which recovers these values for the dimensions.

There is a converse to the mass distribution which gives a measure associated to the Hausdorff dimension. This requires the following version of Frostman's lemma.

Lemma 17. *Assume that* $H^\alpha(X) > 0$ *then there exists a probability measure* μ *on* X *and* $C > 0$ *such that for any* $x \in X$ *we have*

$$\mu(B(x,r)) \leq Cr^\alpha, \text{ for all } r > 0.$$

We omit the proof.

4.4.2 Energy and Hausdorff Dimension

Closely related to the mass distribution principle is the so called potential theoretic approach to Hausdorff dimension, which is based on the notion of "enegery" Let $\alpha > 0$. We can associate to a probability measure μ its α-energy defined by

$$\mathcal{E}_s(\mu) = \int_X \int_X \frac{d\mu(x)d\mu(y)}{\|x-y\|^\alpha} \in [0, +\infty].$$

Theorem 3. *If* μ *is a probability measure with* $\mu(X) = 1$ *and* $\mathcal{E}_s(\mu) < +\infty$ *then* $\dim_H(X) \geq s$

Proof. Let us consider the subset $X_0 \subset X$

$$X_0 = \left\{ x \in X : \limsup_{\epsilon \to 0} \frac{\log \mu(B(x,\epsilon)}{\epsilon^s} > 0 \right\}.$$

Thus by definition for any $x \in X$ we can choose $c > 0$ and a sequence $\epsilon_n \to 0$ such that

$$\mu(B(x, \epsilon_n)) \geq c\epsilon_n^s, \text{ for } n \geq 1.$$

Let us choose a second sequence $t_n = \frac{\epsilon_n + \epsilon_{n+1}}{2}$ so that

$$r_1 > t_1 > r_2 > t_2 > r_3 > t_3 > \cdots$$

and by going to a subsequence in (r_n), if necessary, we can assume that the annulus $A_n = B(x, r_n) \setminus B(x, q_n)$ satisfies

$$\mu(A_n) \geq \frac{c}{4}\epsilon_n^s, \text{ for } n \geq 1.$$

Then we can write for fixed $x \in X_0$:

$$\int_X \int_X \frac{d\mu(y)}{\|x-y\|^\alpha} \geq \sum_{n=1}^{\infty} \int_{A_n} \int_{A_n} \frac{d\mu(y)}{\|x-y\|^s} \geq \sum_{n=1}^{\infty} \frac{c}{4}\epsilon_n^s \epsilon_n^{-s} = +\infty,$$

But considering this as a function of x on X_0 we see from the hypotheses that

$$\int_{X_0} \left(\int_X \frac{d\mu(y)}{\|x-y\|^\alpha} \right) d\mu(x) \leq \int_X \int_X \frac{d\mu(x)d\mu(y)}{\|x-y\|^\alpha} < +\infty$$

which means that $\mu(X_0) = 0$, by Fubini's theorem. We therefore conclude that for almost every $x \in X$ we have that

$$\limsup_{\epsilon \to 0} \frac{\log \mu(B(x, \epsilon)}{\epsilon^s} = 0$$

and by the mass distribution theorem we have that $\dim_H(X) \geq s$, as required. □

This is part of a more general result which gives an alternative definition of the Hausdorff dimension of X.

Theorem 4 (Potential theoretic definition of Hausdorff dimension). *For a compact set X*

$$\dim_H(X) = \sup\{s > 0 : \exists \mu \text{ with } \mu(X) = 1 \text{ and } \mathcal{E}_s(\mu) < +\infty\}$$

4.5 Properties of Hausdorff Dimension

A rather simple, but useful, viewpoint is to think of dimension as being a way to distinguish between sets of zero measure. This is illustrated by the following simple observation.

Lemma 18. *If $X \subset [0, 1]$ has $\dim_H(X) < 1$ then the (d-dimensional) Lebesgue measure of X is zero.*

Proof. Since $\dim_H(X) < 1$ we can choose $\dim_H(X) < \delta < 1$ for which $H^\delta(X) = 0$. In particular, given $\eta > 0$ we can choose $\epsilon > 0$ sufficiently small that $H^\delta_\epsilon(X) < \eta$. In particular, we can then choose a cover $\{U_i\}$ (of intervals) for X with $\sup_i\{\mathrm{diam}(U_i)\} < \epsilon$ such that $\sum_i(\mathrm{diam}(U_i))^\delta < \eta$. But then, in particular, $\sum_i \mathrm{diam}(U_i) < \eta$ which lead to our characterization of sets of zero Lebesgue measure.

□

On the other hand, there exist examples of sets $X \subset [0, 1]$ with zero Lebesgue measure and $\dim_H(X) = 1$.

We can now collect together some basic properties of Hausdorff dimension. The first few are similar to the properties of box dimension.

Lemma 19 (Inclusion)**.** *If $X \subset Y$ then $\dim_H(X) \le \dim_H(Y)$.*

The proof is fairly immediate from the definitions and left as an exercise.

Another useful property is that sets which are the same up to bi-Lipschitz maps have the same dimension (i.e., it is a *invariant* on classifying spaces up to "bi-Lipschitz equivalence"). We have already seen the corresponding result for box dimension.

Lemma 20 (Lipschitz images)**.** *Let $X_1 \subset \mathbb{R}^d$ and $X_2 \subset \mathbb{R}^l$ be bounded sets. If $L : X_1 \to X_2$ is a Lipschitz homeomorphism (i.e., $\exists C > 0$ such that $|L(x) - L(y)| \le C|x - y|$ for all $x, y \in X_1$) then $dim_H(X_1) \le dim_H(X_2)$. In particular, if $L : X_1 \to X_2$ is a bijective bi-Lipschitz map i.e., $\exists C > 0$ such that*

$$\frac{1}{C}|x - y| \le |L(x) - L(y)| \le C|x - y|,$$

then $dim_H(X_1) = dim_H(X_2)$.

Proof. Let $\epsilon > 0$. Consider an open cover $\{U_i\}$ for X_1 with $\sup_i \mathrm{diam}(U_i) \le \epsilon$. Then the collection of images $\{U'_i := L(U_i)\}$ of the open sets under the homeomorphisms now form an open cover for X_2 with $\mathrm{diam}(U'_i) \le C\epsilon$.

Let $\delta > 0$. From the definitions we have $H^\delta_{L\epsilon}(X_2) \le H^\delta_\epsilon(X_1)$. In particular, letting $\epsilon \to 0$ we see that $H^\delta(X_1) \ge H^\delta(X_2)$. Finally, from the definition of Hausdorff dimension we have $\dim_H(X_1) \le \dim_H(X_2)$.

For the second statement, we can apply the first part a second time with L replaced by L^{-1}. □

Is this still true if L is merely continuous and surjective?

Example 25 (Projections and sums of Cantor sets)**.** *Consider X to be the middle third Cantor set. We can consider the cartesian product $X \times X \subset \mathbb{R}^2$ and its image under the projection $\pi : \mathbb{R}^2 \to \mathbb{R}$ defined by $\pi(x, y) = x + y$. We see that that $\pi(X \times X) = X + X$ where*

$$X + X = \{x + y : x, y \in X\}.$$

Moreover, one easily sees that this image $X+X$ *is the interval* $[0,2]$ *(by considering the possible triadic expansions) and thus has dimension* 1*. On the other hand we already saw that* $\dim_H(X\times X) = 2\dim_H(X) = \log 4/\log 3 > 1$*.*

A more general result is the following.

Lemma 21 (Hölder images)**.** *Let* $f : X \to Y$ *be a surjective* α*-Hölder continuous function, i.e., there exists* $K \geq 0$ *such that* $\|f(x) - f(y)\| \leq K\|x-y\|^\alpha$*. Then for any* $t \geq 0$ *we have that* $\dim_H(Y) \leq \alpha \dim_H(X)$*.*

Proof. From the definition of Hausdorff dimension, we see it suffices to show

$$H^s(Y) \leq K^t H^{s/\alpha} H^s(X).$$

To this end Let $\delta \leq 1$. If $\{U_j\}$ be a δ-cover then since $f(U_j) \leq K(U_j)^\alpha \leq K$ we have that $\{f(U_j)\}_j$ is a $(K\delta^\alpha)$-cover of Y.

Thus

$$H^{s/\alpha}_{K\delta^\alpha}(Y) \leq \sum_j (\mathrm{diam}(f(U_j)))^{t/\alpha} \leq \sum_j (K(\mathrm{diam}(f(U_j))^\alpha)^{t/\alpha}$$

and, in particular,

$$H^{s/\alpha}_{K\delta^\alpha}(Y) \leq \inf\{\sum_j K^{s/\alpha}(\mathrm{diam}(U_j))^\alpha : \{U_j\} \text{ is a } \delta\text{-cover for } X\} = K^{s/\alpha}$$

Letting $\delta \to 0$ we have $H^{s/\alpha}(Y)K^{s/\alpha}H^s(X)$.

For any $s > \dim_H(X)$ we have that $H^s(X) = 0$ and thus $H^{s/\alpha}(Y) = 0$. Then we can deduce $\dim_H(Y) \leq \frac{\dim(X)}{\alpha}$. □

Example 26. *The Hölder continuous map between the middle third Cantor set* $C_{1/3}$ *and the middle* λ*-Cantor set* C_λ *is such that* $\dim(C_\lambda) = -\frac{\log 2}{\log \lambda}$*.*

In the case of box dimension we saw the property of finite domination for unions of sets. Not only does the corresponding result hold for Hausdorff Dimension, but unlike the case for box dimension the result *does* extend to countably infinite unions too.

Lemma 22 (Domination)**.** *Given* $X, Y \subset \mathbb{R}^d$ *then*

$$\dim_H(X \cup Y) = \max\{\dim_H(X), \dim_H(Y)\}$$

Moreover, given a countable infinity of sets X_i *we have that* $\dim_H(\cup_i X_i) = \max_i\{\dim_H(X_i)\}$*.*

Proof. It suffices to prove the stronger second statement.

Let $\epsilon > 0$. For each X_i we can choose a cover $\{U_j^{(i)}\}$ with $\mathrm{diam}(U_j^{(i)}) < \epsilon$. We can then take the union of these covers to get a cover $\cup_j \{U_j^{(i)}\}$ for X.

Let $\delta > \sup_i\{\dim_H(X_i)$. We can then write

$$H_\epsilon^\delta(X) \leq \sum_i \left(\sum_j (\mathrm{diam}(U_j^{(i)}))^\delta \right).$$

But we can now individually minimize each of the terms in brackets over all covers of diameter at most ϵ to get

$$H_\epsilon^\delta(X) \leq \sum_i H_\epsilon^\delta(X_i).$$

Letting $\epsilon \to 0$ we then have that

$$H^\delta(X) \leq \sum_i H^\delta(X_i). \tag{1}$$

But from our choice of δ we have that $H^\delta(X_i) = 0$ and so we conclude from (1) that $H^\delta(X_i) = 0$ and thus $\dim_H(X) \leq \delta$. Finally, since $\delta > \max_i\{\sup_H(X_i)\}$ was arbitrary, the result follows.

□

By taking the sets to be single points we immediately have the following corollary.

Corollary 3. *If X is a countable set then* $\dim_H(X) = 0$.

Again, this is in contrast to the case of box dimension.

Finally, we can consider the Hausdorff dimension of products of sets.

Lemma 23 (Products)**.** *If $X, Y \subset \mathbb{R}^d$ are compact then*

$$\dim_H(X) + \dim_H(Y) \geq \dim_H(X \times Y).$$

Proof. This is a simple application of the Frostman lemma. Suppose $\alpha < \dim_H(X)$ and $\beta < \dim_H(Y)$. By Frostman's lemma we can find measures μ_X on X and μ_Y on Y, and a constant $C > 0$, such that for $x \in X$ and $y \in Y$ we can bound

$$\mu_X(B(x,r)) \leq Cr^\alpha, \mu_Y(B(y,r)) \leq Cr^\beta, \text{ for all } r > 0.$$

In particular, we can consider the product measure $\mu_X \times \mu_Y$ on $X \times Y$ then

$$(\mu_X \times \mu_Y)(B(x,r) \times B(y,r)) \leq C^2 r^{\alpha+\beta}$$

But since $B(x,r)\times B(y,r)$ is contained in a ball $B((x,y),\sqrt{2d})$ in the product space of radius comparable to $r>0$ then by the Mass Distribution Theorem

$$\dim_H(X\times Y)\geq \alpha+\beta.$$

But since we can choose α and β arbitrarily close to the relative dimensions we have

$$\dim_H(X\times Y)\geq \dim_H(X)+\dim_H(X)$$

as required. □

Example 27. *Let C be the middle third Cantor set. Since $\dim_H(C)=\log 2/\log 3$ we see from our earlier estimate*

$$\dim_H(C\times C)=2\log 2/\log 3=2\dim_H(C)$$

i.e., we get equality in this case.

However, there exist examples for which there is a strict inequality

$$\dim_H(X)+\dim_H(Y)>\dim_H(X\times Y).$$

4.5.1 Generic maps

Although the examples for which the different notions of dimension are slightly artificial, in some respects it is the typical case.

Let X be a compact d-dimensional manifold and let $n>d$. We can consider any continuous map $f:X\to\mathbb{R}^n$ and its image $f(X)$. The space of such functions has a natural metric coming from

$$\|f-g\|=\sup_{x\in X}\|f(x)-f(y)\|_2$$

where $\|\cdot\|_2$ if the usual Euclidean norm on $\mathbb{R}^n$

The following amusing result appears in (unpublished?) lecture notes of Milnor.

Lemma 24 (Milnor)**.** *For a dense G_δ set* [3] *of continuous maps $f:X\to\mathbb{R}^n$ we have that the image $Y=f(X)$ satisfies*

$$\dim_H(Y)=\underline{\dim}_B(Y)=d<n=\overline{\dim}_B(Y).$$

Proof. Given a compact set $Y\subset\mathbb{R}^n$ and $\epsilon>0$, let $S_\epsilon(Y)$ denote the largest cardinality of a finite subset $F\subset X$ such that $d(x,y)>\epsilon$ for $x,y\in F$ with $x\neq y$.

[3] By Baire's theorem this is a countable intersection of open dense sets

Given $k > 0$ let $V_k \subset C(X, \mathbb{R}^n)$ consist of those maps $f \in C(X, \mathbb{R})$ for which there exists $\epsilon < \frac{1}{k}$ with

$$\frac{\log S_\epsilon(f(X))}{\log(1/\epsilon)} \geq n - \frac{1}{k}. \tag{1}$$

Each set V_k $(k \geq 1)$ is easily seen to be open.

We claim V_k $(k \geq 1)$ are also dense. We see this, we proceed as follows. Let $p > 0$. Fix $\epsilon = p^{-kn/(kn-1)} > 0$ Given $f_0 \in C(X, \mathbb{R}^n)$ let $x_i \in X$ $(i = 1, \cdots, p^n)$ be (nearby) points whose images $f(x_i)$ are all close to $\underline{x}_0 \in \mathbb{R}^d$. We can deform f_0 to f which maps the same points to lie on p^n points in a (2ϵ)-grid, i.e., for some $\underline{x} \in \mathbb{R}$

$$\underline{x}_0 + (2\epsilon i_1, \cdots, 2\epsilon i_n) \text{ for } i_1, \cdots, i_n \in \{0, 1, \cdots, p-1\}$$

in $\mathbb{R}^n$. Then

$$\log S_\epsilon(f(X)) \geq n \log p = (n - 1/k) \log(1/\epsilon) \tag{2}$$

and so (1) holds. Moreover, we can assume that $\|f - f_0\| = O(p\epsilon) = O(p^{-1/(nk-1)})$, since we are changing f in a neighbourhood of size $O(p\epsilon)$ which can be made arbitrarily small by choosing p larger. Thus each V_k is dense, as claimed.

To complete the first part of the proof we observe that, if $f \in \cap_k V_k$ (which we have just shown is a dense G_δ set, then by (1) $\overline{\dim}_B(f(X)) = n$.

Let U_k consist of those $f \in C(X, \mathbb{R}^n)$ such that there exists $0 < \epsilon < \frac{1}{k}$

$$\frac{\log S_\epsilon(f(K))}{\log(1/\epsilon)} < d + \frac{1}{k}. \tag{3}$$

To see that U_k is open observe that for $f_0 \in U_k$ and $\|f - f_0\| < \eta/2$ then $S_{\epsilon+\eta}(f) \leq S_\epsilon(f_0)$. Thus for η sufficiently small, $f \in U_k$. We also claim that U_k is dense.

Given $f \in C(X, \ltimes)$ choose a cover for K by small open sets $W_1, \cdots, W_p$, say, and points $b_i \in \mathbb{R} * n$ close to $f_0(W_i)$ $(i = 1, \cdots, p)$. Let $\psi : X \to [0, 1]$ be a partition of unity (with ψ supported in W_i). The new function

$$f(x) = \phi_1(x) b_1 + \cdots + \phi_p(x) b_p$$

approximates f_0 and maps X to a d-dimensional simplex in $\mathbb{R}^n$. Thus $\underline{\dim}_B(f(X)) \leq d$. Moreover, $f \in V_k$ for all k.

□

4.5.2 translations of Cantor sets

4.5.3 examples

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Frontiers of different European countries

Example 28 (Snowflake/von Koch curve)**.** *The von Koch curve X is a standard fractal construction. Starting from the interval $X_0 = [0,1]$ we associate to each piecewise linear curve X_n in the plane (which is a union of 4^n segments of length 3^{-n}) a new one X_{n+1}. This is done by replacing the middle third of each line segment by the other two sides of an equilateral triangle bases there. Alternatively, one can start from an equilateral triangle and apply this iterative procedure to each of the sides one gets a "snowflake curve".*

=3.25in snowflake.eps

The top third of this snowflake is the von Koch curve.

Example 29 (Middle third Cantor set and E_2)**.** . *Let X denote the middle third Cantor set. This is the set of closed set of points in the unit interval whose triadic expansion does not contain any occurrences of the the digit 1, i.e.,*

$$X = \left\{ \sum_{k=1}^{\infty} \frac{i_k}{3^k} : i_k \in \{0,2\}, k \geq 1 \right\}$$

Proposition 6. *the middle third Cantor set both the Box dimension and the Hausdorff dimension are $\frac{\log 2}{\log 3} = 0.690...$.*

Proof. When $\epsilon_n = \frac{1}{3^n}$ it is possible to cover the set of X by the union of 2^n intervals

$$X_n = \left\{ \sum_{k=1}^{n} \frac{i_k}{3^k} + \frac{t}{3^n} : i_k \in \{0,2\}, k \geq 1, \text{ and } 0 \leq t \leq 1 \right\}$$

of length $\frac{1}{3^n}$. Therefore, we deduce that $N(\epsilon_n) \leq 2^n$.

Moreover, it is easy to see that any interval of length ϵ_n intersecting X can intersect at most two intervals from X_n, and thus $N(\epsilon_n) \geq 2^{n-1}$. For any $\epsilon > 0$ we can choose $\epsilon_{n+1} \leq \epsilon < \epsilon_n$ and we know that $N(\epsilon_n) \leq N(\epsilon) \leq N(\epsilon_{n+1})$. Then

$$\frac{n-1}{(n+1)} \frac{\log 2}{\log 3} \leq \frac{\log N(\epsilon_n)}{\log(\frac{1}{\epsilon_{n+1}})} \leq \frac{\log N(\epsilon)}{\log(\frac{1}{\epsilon})} \leq \frac{\log N(\epsilon_{n+1})}{\log(\frac{1}{\epsilon_n})} \leq \frac{(n+1)}{n} \frac{\log 2}{\log 3}.$$

Letting $n \to +\infty$ shows that $\dim_B(X) = \lim_{\epsilon \to 0} \frac{\log N(\epsilon)}{\log(\frac{1}{\epsilon})} = \frac{\log 2}{\log 3}$. We again postpone the proof that $\dim_B(X) = \dim_H(X)$ until later, when we shall show a more general result. □

The set E_2 is the set of points whose continued fraction expansion contains only the terms 1 and 2. Unlike the Middle third Cantor set, the dimension of this set is not explicitly known in a closed form and can only be numerically estimated to the desired level of accuracy.

Example 30 (Sierpinski carpet)**.** . *Let*

$$X = \left\{ \left(\sum_{n=1}^{\infty} \frac{i_n}{3^n}, \sum_{n=1}^{\infty} \frac{j_n}{3^n} \right) \ : (i_n, j_n) \in S \right\}$$

where $S = \{0,1,2\}\times\{0,1,2\}-\{(1,1)\}. Thisisaconnectedsetwithoutinterior. We call X a Sierpinski carpet.$

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The Sierpinski Carpet

Proposition 7. *For the Sierpinski carpet both the Box dimension and the Hausdorff dimension are equal to* $\frac{\log 8}{\log 3} = 1.892...$

Proof. When $\epsilon_n = \frac{1}{3^n}$ it is possible to cover the set of X by 8^n boxes of size $\frac{1}{3^n}$:

$$X_n = \left\{ \left(\sum_{k=1}^{n} \frac{i_k}{3^k} + \frac{s}{3^n}, \sum_{k=1}^{n} \frac{j_k}{3^k} + \frac{t}{3^n} \right) \ : (i_k, j_k) \in S \text{ and } 0 \le s, t \le 1 \right\}$$

Moreover, it is easy to see that there is no cover with less elements. For any $\epsilon > 0$ we can choose $\epsilon_{n+1} \le \epsilon < \epsilon_n$ and we know that $N(\epsilon_n) \le N(\epsilon) \le N(\epsilon_{n+1})$. Then

$$\frac{n}{(n+1)} \frac{\log 8}{\log 3} = \frac{\log N(\epsilon_{n+1})}{\log(\frac{1}{\epsilon_n})} \le \frac{\log N(\epsilon)}{\log(\frac{1}{\epsilon})} \le \frac{\log N(\epsilon_n)}{\log(\frac{1}{\epsilon_{n+1}})} = \frac{(n+1)}{n} \frac{\log 8}{\log 3}.$$

Letting $n \to +\infty$, gives that $\dim_B(X) = \frac{\log 8}{\log 3}$. We postpone the proof that $\dim_B(X) = \dim_H(X)$ until later, when we shall show a more general result. □

4.6 Thickness and Cantor sets

There is an alternative notion of *size* for Cantor sets X which we briefly recall.

We can consider the gaps for $X \subset [a,b]$, which are the maximal open connected sets $U = (c,d)$ where $c,d \in X$ and $U \cap K = \emptyset$.

Example 31. *For the middle third Cantor set the gaps* $(\frac{1}{3}, \frac{2}{3})$, $(\frac{1}{9}, \frac{2}{9})$, $(\frac{7}{9}, \frac{8}{9})$, *etc.*

Given a gap U, with a boundary point $u \in \partial U$, we call a *bridge* a maximal interval J with endpoint $u \in \partial J$ and doesn't intersect a gap U' with $\ell(U') \ge \ell(U)$.

We can now associate the idea of the thickness at a point:

Definition 9. *The thickness of X at u is defined by $\tau(X, u) = \ell(J)/\ell(U)$.*

We can associate a single value for X:

Definition 10. *The thickness of X is defined to be*

$$\tau(X) = \inf\{\tau(K, u) : u = \text{ boundary points } u \text{ of (bounded) gaps}\}$$

This definition is taken from the book of Palis and Takens. There is an equivalent earlier definition (introduced by Hall (in 1947) and Newhouse (in 1979).

We begin by enumerating the countable collection of gaps. Let $\mathcal{U} = \{U_n\}_{n=1}^{\infty}$ be a fixed labelling of the gaps of X. Given n and $u \in \partial U$ let C be the connected component of $[a, b] \setminus \cup_{i=1}^{n} U_i$ containing u (where $[a, b] \supset X$ is the smallest interval containing X). We can then denote

$$\tau(X,\mathcal{U}, u) := \ell(C)/\ell(U_n).$$

This gives the alternative (equivalent) definition of the thickness.

Lemma 25. . *We can write*

$$\tau(X) = \sup_{\mathcal{U}} \inf_{u} \tau(X,\mathcal{U}, u)$$

where the infimum is taken over all boundary points of finite gaps of X and the supremum is taken over all orderings $\mathcal{U}$.

Proof. For any $\mathcal{U} = \{U_n\}$ with $\ell(U_n) \leq \ell(U_m)$ for all $n > m$ the supremum is achieved. □

The following inequality relates the Hausdorff Dimension dimension and the thickness of Cantor sets.

Lemma 26. *If $X \subset \mathbb{R}$ has thickness τ then*

$$\dim_H(X) \geq \frac{\log 2}{\log(2 + 1/\tau)}.$$

In particular, we have the

Lemma 27. *If X is a dynamically defined Cantor set then $0 < \dim_H(X) < 1$.*

Chapter 5

Iterated Function Schemes

In earlier chapters we have introduced many examples and introduced two different notions of dimension. Now we will bring these two themes together for a simple class of sets covering many of the earlier examples.

In this chapter we introduce one of the basic constructions, that of *iterated function schemes* They appear in a surprisingly large number of familiar settings, including several that we have already described in the chapter 2. Moreover, those sets X for which we stand most chance of computing the dimension are those which exhibit some notion of *self-similarity* (for example, the idea that if you magnify a piece of the set enough then somehow it looks roughly the same). Often, if we have a local distance expanding map on a compact set we can view the natural associated invariant set as the limit set of an iterated function scheme of the inverse branches of this map (e.g., hyperbolic Julia sets, etc.). We can think of X as being the associated *limit set* Λ given in the following result.

In the case of many linear maps, the dimension can be found implicitly in terms of an expression involving only the rates of contraction. In the non-linear case, the corresponding expression involves the so called pressure function.

5.1 Definitions

Recall that $\mathbb{R}^d$ is equipped with the usual Euclidean metric

$$\|x - y\| = \sqrt{\sum_{i=1}^{d} (x_i - y_i)^2}$$

where $x = (x_1, \cdots, x_n)$ and $y = (y_1, \cdots, y_n)$.

Another familar definition is the following

Definition 11. *A map $T : \mathbb{R}^d \to \mathbb{R}^d$ is a contraction if there exists $0 \le c < 1$*

such that

$$\|Tx - Ty\| \leq c\|x - y\|$$

More generally, Let $U \subset \mathbb{R}^d$ *be an open set. We say that* $S : U \to U$ *is a contraction if there exists* $0 < \alpha < 1$ *such that*

$$||S(x) - S(y)|| \leq \alpha||x - y|| \text{ for all } x, y \in U.$$

(Here $|| \cdot ||$ *denotes the induced Euclidean norm on* U*.)*

In particular, a contraction is a special case of a Lipschitz map, for which the Lipschitz constant is strictly smaller than unity.

One of the most useful approachs to systematically construct examples is to use Iterated Function Schemes. Therefore following definition is fundamental to what follows.

Definition 12. *An iterated function scheme on an open set* $U \subset \mathbb{R}^d$ *consists of a family of contractions* $T_1, \ldots, T_k : U \to U$*.*

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The images of U under the maps $T_1, \ldots, T_4$ in an iterated function scheme.

Notation. Let $0 < c_1, \cdots, c_k < 1$ be the contraction constants associated to the maps $T_1, \cdots, T_k$ and let us denote by $0 < c = \max_{1 \leq i \leq k} c_i < 1$ the contraction constant for the iterated function scheme.

In fact, in some examples it is convenient to broaden even slightly more the definition of an iterated function scheme. More precisely, we might want want to consider contractions $T_i : U_i \to U$ which are only defined on part of the domain U. In this case, we consider only those sequences $(x_n)_{n=0}^{\infty}$ such that $U_{x_n} \supset T_{x_{n-1}}(U_{x_{n-1}})$.

5.1.1 Open set condition

We want to introduce a very useful assumption.

Definition 13 (Open set condition)**.** *An iterated function scheme consisting of contractions* $T_1, \cdots, T_k$ *is said to satisfy the open set condition if exists an open set* $V \subset \mathbb{R}^k$ *such that:*

1. $T_1(V), T_2(V), \cdots, T_k(V) \subset V$*; and*

2. $T_i(V) \cap T_j(U) = \emptyset$ *for* $i \neq j$*.*

A very simple example is the following.

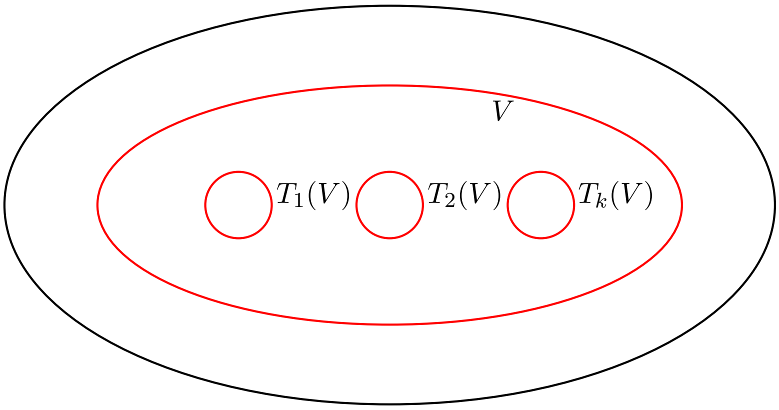

Figure 5.1: Open set condition

Example 32. *Consider the maps $T_1, T_2 : \mathbb{R} \to \mathbb{R}$ defined by*

$$T(x) = \frac{x}{3} \text{ and } T(x) = \frac{x}{3} + \frac{2}{3}.$$

If we let $V = (0, 1)$ then

$$T_1(V) = \left(0, \frac{1}{3}\right) \text{ and } T_2(V) = \left(\frac{2}{3}, 1\right)$$

and we see that $T_1(V) \cap T_2(V) = \emptyset$. Thus this iterated function scheme satisfies the open set condition.

A stronger property is the *strong separation condition.*

5.2 Hutchinson's theorem

We will now describe a general method to associate to a finite number of contractions (i.e., an iterated function scheme) a fractal set.

We can denote by $\mathcal{K}$ the space of compact subsets $K \subset U \subset \mathbb{R}^d$. This is equipped with the Hausdorff metric, which we already encountered in Chapter 2. We can now associate to a given iterated function scheme a single map on $\mathcal{K}$.

Definition 14. *Given contractions $T_1, \cdots, T_k : U \to U$ we can define a map $\mathcal{T} : \mathcal{K} \to \mathcal{K}$ by*

$$\mathcal{T} : K \mapsto \cup_{i=1}^{k} T_i(K).$$

He we are merely using that continuous images of compact sets are compact and finite unions of compact sets are again compact.

Here is a very simple example.

Example 33 (previous example revisited)**.** *With the previous example, we can start with $K = [0, 1]$ and then*

$$\mathcal{T}([0,1]) = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$$

and

$$\mathcal{T}^2([0,1]) = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]$$

It is no coincidence that these are the same intervals in the construction of the middle third Cantor set.

The next result shows that this map on compact sets is a contraction.

Proposition 8. *The map $\mathcal{T}$ is a contraction in the Hausdorff metric (i.e., $d(\mathcal{T}K_1, \mathcal{T}K_2) \leq cd(K_1, K_2)$ for any $K_1, K_2 \in \mathcal{K}$).*

Proof. Given two (compact) sets $X, Y \subset U \subset \mathbb{R}^d$ we want to show that

$$d(\mathcal{T}X, \mathcal{T}Y) \leq cd(X, Y).$$

For any $t > d(X, Y)$ we have from the definition of the Hausdorff dimension that $X \subset B(Y, t)$ and $Y \subset B(X, t)$.

Thus for any $x \in X$ there exists $y \in Y$ with $\|x-y\| < t$ and for any $y' \in Y$ there exists $x' \in X$ with $\|x' - y'\| < t$). Therefore, for each $i = 1, \cdots, k$,

1. for any $T_i(x) \in T_i(X)$ we have $T_i(y) \in T(Y)$ with $\|T_i(x)-T_i(y)\| < c_i t$, and
2. for any $T_i(y') \in T(Y)$ we have $T_i(x') \in T(X)$ with $\|T_i(x') - T_i(y')\| < c_i t$

From the definition of the Hausdorff metric we see that $d(\mathcal{T}X, \mathcal{T}Y) \leq cd(X, Y)$ where $c = \max_i c_i < 1$, as required. □

We want to use the contraction mapping theorem to find a fixed point $X = \mathcal{T}(X)$. The missing ingredient is the following property of $\mathcal{K}$.

Proposition 9. *The Hausdorff metric on the space $\mathcal{K}$ of compact sets in U is complete.*

Proof. Let $(X_n)_{n=1}^{\infty} \subset \mathcal{K}$ be a Cauchy sequence of compact sets (i.e., $d(X_n, X_m) \to 0$ as $n, m \to +\infty$). We want to show that there exists a compact set $X \in \mathcal{K}$ such that $d(X, X_n) \to 0$ as $n \to +\infty$.

Step 1: A subsequence. By going to a subsequence, if necessary, we can assume that

$$d(X_{n+1}, X_n) \leq \frac{1}{2^n}. \tag{1}$$

Moreover, since the original sequence is Cauchy its convergence is equivalent to that of the subsequence so we don't lose anything by this assumption.

Step 2:New sets Y_n: We can now associate new closed sets $(Y_n)_{n=1}^{\infty}$ defined by

$$Y_n := \mathrm{cl}\left(\cup_{k=n}^{\infty} X_k\right), \quad n \geq 1,$$

and observe that:

1. the new sequence is nested, i.e.,

$$Y_1 \supset Y_2 \supset Y_3 \supset \cdots \supset Y_n \supset \cdots \text{; and}$$

2. by the triangle inequality and (1):

$$d(X_n, X_1) \leq \sum_{k=1}^{n-1} d(X_k, X_{k+1}) \leq \sum_{k=1}^{n-1} \frac{1}{2^k} \leq 1, \quad \forall n \geq 1.$$

   In particular, the sets Y_n ($n \geq 1$) are all (uniformly) bounded and, since they are closed, they are thus compact.

3. Since $X_n \subset Y_n$ and $Y_n \subset B(X_n, \epsilon_n)$ where

$$\epsilon_n = \sum_{k=n+1}^{\infty} d(X_k, X_{k+1}) \leq \frac{1}{2^n},$$

   again using (1), we deduce that $d(X_n, Y_n) \to 0$ as $n \to +\infty$.

Step 3: Identifying the limit X: We can now identify the limit as $X := \cap_{n=1}^{\infty} Y_n$, which is compact and non-empty (since it is the intersection of a nested sequence of compact non-empty sets).

Step 4: The sequence (Y_n) is Cauchy. We first observe that

$$d(Y_n, Y_m) \leq d(Y_n, X_n) + d(X_n, X_m) + d(X_n, Y_m) \to 0 \text{ as } n, m \to +\infty$$

by 3. above and the fact that $(X_n)_{n=1}^{\infty}$ was Cauchy. Given $\epsilon > 0$ choose $(N_k)_{k=1}^{\infty}$ such that

$$d(Y_n, Y_m) \leq \epsilon/2^k \text{ for all } n, m \geq N_k.$$

Step 5: End of the proof. To complete the proof its suffices to show that $d(Y_n, X) \to 0$ as $n \to +\infty$, since by 3. above this implies $d(X_n, X) \to 0$ as $n \to +\infty$, i.e., that the Cauchy sequence $(X_n)_{n=1}^{\infty}$ converges to X. We therefore proceed as follows.

We choose an arbitrary point $y_1 \in Y_{N_1}$ and from the definition of the Hausdorff metric choose $y_2 \in Y_{N_2}$ with $\|y_1 - y_2\| \leq \frac{\epsilon}{2}$. Proceeding iteratively, we can choose points $y_k \in Y_{N_k}$ with $\|y_k - y_{k+1}\| \leq \frac{\epsilon}{2^k}$. The sequence $(y_k)_{k=1}^{\infty}$ will converge to a point $y \in X$ (by completeness of $\mathbb{R}^d$) and (by the Euclidean triangle inequality)

$$\|y_1 - y\| \leq \frac{\epsilon}{2} + \frac{\epsilon}{4} + \frac{\epsilon}{4} + \cdots \leq \epsilon.$$

Since $y_1 \in Y_{N_1}$ was arbitrary we see that $Y \subset Y_{N_1} \subset B(Y, 2\epsilon)$ and thus from the definition of the Hausdorff metric we see that $d(Y_{N_1}, Y) \leq 2\epsilon$. Since N_1 can be replaced by any value $n \geq N_1$ the result follows.

□

We can now deduce the main result of this chapter.

Theorem 5 (Hutchinson)**.** *Let $T_1, \cdots, T_k : U \to U$ $(n \geq 2)$ be a finite family of contractions then there is a unique non-empty compact set $X \subset \mathbb{R}^d$ such that*

$$X = \cup_{i=1}^{k} T_i(X). \tag{2}$$

Proof. We need only collect together the pieces of the proof. We associate to $T_1, \cdots, T_k : U \to U$ $(n \geq 2)$ the contraction $\mathcal{T} : \mathcal{K}. \to \mathcal{K}$. The space $\mathcal{K}$ is a complete metric space and thus by applying the contraction mapping theorem we have that there exists a unique fixed point $\mathcal{T}(X) = X$. This is equivalent to the conditon (2) □

Remark 9. *Another consequence of the contraction mapping principle if that if we take any compact set $K \subset \mathbb{R}^d$ then we have exconential converge $d(\mathcal{T}^n K, X) \leq C.c^n$, for $n \geq 1$, where $C = d(K, X)$. In the case of the middle third Cantor set example, this is well illustrated by taking $K = [0, 1]$*

5.3 Examples

Many of the examples we have previously studied are examples of the construction we discussed above. The main point is to try to find the associated contractions for which the fractal set is a fixed point for the associated contraction $\mathcal{T} : \mathcal{K} \to \mathcal{K}$.

We begin with the standard examples of Cantor sets

5.3.1 Middle third Cantor set

Let $T_1, T_2 : [0, 1] \to [0, 1]$ be contractions of the interval defined by

$$T_1(x) = \frac{x}{3} \text{ and } T_2(x) = \frac{x}{3} + \frac{2}{3}.$$

The middle third Cantor set X can we written in terms of triadic expansions as

$$X = \left\{ \sum_{n=1}^{\infty} \frac{x_n}{3^n} : x_1, x_2, \cdots \in \{0, 2\} \right\},$$

However, the images of.a typical point in X under T_1 and T_2 ate

$$T_1\left(\sum_{n=1}^{\infty}\frac{x_n}{3^n}\right) = \frac{0}{3} + \sum_{n=1}^{\infty}\frac{x_n}{3^{n+1}} \text{ and } T_2\left(\sum_{n=1}^{\infty}\frac{x_n}{3^n}\right) = \frac{2}{3} + \sum_{n=1}^{\infty}\frac{x_n}{3^{n+1}}$$

respectively. From this it is easy to see that $X = T_1(X) \cup T_2(X)$ is the unique fixed point for $\mathcal{T}$. Moreover, if we start with $K = [0,1]$ then the sets $\mathcal{T}^n([0,1])$ which converge to the middle third Cantor set X correspond to the nth stage of the construction with 2^n intervals.

5.3.2 Middle λ-Cantor set

We can also consider the more general examples of Cantor sets In this case, let $T_1, T_2 : [0,1] \to [0,1]$ be defined by

$$T_1(x) = \left(\frac{1-\lambda}{2}\right)x \text{ and } T_2(x) = \left(\frac{1-\lambda}{2}\right)x + \left(\frac{1+\lambda}{2}\right).$$

Again, it is easy to see that $X = T_1(X) \cup T_2(X)$, i.e., that X is the unique fixed point of $\mathcal{T}$. Similarly, if we start with $K = [0,1]$ then the sets $\mathcal{T}^n([0,1])$ which converge to the middle λ Cantor set X correspond to the nth stage of the construction with 2^n intervals.

5.3.3 von Koch curve

Let us consider the four contractions $T_1, T_2, T_3, T_4 : \mathbb{R}^2 \to \mathbb{R}^2$ of the plane defined by

$$\begin{aligned}
T_1(x,y) &= \left(\frac{x}{3}, \frac{y}{3}\right)\\
T_2(x,y) &= \left(\frac{x}{3}\cos(\pi/3) + \frac{y}{3}\sin(\pi/3) + \frac{1}{3}, \frac{x}{3}\sin(\pi/3) - \frac{y}{3}\cos(\pi/3)\right)\\
T_3(x,y) &= \left(\frac{x}{3}\cos(\pi/3) - \frac{y}{3}\sin(\pi/3) + \frac{1}{2}, -\frac{x}{3}\sin(\pi/3) - \frac{y}{3}\cos(\pi/3) + \frac{1}{6\sqrt{3}}\right)\\
T_4(x,y) &= \left(\frac{x}{3}, \frac{y}{3} + \frac{2}{3}\right)
\end{aligned}.$$

The limit set X corresponds to one third of the von Koch snowflake. If we let $K = [0,1] \times \{0\} \subset \mathbb{R}^2$ then $\mathcal{T}^n(K)$ represents the nth step in the construction.

5.3.4 Sierpinski Triangle

We can consider the three contractions $T_1, T_2, T_3 : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$\begin{aligned} T_1(x,y) &= \left(\frac{x}{2}, \frac{y}{2}\right) \\ T_2(x,y) &= \left(\frac{1}{2}, 0\right) + \left(\frac{x}{2}, \frac{y}{2}\right) \\ T_3(x,y) &= \left(\frac{1}{4}, \frac{\sqrt{3}}{4}\right) + \left(\frac{x}{2}, \frac{y}{2}\right) \end{aligned}$$

The limit set X corresponds to the Sierpinski triangle. If we let K be an equilateral triangle with vertices $(0,1)$, $(1,0)$ and $(\frac{1}{2}, \frac{\sqrt{3}}{2})$ then $\mathcal{T}^n(K)$ represents the nth step in the construction.

5.3.5 Bedford-McMullen sets

Given $S \subset \{0, \cdots, n-1\} \times \{0, \cdots, m-1\}$ we can consider the contractions $T_{i,j} : \mathbb{R}^2 \to \mathbb{R}^2$ with $(i,j) \in S$ defined by

$$T_{i,j}(x,y) = \left(\frac{x}{n}, \frac{y}{m}\right) + \left(\frac{i}{n}, \frac{j}{m}\right)$$

The limit set X corresponds to the Bedord-McMullen carpet. If we let $K = [0,1] \times \{0\} \subset \mathbb{R}^2$ then $\mathcal{T}^n(K)$ represents the nth step in the construction.

5.3.6 Apollonian circle packings

Given the four tangent circles we can consider four complimentary circles K_i each of which passes through three points from the four tangency points (where pairs of circles touch). In each of these circles $K_i = \{x \in \mathbb{C} : |z - c_i| - r_i\}$, where $z_i \in \mathbb{C}$ and $r_i > 0$, we can associate a map $S_i : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ defined by

$$S_i(z) = r_i^2 \frac{(z - z_i)}{|z - z_i|}, \quad i = 1, \cdots, 4.$$

We can then define maps $T_{i,j} = S_i \circ S_j : D \to D$ with $i \neq j$ on the unit disk D. These maps $T_{i,j}$ aren't quite contractions, and so the Hutchinson theorem doesn't quite apply directly. [1]

5.3.7 Julia sets

Let us consider the polynomial maps $T : \mathbb{C} \to \mathbb{C}$ defined by $T(z) = z^2 + c$, for some $c \in \mathbb{C}$.

[1] This can be overcome by *inducing* which corresponds to iterating the maps to pick up the strict contraction. However, this means we end up with finitely many contractions, which still needs our definitions broadening in any case

When $c = 0$ then the Julia set is the unit circle. We can consider neighbourhoods

$$U^+ \supset \{e^{i\theta} : 0 \leq \theta \leq \pi\} \text{ and } U^- \supset \{e^{i\theta} : \pi \leq \theta \leq 2\pi\}$$

We can consider the maps $T_1 : U^\pm \to U^+$ and $T_2 : U^\pm \to U^-$ defined by $T_1(e^{i\theta}) = e^{i\theta/2}$ and $T_2(e^{i\theta}) = e^{i\theta/2+\pi}$. For c close to zero the Julia set is close to the unit circle and we can consider contractions in a neighbourhood of the Julia set of the form $T_1(z) = \sqrt{z-c}$ and $T_2(z) = -\sqrt{z-c}$. Whereas these maps are contractions, there are multiple domains and so the Hutchinson theorem needs to be adapted to this situation (as mentioned in a previous remark).

5.4 Similarities

The Iterated Function Scheme construction of fractals has the additional benefit that it gives us the possibility to estimate the dimension of the set. The most successful setting for this is that of similarities.

Definition 15. *We say that a (contraction) $T : U \to U$ are similarities if there exists $0 < c < 1$ such that*

$$\|T(x) - T(y)\| = c\|x - y\|, \quad x, y \in U$$

In particular, when $0 < c < 1$ this is s special case of a contraction.

We are particularly interested in the case of Iterated Function schemes consisting of contractions $T_1, \cdots, T_k$ each of which is a similarity. We can check this in the case of our examples

- *Middle third Cantor set:* The two contractions here are similarities with $c = \frac{1}{2}$.
- *Middle λ-Cantor set:* The two contractions here are similarities with $c = \frac{1-\lambda}{2}$.
- *von Koch snowflake:* The four contractions here are similarities with $c = \frac{1}{3}$.
- *Sierpinski Triangle:* The three contractions here are similarities with $c = \frac{1}{2}$.

But the Bedford-McMullen carpets have contractions which are not similarities since the contraction is by different amounts (namely $\frac{1}{n}$ and $\frac{1}{m}$). Moreover, the Apollonian circle packing maps and Julia sets have non-linear contractions, which cannot be similarities either.

5.5 Moran's Theorem

Now that we have described a more systematic approach to constructing examples of fractal sets X, we want to describe an associated (implicit) expression for the dimension(s).

As a preliminary, we begin with a little calculus.

Lemma 28. *Let* $0 < c_1, \cdots, c_k < 1$ *with* $c_1^d + \cdots + c_k^d < 1$. *The function* $f : [0, k] \to \mathbb{R}$ *defined by*

$$f(t) = \sum_{i=1}^{k} c_i^t - 1$$

is a monotone decreasing function and there exists a unique value $0 < t_0 < 1$ *with* $f(t_0) = 0$.

Proof. We observe that the $f(t)$ is continuously differentiable and the derivative satisfies

$$f'(t) = \sum_{i=1}^{k} (\log c_i) c_i^t < 0.$$

This shows that $f(t)$ is monotone decreasing. Moreover, since $f(0) = k-1 > 0$ and $f(d) = c_1^d + \cdots + c_n^d - 1 < 0$ the intermediate value theorem gives the existence of a solution $0 < t_0 < d$ to $f(t_0) = 0$. ‘ □

Now we see how to find the dimension of the limit set X associated to an iterated function scheme consisting of contractions $T_1, \cdots, T_k$. [2]

Theorem 6 (Moran)**.** *Let* $\{T_1, \cdots, T_k\}$ *be an iterated function scheme of similarities satisfying the open set condition with contraction constants* $0 < c_1, \cdots, c_k < 1$. *The associated limit set* X *has* $\dim_H(X)$ *given by solution* $0 < D \leq d$ *to*

$$\sum_{i=1}^{k} c_i^D = 1. \tag{1}$$

This will follow from the next two lemmas.

Lemma 29. *Let* $\{T_1, \cdots, T_k\}$ *be an iterated function scheme of contractions with contraction constants* $0 < c_1, \cdots, c_k < 1$. *The associated limit set* X *satisfies* $\dim_H(X) \leq D$, *where* $0 < D < d$ *is the unique solution to (1).*

[2] It is a common misconception that Patrick Moran was a student of Besicovitch. In fact, he took courses at Cambridge from 1937-1939 (including those from Besicovitch) and while it appears he wasn't very successful at mathematics, he did used to take Mrs. Besicovitch occasionally to the cinema. From 1939-1945 he did war work (during which period he proved this result). He was given a studentship at Cambridge in 1945, but Besicovitch declined to supervise him and he ended up as a student of Smithies instead. In any event, Moran couldn't solve the research problem he was then given and so never received a PhD.

Proof. Since $\mathcal{T}(X) = X$, we can write

$$X = \mathcal{T}^N X = \cup_{i_1,\cdots,i_N} T_{i_1} \circ \cdots \circ T_{i_N}(X)$$

for any $N > 0$. Moreover, since the maps T_i are contractions then we can write

$$\mathrm{diam}(T_{i_1} \circ \cdots \circ T_{i_N}(X)) \leq c_{i_1} c_{i_2} \cdots c_{i_N} \mathrm{diam}(X)$$

Furthermore, we can cover each of these closed sets $T_{i_1} \circ \cdots \circ T_{i_N}(X)$ by an open ball of radius

$$c_{i_1} c_{i_2} \cdots c_{i_n} \mathrm{diam}(X) \leq c^N \mathrm{diam}(X)$$

where $c := \max_i c_i$. In particular, this then gives us a cover of X by open sets of diameter at most $c^N \mathrm{diam}(X)$ and thus by definition we can write

$$\begin{aligned} H^D_{c^N \mathrm{diam}(X)}(X) &\leq \sum_{i_1,\cdots,i_N} (c_{i_1} c_{i_2} \cdots c_{i_N})^D (\mathrm{diam}(X)^D \\ &= \underbrace{(c_1^D + \cdots + c_k^D)^N}_{=1} (\mathrm{diam}(X))^D \\ &= (\mathrm{diam}(X))^D. \end{aligned}$$

Letting $N \to +\infty$ we see that $c^N \mathrm{diam}(X) \to 0$ and $H^D(X) \leq (\mathrm{diam}(X))^D < +\infty$. Thus we deduce that $\dim_H(X) \leq D$.

□

The above lemma doesn't require the contractions to be similarities, and without further assumptions we cannot expect equality.

Example 34. *Fix $0 \leq t \leq \frac{2}{3}$ and let $T_1, T_2 : [0,1] \to [0,1]$ be defined by*

$$T_1(x) = \frac{x}{3} \text{ and } T_2(x) = \frac{x}{3} + t$$

If $0 < t \leq \frac{2}{3}$ then the limit set X is a scaled copy of the middle third Cantor set and thus has dimension $\frac{\log 2}{\log 3}$. On the other habd, when $t = 0$ then $T_1 = T_2$ and then $X = \{0\}$ is a single point and so has dimension equal to 0.

To get an equality in Moran's Theorem we need to assume that the contractions are similarities and satisfy the Open Set Condition

Lemma 30. *Assume that each T_i is a similarity with constant $0 < c_i < 1$ $(i = 1, \cdots, k)$ and they satisfy the open set condition. Then $D \leq \dim_H(X)$ where D is the unique solution to $\sum_{j=1}^k c_j^D = 1$.*

Proof. Given the previous lemma, it suffices to show that $\dim_H(X) \geq D$. In particular, we will use the Mass Distribution Principle. [3]

Constructing the measure. By the Open Set Condition (with the open set V and a little induction argument) we can assume that the sets $T_{i_1} \cdots T_{i_n}(V)$ are disjoint, for $i_1, \cdots, i_n \in \{1, \cdots, k\}$. We define a probability measure μ which associates to the nth level sets the measures

$$\mu(T_{i_1} T_{i_2} \cdots T_{i_n}(\mathrm{cl}(V)) = (c_{i_1} c_{i_2} \cdots c_{i_n})^D.$$

Special image sets. Fix an arbitrary open ball $B(x_0, \epsilon)$. If we denote

$$\theta = \min_{1 \leq i,j \leq} \left\{ \frac{c_i}{c_j} \right\}$$

then we can consider the family of sets

$$\left\{ T_{i_1} T_{i_2} \cdots T_{i_n}(\mathrm{cl}(V)) : \theta\epsilon \leq \underbrace{\mathrm{diam}(T_{i_1} T_{i_2} \cdots T_{i_n}(V))}_{= c_{i_1} c_{i_2} \cdots c_{i_n} \mathrm{diam}(V)} \leq \epsilon \right\}$$

of comparable diameter. (The value of n can vary, so long as the condition on the diameter is satisfied). In order to estimate $\mu(B(x_0, \epsilon))$ we want to consider those for which $T_{i_1} T_{i_2} \cdots T_{i_n}(\mathrm{cl}(V)) \cap B(x_0, \epsilon) \neq \emptyset$.

Volume estimates. Next choose an open ball in $B(y_0, r\mathrm{diam}(V)) \subset V$ of radius equal to $r\mathrm{diam}(V)$, say, for some sufficiently $0 < r < 2$ (and some point y_0). Then for each image

$$T_{i_1} T_{i_2} \cdots T_{i_n}(V) \supset T_{i_1} T_{i_2} \cdots T_{i_n}(B(y_0, r\mathrm{diam}(V))) \tag{1}$$

and since the maps are similarities this image of $B(y_0, r\mathrm{diam}(V))$ is itself an open ball now

a) centred at $T_{i_1} T_{i_2} \cdots T_{i_n}(y_0)$, and

b) radius $r c_{i_1} c_{i_2} \cdots c_{i_n} \mathrm{diam}(V)$.

In particular, the volume of $T_{i_1} T_{i_2} \cdots T_{i_n}(B(x_0, r\mathrm{diam}(V)))$ is

$$\lambda(T_{i_1} T_{i_2} \cdots T_{i_n}(B(y_0, r\mathrm{diam}(V)))) = \lambda(B(0,1)) \left(r c_{i_1} c_{i_2} \cdots c_{i_n} \mathrm{diam}(V) \right)^d \tag{2}$$

where $\lambda(\cdot)$ represents d-dimensional volume (and, in particular, $\lambda(B(0,1))$ is the volume of the unit ball in $\mathbb{R}^d$. [4]) Therefore, by the inclusion (1) and

[3] This can be avoided, as in the book of Climenhaga-Pesin

[4] There is an explicit formula $\lambda(B(0,1)) = \pi^{d/2}/\Gamma(1 + d/2)$ which we will make absolutely no use of

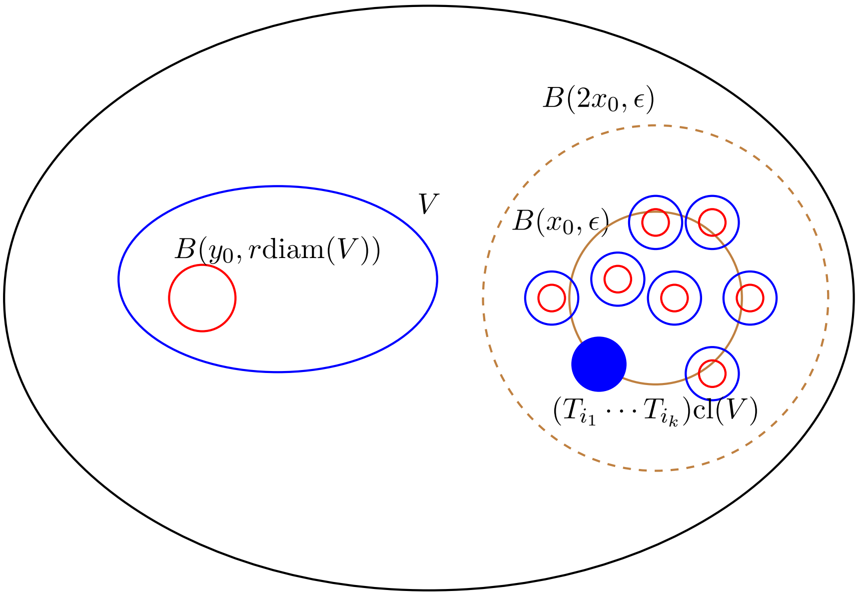

Figure 5.2: We estimate the number of (disjoint) images $T_{i_1}T_{i_2}\cdots T_{i_n}(\mathrm{cl}(V))$ intersecting $B(x_0, \epsilon)$ by comparing volumes

the equality (2) we have a *lower bound* on the volumes of each of the images of V of the form

$$\begin{aligned}\lambda(T_{i_1}T_{i_2}\cdots T_{i_n}(V)) &\geq \lambda(B(T_{i_1}T_{i_2}\cdots T_{i_n}(y_0), rc_{i_1}c_{i_2}\cdots c_{i_n}\mathrm{diam}(V))) \\ &= \lambda(B(0,1))\,(rc_{i_1}c_{i_2}\cdots c_{i_n}\mathrm{diam}(V))^d \\ &\geq \lambda(B(0,1))(r\theta)^d\epsilon^d.\end{aligned} \tag{1}$$

To get an upper bound on the μ-measure of $B(x_0, \epsilon)$ we can further restrict attention to those images of V that satisfy

$$T_{i_1}T_{i_2}\cdots T_{i_n}(\mathrm{cl}(V)) \cap B(x_0, \epsilon) \neq \emptyset,$$

but then since $\mathrm{diam}(T_{i_1}T_{i_2}\cdots T_{i_n}(\mathrm{cl}(V))) \leq \epsilon$ we see that

$$T_{i_1}T_{i_2}\cdots T_{i_n}(\mathrm{cl}(V)) \subset B(x_0, 2\epsilon).$$

In particular, we then have a trivial *upper bound* of the union of all such images of the form

$$\begin{aligned}\lambda\left(\bigcup_{(T_{i_1}\cdots T_{i_k})(\mathrm{cl}(V))\cap B(x_0,\epsilon)\neq\emptyset} (T_{i_1}\cdots T_{i_k})(\mathrm{cl}(V))\right) &\leq \lambda(B(x_0, 2\epsilon)) \\ &= \lambda(B(0,1))\,(2\epsilon)^d\end{aligned} \tag{2}$$

Counting images of $\mathrm{cl}(V)$ intersecting $B(x_0, \epsilon)$. We can now use the volumes to get an upper bound on the number of open sets of the form $(T_{i_1} \cdots T_{i_k})(V)$ which intersected $B(x_0, \epsilon)$, which in turn gives an estimate on its measure.

By the open set condition, two distinct sets of the form $(T_{i_1} \cdots T_{i_k})(V)$ are disjoint and so by (1) and (2) we have an upper bound on the number of such sets intersecting the ball $B(y_0, \epsilon)$ of the form

$$\begin{aligned}
&\mathrm{Card}\{(T_{i_1} \cdots T_{i_n})(\mathrm{cl}(V)) : (T_{i_1} \cdots T_{i_n})(\mathrm{cl}(V)) \cap B(x_0, \epsilon) \neq \emptyset\} \\
&\leq \frac{\lambda\left(\bigcup_{(T_{i_1} \cdots T_{i_n})(\mathrm{cl}(V)) \cap B(x_0,\epsilon)\neq\emptyset} (T_{i_1} \cdots T_{i_n})(\mathrm{cl}(V))\right)}{\min\left\{\lambda(T_{i_1} T_{i_2} \cdots T_{i_n}(V))\right\}} \\
&\leq \frac{\lambda(B(0,1)) 2^d \epsilon^d}{\lambda(B(0,1))(r\theta)^d \epsilon^d} \\
&= \left(\frac{2}{r\theta}\right)^d,
\end{aligned} \tag{3}$$

In particular, this bound is independent of the ball $B(x_0, \epsilon)$.

Mass distribution principle. We can now bound the measure $\mu(B(x_0, \epsilon))$ as follows

$$\begin{aligned}
\mu(B(x_0, \epsilon)) &\leq \sum_{(T_{i_1} \cdots T_{i_n})(\mathrm{cl}(V)) \cap B(x_0,\epsilon)\neq\emptyset} \mu(T_{i_1} \cdots T_{i_n})(\mathrm{cl}(V)) \\
&= \sum_{(T_{i_1} \cdots T_{i_n})(\mathrm{cl}(V)) \cap B(x_0,\epsilon)\neq\emptyset} (c_{i_1} \cdots c_{i_n})^D \\
&\leq \mathrm{Card}\{(T_{i_1} \cdots T_{i_n})(\mathrm{cl}(V)) : (T_{i_1} \cdots T_{i_n})(\mathrm{cl}(V)) \cap B(x_0, \epsilon) \neq \emptyset\} \times \epsilon^D \\
&\leq \left(\frac{2}{r\theta}\right)^d \epsilon^D
\end{aligned} \tag{4}$$

by (3). In particular, this is enough to apply the Mass Distribution and deduce that $\dim_H(X) = D$.

□

Finally, we can related this to the box dimension $\dim_B(X)$

Lemma 31. *Let $\{T_1, \cdots, T_k\}$ be an iterated function scheme of similarities satisfying the open set condition amd with contraction constants $c_1, \cdots, c_k$. The associated limit set X satisfies $\dim_H(X) = \dim_B(X) = D$, where $0 < D < d$ is the unique solution to (1).*

Proof. Consider a small ball $B(y_0, \delta) \subset V$, where V is the open set used in the open set condition. By the similarities condition, the images $(T_{i_1} \cdots T_{i_n})(B(y_0, \delta))$ of the ball will again be balls, now of radius $c_{i_1} \cdots c_{i_n} \delta$.

Given $\epsilon > 0$, we can then restrict to those images such that

$$\epsilon\theta \leq c_{i_1} \cdots c_{i_n} \delta \leq \epsilon \tag{1}$$

Because of the open set condition these sets will all be disjoint. We can now associate a measure ν such that $\nu((T_{i_1} \cdots T_{i_n})B(y_0, \delta)) = (c_{i_1} \cdots c_{i_n})^D$. Then we can write

$$1 \geq \sum_{\epsilon\theta \leq c_{i_1} \cdots c_{i_n} \delta \leq \epsilon} \nu((T_{i_1} \cdots T_{i_n})B(y_0, \delta)) = \sum_{\epsilon\theta \leq c_{i_1} \cdots c_{i_n} \delta \leq \epsilon} (c_{i_1} \cdots c_{i_n})^D \tag{1}$$

where the summation is restricted to those images satisfying (1) above. But for this particular disjoint collection of open sets

$$(T_{i_1} \cdots T_{i_n})(B(y_0, \delta)) = B\left((T_{i_1} \cdots T_{i_n})(y_0), c_{i_1} \cdots c_{i_n} \delta\right)$$

of radius $c_{i_1} \cdots c_{i_n} \delta$ the inequality (1) a bound on their cardinality given by

$$1 \geq \mathrm{Card}\{(T_{i_1} \cdots T_{i_n})(B(y_0, \delta)) : \theta\epsilon \leq c_{i_1} \cdots c_{i_n} \delta \leq \epsilon\} \times (\theta\epsilon)^D. \tag{2}$$

Given *any* open cover by balls of radius $\frac{\epsilon\theta}{2}$ there must be at least one ball contained inside each of the (disjoint) balls $(T_{i_1} \cdots T_{i_n})(B(y_0, \delta))$ (covering its centre, for example). We can then deduce from (2) that

$$N\left(X, \frac{\epsilon\theta}{2}\right) \leq (\theta\epsilon)^{-D}.$$

Therefore

$$\overline{\dim}_B(X) := -\limsup_{\epsilon \to 0} \frac{\log N(X, \epsilon)}{\log \epsilon} \leq -\limsup_{\epsilon \to 0} \frac{\log(1/\epsilon^D)}{\log \epsilon} = D.$$

Since $D = \dim_H(X) \leq \underline{\dim}_B(X) :\leq \overline{\dim}_B(X)$ (by the previous lemma and standard bounds) we deduce the result. □

Without the open set condition, things can go hideously wrong!

Example 35 (An example without the open set condition)**.** *Consider, as an example, the maps $T_i x = \lambda x + i$, for $i = 0, 1, 3$ and let Λ_λ be the limit set*

1. *For almost all $1/4 < \lambda < 1/3$ we have that $dim_H(\Lambda_\lambda) = \frac{\log 3}{\log(1/\lambda)}$ (as expected); However,*

2. *For a dense set of values λ we have that $dim_H(\Lambda_\lambda) < \frac{\log 3}{\log(1/\lambda)}$.*

In particular, the dimension of the set Λ_λ is not continuous in λ. We shall return to this example later.

5.6 Examples of Moran's Theorem

For the present, let us just see how Moran's theorem allows us to deduce the dimensions of the limits sets in three familiar simple examples.

5.6.1 Middle third Cantor set

Consider the middle third Cantor set. We have $\alpha_1 = \alpha_2 = \frac{1}{3}$ and observe that that with $D = \frac{\log 2}{\log 3}$ we have

$$1 = \left(\frac{1}{3}\right)^{\frac{\log 2}{\log 3}} + \left(\frac{1}{3}\right)^{\frac{\log 2}{\log 3}}.$$

In particular, we recover $\dim_H(X) = \frac{\log 2}{\log 3} = 0.63093$.

5.6.2 Sierpinski Carpet

Consider the Sierpinski Carpet. Consider the eight contractions defined by

$$T_{(i,j)}(x,y) = \left(\frac{x+i}{3}, \frac{y+j}{3}\right)$$

where $0 \leq i, j \leq 2$, and $(i,j) \neq (1,1)$. We can then identify the Sierpinski gasket as the limit set $\Lambda = \Lambda(T_{(0,0)}, \cdots, T_{(2,2)})$. We have $\alpha_{ij} = \frac{1}{3}$ for $(i,j) \in S$ and observe that with $D = \frac{\log 8}{\log 3}$ we have

$$1 = \underbrace{\left(\frac{1}{3}\right)^{\frac{\log 8}{\log 3}} + \ldots + \left(\frac{1}{3}\right)^{\frac{\log 8}{\log 3}}}_{\times 8}$$

thus $\dim_H(X_S) = \frac{\log 8}{\log 3} = 1.89279\ldots$.

5.6.3 von Koch curve

We consider again the Koch Curve. We can consider four affine contractions

$$\begin{aligned}
T_1 &: (x,y) \mapsto \left(\frac{x}{3}, \frac{y}{3}\right) \\
T_2 &: (x,y) \mapsto \left(\frac{1}{3} + \frac{x}{6}, \frac{y}{2\sqrt{3}}\right) \\
T_3 &: (x,y) \mapsto \left(\frac{1}{2} + \frac{x}{6}, \frac{1}{2\sqrt{3}} - \frac{y}{2\sqrt{3}}\right) \\
T_4 &: (x,y) \mapsto \left(\frac{2}{3} + \frac{x}{3}, \frac{y}{3}\right).
\end{aligned}$$

Each branch contracts by $\frac{1}{3}$ the limit figure and observe that with $D = \frac{\log 4}{\log 3}$ we have

$$1 = \left(\frac{1}{3}\right)^{\frac{\log 4}{\log 3}} + \left(\frac{1}{3}\right)^{\frac{\log 4}{\log 3}} + \left(\frac{1}{3}\right)^{\frac{\log 4}{\log 3}} + \left(\frac{1}{3}\right)^{\frac{\log 4}{\log 3}},$$

thus we recover $\frac{\log 4}{\log 3} = 1.2619....$

The situation becomes interesting when we drop the assumption that the iterated function scheme is made up of similarities. (However, dropping the conformal assumption or the open set condition is, for the moment, something we prefer not even to contemplate!)

5.7 A simple special case

We can consider the special case of two similarities and a simpler proof.

Theorem 7. *If $T_1, T_2 : \mathbb{R} \to \mathbb{R}$ are similarities satisfying the open set condition, then the dimension is the unique solution $s = dim_H(\Lambda)$ to the identity*

$$1 = (\alpha_1)^s + (\alpha_2)^s,$$

where α_1 and α_2 are the associated contraction rates.

Proof. For simplicity, we consider the case of just two maps $T_1, T_2 : \mathbb{R} \to \mathbb{R}$ with limit set Λ. It is also convenient to write the two contractions as

$$\lambda := |\lambda_1| \lambda^\alpha := |\lambda_2|, \text{for some } 0 < \alpha < 1,$$

say. We can assume, for simplicity, that the open set in the open set condition is a ball $U = \{x \in \mathbb{R}^2 : ||x|| < r\}$.

Given $k > 1$ we can consider a cover for Λ by all balls of the form

$$T_{i_1} \dots T_{i_m} U \text{ where } M \text{ is chosen with} \frac{\lambda}{k} \leq |\lambda_{i_1}| \dots |\lambda_{i_m}| \leq \frac{1}{k} \tag{2.1}$$

Let M_k be the total number of such disks, and let $N_k = N(1/k)$.

It is easy to see that there are constants $C_1, C_2 > 0$ with $C_1 N_k \leq M_k \leq C_2 N_k$.

For example, we are considering

$$\underbrace{T_1 T_1 \dots T_1}_{\times n} U T_2 \underbrace{T_1 \dots T_1}_{\times (n-1)} U T_1 T_2 \underbrace{T_1 \dots T_1}_{\times (n-2)} U \cdots \underbrace{T_2 \dots T_2}_{\times [\alpha n]}$$

(where $[\alpha n]$ is the largest integer smaller than $[\alpha n]$).

If T_1 occurs $[(1-\beta)n]$ times, for some $0 < \beta < 1$, then for (2.1) to be satisfied we require that T_2 occurs approximately $[\beta \alpha n]$ times. Moreover, then number of contributions to the above list depends on their ordering, which is approximately $\binom{[(1-\beta+\alpha\beta)n]}{[\beta\alpha n]}$.

The total number M_k of disks satisfies:

$$\max_\beta \binom{[(1-\beta+\alpha\beta)n]}{[\beta\alpha n]} \leq M_k \leq n\left(\max_\beta \binom{[(1-\beta+\alpha\beta)n]}{[\beta\alpha n]}\right)$$

and to esimate this we need to maximize $\binom{[(1-\beta+\alpha\beta)n]}{[\beta\alpha n]}$ in β.

By Stirling's formula we know that $\log n! \sim n\log n$, as $n \to +\infty$. Thus

$$\log \binom{[(1-\beta+\alpha\beta)n]}{[\beta\alpha n]} = \log\left(\frac{[(1-\beta+\alpha\beta)]!}{[\beta\alpha n]![(1-\beta]!}\right) \sim n\left((x+y)\log(x+y) - x\log x - y\log y\right)$$

where $x = \alpha\beta$ and $y = (1-\beta)$. Writing $f(x,y) = (x+y)\log(x+y) - x\log x - y\log y$, we have a problem of maximizing this function subject to the condition $g(x,y) = x + \alpha y = \alpha$. Using a Lagrange multiplier γ this reduces to solving

$$\nabla f = (\log(x+y) - \log x, \log(x+y) - \log y) = \gamma\nabla g = \gamma(1,\alpha)$$

In particular, we get $\left(\frac{x}{x+y}\right)^\alpha = \left(\frac{x}{x+y}\right)$ and so setting $\lambda^d := \frac{x}{x+y}$ solves $\lambda^d + (\lambda^\alpha)^d = 1$. Thus

$$d = \lim_{k\to+\infty} \frac{\log N_k}{\log k} = \lim_{k\to+\infty} \frac{\log\left(\frac{1}{\lambda^d}\right)^k}{\log\left(\lambda^k\right)}$$

as required. □

5.8 non-conformal maps

Let us now return to the problem of Hausdorff dimension for non-conformal maps, and examples of where number theoretic properties of parameters can lead to complicated behaviour.

Consider a family of affine maps $T_i x = a_i x + b_i$, $i = 1, \ldots, k$, on $\mathbb{R}^2$. In particular, a_i is a $d \times d$ matrix and b_i is a vector in $\mathbb{R}^d$. Let Λ denote the limit set of this family of maps, defined precisely as before.

There are simple examples of affine maps where the dimension disagrees. The following is a simple illustration.

Example 36. *(Bedford-McMullen) Consider the following three affine maps of $\mathbb{R}^2$:*

$$T_i : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} c_i \\ d_i \end{pmatrix}, \quad i = 1, 2, 3,$$

where

$$\begin{pmatrix} c_1 \\ d_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} c_2 \\ d_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{2} \end{pmatrix}, \begin{pmatrix} c_3 \\ d_3 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} \\ 0 \end{pmatrix}.$$

=1.5in bedford.eps

The first two steps in the Bedford-McMullen example The limit set takes the form

$$\Lambda = \left\{ \left(\sum_{n=1}^{\infty} \frac{i_n}{3^n}, \sum_{n=1}^{\infty} \frac{j_n}{2^n} \right) \ : (i_n, j_n) \in \{(0,0),(1,1),(2,0)\} \right\},$$

and is closely related to what is called Hironaka's curve. The Box dimension and the Haudorff dimension of the limit set Λ can be explicitly computed in such examples, and be show to be different. More precisely,

$$dim_H(\Lambda) = \log_2(1 + 2^{\log_3 2}) = 1.34968 \ldots < dim_B(\Lambda) = 1 + \log_3(\frac{3}{2}) = 1.36907 \ldots.$$

This is part of more general result.

Theorem 8 (Bedford-McMullen Theorem)**.** *Let $l > m \geq 2$ be integers. Given $S \subset \{0,1,\ldots,m-1\} \times \{0,1,\ldots,l-1\}$ we can associate an affine "Sierpinski carpet":*

$$\Lambda = \left\{ \left(\sum_{n=1}^{\infty} \frac{i_n}{l^n}, \sum_{n=1}^{\infty} \frac{j_n}{m^n} \right) \ : (i_n, j_n) \in S \right\}$$

Assume that every row contains a rectangle. If we denote $t_j = Card\{i : (i,j) \in S\}$, and $a = Card(S)$ then

$$dim_H(\Lambda) = \log_m \left(\sum_{j=0}^{m-1} t_j^{\log_l m} \right), \ \ and \ dim_B(\Lambda) = 1 + \log_l \left(\frac{a}{m} \right)$$

=2.0in bedford-rev.eps

The generalized construction of Bedford-McMullen

Proof. At the j the level of the construction we have S^j rectangles of size $l^{-j} \times m^{-j}$. Moreover, we can cover each rectangle by approximately $(l/m)^j$ squares of size m^{-j}. Moreover, because no rows are empty this many are needed.

Thus for $\epsilon = l^{-j}$ we have that $N(l^{-j}) = a^j (l/m)^j$. Thus

$$\begin{aligned}
\dim_B(\Lambda) &= \lim_{\epsilon \to 0} -\frac{\log N(\epsilon)}{\log \epsilon} \\
&= \lim_{j \to +\infty} \frac{\log(a(l/m)^j)}{\log l^j} \\
&= \frac{\log a}{\log l} + 1 - \frac{\log m}{\log l} \\
&= 1 + \log_l \frac{a}{m},
\end{aligned}$$

as required. The calculation of $\dim_H(\Lambda)$ is a little more elaborate (and postponed). Let $0 < \alpha = \log_l m < 1$. To get a measure on Λ we take the bernoulli measure $\mu = (p_1, \dots, p_a)^{\mathbb{Z}^+}$, where $p_{i,j} = {t_j}^{\alpha-1} / \sum_{(i,j)\in S} {t_j}^{\alpha-1}$.

We can consider a cover by squares given by the union of rectangles $[x_0, \dots, x_l, \dots, x_m]$ over all $x_l, \dots, x_m$. These are a^{m-1} rectangles of □

Example 37. *One can consider "genericity" in the linear part of the affine map (rather than the translation). Consider contractions $T_1, T_2 : \mathbb{R}^2 \to \mathbb{R}^2$ defined by*

$$T_i : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} c_i \\ d_i \end{pmatrix}, \quad i = 1, 2,$$

where $\lambda_1 < \lambda_2$.

=2.5in boxes.eps

Two affine contractions There are the following estimates on the Hausdorff and Box dimensions of the limit set.

Theorem 9. *For any choices $c_i, d_i \in \mathbb{R}$ $(i = 1, 2)$ we have:*

1. *For $0 < \lambda_1 < \lambda_2 < \frac{1}{2}$, $dim_H(\Lambda) = dim_B(\Lambda) = -\frac{\log 2}{\log \lambda_2}$;*

2. *For $0 < \lambda_1 < \frac{1}{2} < \lambda_2 < 1$, $dim_B(\Lambda) = -\frac{\log\left(\frac{2\lambda_2}{\lambda_1}\right)}{\log \lambda_1}$ and*

$$dim_H(\Lambda) \begin{cases} = -\frac{\log\left(\frac{2\lambda_2}{\lambda_1}\right)}{\log \lambda_1} & \text{for almost every } \lambda_2, \text{ but} \\ < -\frac{\log\left(\frac{2\lambda_2}{\lambda_1}\right)}{\log \lambda_1} & \text{whenever } 1/\lambda_1 \text{ is a Pisot number.} \end{cases}$$

$$dim_B(\Lambda) = -\frac{\log 2}{\log \lambda_2} \text{ if } 0 < \lambda_2 \le \frac{1}{2} - \frac{\log\left(\frac{2\lambda_1}{\lambda_2}\right)}{\log \lambda_1} \text{ if } \frac{1}{2} \le \lambda_2 < 1 \text{ and}$$

$$dim_B(\Lambda) = -\frac{\log 2}{\log \lambda_2} \text{ if } 0 < \lambda_2 \le \frac{1}{2} - \frac{\log\left(\frac{2\lambda_2}{\lambda_1}\right)}{\log \lambda_1} \text{ if } \frac{1}{2} \le \lambda_2 < 1 \text{ for almost every } \lambda_2$$

but $dim_B(\Lambda) < -\frac{\log\left(\frac{2\lambda_2}{\lambda_1}\right)}{\log \lambda_1}$ whenever λ_1 is a Pisot number.

A Pisot number is an algebraic number for which all the other roots of the integer polynomial defining it have modulus less than one. For example, $\frac{\sqrt{5}-1}{2}$ is a Pisot number.

Example 38. *For part (1), observe that since $\lambda_1 < \lambda_2 < \frac{1}{2}$ the projection onto the vertical axis is a homeomorphic to a Cantor set C in the line generated by two contractions with $\lambda_2 < \frac{1}{2}$. In particular, $dim_H(\Lambda) \geq dim_H C \geq \frac{\log 2}{\log \lambda_2}$. On the other hand, when $\lambda_2^{n-1} < \epsilon \leq \lambda_2^n$ we can cover Λ by 2^n ϵ-balls. In particular, $N(\epsilon) \leq 2^n$ and thus*

$$dim_H(\Lambda) \leq dim_B(\Lambda) = \lim_{\epsilon \to 0} -\frac{\log N(\epsilon)}{\log \epsilon} \leq -\frac{\log 2}{\log \lambda_2}$$

The proof of the second part is postponed. □

In particular, we conclude that

Corollary 4. *$dim_B(\Lambda)$ is continuous in λ_1, λ_2, but $dim_H(\Lambda)$ isn't.*

These examples are easily converted into estimates on limit sets for invertible maps (Smale horsehoes) in three dimensions, by "adding" a one dimensional expanding direction.

Example 39. *We can also consider the case of more contractions. Assume that $T_i : \mathbb{R}^2 \to \mathbb{R}^2$, $i = 1, 2, 3, 4$ are defined by*

$$T_i : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} c_i \\ d_i \end{pmatrix}, \quad i = 1, 2, 3, 4$$

where $\lambda_1 < \lambda_2 < \frac{1}{4}$. If we let

$$\begin{pmatrix} c_1 \\ d_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} c_2 \\ d_2 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{4} \end{pmatrix}, \begin{pmatrix} c_3 \\ d_3 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix}, \begin{pmatrix} c_4 \\ d_4 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{3}{4} \end{pmatrix},$$

then the limit set is the product of a point on the x-axis with a Cantor set on the y-axis (with Hausdorff dimension $-\log 4/\log \lambda_2$). In particular, $\dim_H(\Lambda) = -\log 4/\log \lambda_2$. *On the other hand, if we let*

$$\begin{pmatrix} c_1 \\ d_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} c_2 \\ d_2 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix}, \begin{pmatrix} c_3 \\ d_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}, \begin{pmatrix} c_4 \\ d_4 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix},$$

then the limit set is the product of a Cantor on the x-axis (of Hausdorff dimension $-\log 2/\log \lambda_1$) with a Cantor set on the y-axis (of Hausdorff dimension $-\log 2/\log \lambda_2$). In particular, $\dim_H(\Lambda) = -\log 2/\log \lambda_1 - \log 2/\log \lambda_2$.

Since $\lambda_1 \neq \lambda_2$, the dimensions of these two different limit sets do not agree, and we conclude that $\dim_H(\Lambda)$ *depends not only on the contraction rates but also on the translational part of the affine maps.*

Chapter 6

Conformal systems and thermodynamic formalism

In the previous chapter we consider the construction of limit sets for iterated function systems by similarities. We will now consider a generalization to the case of contractions which are conformal maps.

6.1 Coding limit sets

An alternative approach to constructing the limit set is as follows.

Definition 16. *Consider a family of contractions $T_1, \ldots, T_k : U \to U$. Fix any point $z \in U$ then we define the limit set X by the set of all limit points of sequences:*

$$X = \left\{ \lim_{n \to +\infty} T_{x_0} \circ T_{x_1} \circ \ldots \circ T_{x_n}(z) \, : \, x_0, x_1, \ldots \in \{1, \ldots, k\} \right\}$$

It is easy to see that the individual limits exist. More precisely, given a sequence $(x_n)_{n=0}^{\infty}$ we can denote $\Lambda_k := T_{x_0} \circ \ldots \circ T_{x_k}(\Lambda)$, for each $k \geq 0$. Since this is a nested sequence of compact sets the intersection is non-empty. Moreover, since all of the maps T_i are contracting it is easy to see that the limit consists of a single point.

Lemma 32. *The limit set Λ agrees with the attractor defined above. In particular, it is independent of the choice of z.*

Proof. The set of limit points defined above is clearly mapped into itself by $T : X \to X$. Moreover, it is easy to see that it is fixed by T. Since Λ was the unique fixed point (by the contraction mapping theorem) this suffices to show that the two definitions of limit sets coincide. □

This second point of view has the additional advantage that every point is coded by some infinite sequence. We can define a metric on the space of *sequences* $\{1, \dots, k\}^{\mathbb{Z}^+}$ *as follows. Given distinct sequences* $\underline{x} = (x_n)_{n=0}^{\infty}, \underline{y} = (y_n)_{n=0}^{\infty} \in \{1, \dots, k\}^{\mathbb{Z}^+}$ *we denote*

$$n(\underline{x}, \underline{y}) = \min\{n \geq 0 \; : \; x_i = y_i \text{ for } 0 \leq i \leq k, \; \text{ but } x_k \neq y_k\}.$$

We then define the metric by

$$d(\underline{x}, \underline{y}) = 2^{-n(\underline{x}, \underline{y})} \text{ if } \underline{x} \neq \underline{y} 0 \text{ otherwise}$$

It is easy to check that this is a metric. We can define a continuous map $\pi : \{1, \dots, k\}^{\mathbb{Z}^+} \to \mathbb{R}^d$ *by*

$$\pi(x) := \lim_{n \to +\infty} T_{x_0} \circ T_{x_1} \circ \dots \circ T_{x_n}(z)$$

Lemma 33. *The map* π *is Hölder continuous (i.e.,* $\exists C > 0, \beta > 0$ *such that* $||\pi(\underline{x}) - \pi(\underline{y})|| \leq C d(\underline{x}, \underline{y}))^{\beta}$ *for any* $\underline{x}, \underline{y}$. *)*

Proof. By definition, if $d(\underline{x}, \underline{y}) = 2^{-n}$, say, then $\pi(\underline{x}), \pi(\underline{y}) \in T_{x_0} \circ \dots \circ T_{x_n}(\Lambda)$. However,

$$||\pi(\underline{x}) - \pi(\underline{y})|| \leq \text{diam } (T_{x_0} \circ \dots \circ T_{x_n}(\Lambda)) \leq \alpha^n \text{diam } (\Lambda) \leq (d(\underline{x}, \underline{y}))^{\beta} \text{diam } (\Lambda)$$

where $\beta = \log \alpha / \log(1/2)$. $\square$

We shall assume for this chapter that $T_1, \dots, T_k$ *are conformal, i.e., the contraction is the same in each direction. Of course, for contractions on the line this is automatically satisfied, and is no restriction. In the one dimensional setting, such iterated function schemes are often called cookie cutters.*

If we identify $\mathbb{R}^2$ *with* $\mathbb{C}$ *then this naturally leads to simple and familiar examples of conformal maps.*

Example 40. *We can consider two natural examples of conformal maps.*

1. *Any linear fractional transformation* $T : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ *on the Riemann sphere* $\widehat{\mathbb{C}}$ *is conformal. Moreover, if* $Tz = (az + b)/(cz + d)$ *where* $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2, \mathbb{C})$ *then* $T'(z) = 1/(cz + d)^2$. *(More generally, Mobius tranformations* $T : \mathbb{S}^d \to \mathbb{S}^d$ *are conformal.)*

2. *Any analytic function* $T : U \to \mathbb{C}$, *where* $U \subset C$ *is conformal. For example, we could consider* T *to be a rational map on a neighbourhood of* U *of the hyperbolic Julia set.*

In addition, we shall also generalize the

Definition 17. *We say that a family of maps satisfies the open set condition if there exists an open set* $U \subset \mathbb{R}^d$ *such that the sets* $T_1(U), \dots, T_k(U)$ *are all contained in* U *and are disjoint.*

The next result shows that for conformal iterated function schemes, the Hausdorff dimension and Box dimension of the limit set actually coincide.

Lemma 34. *For conformal iterated function schemes satisfying the open set condition* $dim_B(\Lambda) = dim_H(\Lambda)$.

Proof. We need to show that $\dim_B(\Lambda) \leq \dim_H(\Lambda)$. This is down using the Mass Distribution Principle. Let us denote $d = \dim_B(\Lambda)$. In order to employ this method, we want to show that there is a probability measure μ on Λ and constants $C_1, C_2 > 0$ such that

$$C_1 \text{diam } (T_{x_0} \circ \ldots \circ T_{x_n}(\Lambda))^d \leq \mu(T_{x_0} \circ \ldots \circ T_{x_n}(\Lambda)) \leq C_2 \text{diam } (T_{x_0} \circ \ldots \circ T_{x_n}(\Lambda))^d.$$

In fact, the existence of such a measure is due to ideas from *Thermodynamic Formalism*, which we shall discuss later. In particular, if $x = \pi((x_n)_{n=0}^{\infty})$ then

$$\lim_{\epsilon \to 0} \frac{\log \mu(B(x,\epsilon))}{\log \epsilon} = \lim_{n \to +\infty} \frac{\log \mu(T_{x_0} \circ \ldots \circ T_{x_n}(\Lambda))}{\log \text{diam } (T_{x_0} \circ \ldots \circ T_{x_n}(\Lambda))} = d.$$

Thus by the Mass distribution principle we have that $\dim_B(\Lambda) \geq d = \dim_H(\Lambda)$. □ □

In particular, this applies to two of our favorite examples.

Corollary 5. *For hyperbolic Julia sets and Schottky group limit sets the Hausdorff dimension and the Box dimension coincide.*

We now turn the issue of calculating the dimension of limit sets. We begin with a special case, and then subsequently consider the more general case.

6.1.1 Expanding maps and conformal iterated function schemes

In many of our examples, the iterated function scheme arises from the inverse branches of an expanding map. Let $T : X \to X$ *be a* C^1 *conformal expanding map (i.e., the derivative is the same in all directions and* $|T'(x)| \geq \lambda > 1$*) on a compact space.*

Example 41. *For the set* $E_2 \subset [0,1]$ *consisting of numbers whose continued fraction expansions contains only* 1*s or* 2*s, we can take* $T : E_2 \to E_2$ *to be* $T(x) = \frac{1}{x} - [\frac{1}{x}]$. *We can consider the local inverses* $T_1 : [0,1] \to [0,1]$ *and* $T_2 : [0,1] \to [0,1]$ *defined by* $T_1(x) = 1/(1+x)$ *and* $T_1(x) = 1/(2+x)$. *We can then view* E_2 *as the limit set* $\Lambda = \Lambda(T_1, T_2)$.

More generally, to associate an iterated function scheme, we want to introduce the idea of a Markov Partition. The contractions in an associated iterated function scheme will then essentially be the inverse branches to the expanding maps. Let $T : X \to X$ *be a* $C^{1+\alpha}$ *locally expanding map on* $X \subset \mathbb{R}^d$.

Definition 18. *We call a finite collection of closed subsets $P = \{P_i\}_{i=1}^k$ a Markov Partition if it satisfies the following:*

1. *Their union is X (i.e., $\cup_{i=1}^k P_i = X$);*
2. *The sets are proper (i.e., each P_i is the closure of their interiors, relative to X);*
3. *Each image TP_i, for $i = 1, \ldots, k$, is the union of finitely many elements from P and $T : P_i \to TP_i$ is a local homeomorphism.*

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The set X is partitioned into pieces $P_1, \ldots, P_k$ each of which is mapped under T onto X.

In many examples we consider, each image $TP_i = X$, for $i = 1, \ldots, k$, in condition (iii). (Such partitions might more appropriately be called Bernoulli Partitions.)

We shall want to make use of the following standard result.

Lemma 35. *For $T : X \to X$ a $C^{1+\alpha}$ locally expanding map, there exists a Markov Partition.*

The proof of this result will be outlined in a later Appendix.

The usefulness of this result is that we can now consider the local inverses $T_i : TP_i \to P_i$, i.e., $T \circ T_i(x) = x$ for $x \in TP_i$, (extended to suitable open neighbourhoods) to be an iterated function scheme for which X is the associated limit set.

Example 42 (Hyperbolic Julia sets)**.** *Let $T : J \to J$ be a linear fractional transformation on the Julia set. Assume that the transformation $T : J \to J$ is hyperbolic (i.e., $\exists C > 0, \lambda > 1$ such that $|(T^n)'(x)| \geq C\lambda^n$, for all $x \in J$ and $n \geq 1$). Then Proposition 2.3.1 applies to give a Markov partition.*

If we consider the particular case of a quadratic map $Tz = z^2 + c$, with $|c|$ small then we can define the local inverses by

$$T_1(z) = +\sqrt{z-c} \text{ and } T_2(z) = -\sqrt{z-c}$$

Of course, in order for these maps we well defined, we need to define them on domains carefully chosen relative to the cut locus.

Example 43. *Limit sets for Kleinian groups. We will mainly be concerned with the special case of Schottky groups. In this case, we have $2n$ pairs of* *disjoint disks $D_i^+, D_i{}^-$, with $0 \le i \le n$, whose boundaries are the isometric circles associated to the generators $g_1, \dots, g_n$ (and there inverses). In particular, we can define $T : \Lambda \to \Lambda$ by*

$$T(z) = g_i(z) \text{ if } z \in D_i^+ g_i^{-1}(z) \text{ if } z \in D_i^-$$

If all of the closed disks are disjoint then $T : \Lambda \to \Lambda$ is expanding.

We now want to state the generalization of Moran's theorem to the non-linear setting . The main ingredient that we require if the following:

Example 44. *Given any continuous function $f : X \to \mathbb{R}$ we define its pressure $P(f)$ (with respect to T) as*

$$P(f) := \limsup_{n\to+\infty} \frac{1}{n} \log \underbrace{\left(\sum_{\substack{T^n x = x \\ x\in X}} e^{f(x)+f(Tx)+\ldots+f(T^{n-1}x)} \right)}_{\text{Sum over periodic points}}$$

(As we shall presently see, the limit actually exists and so the "$\limsup$" can actually be replaced by a "$\lim$".) In practise, we shall mainly be interested in a family of functions $f_t(x) = -t \log |T'(x)|$, $x \in X$ and $0 \le t \le d$, so that the above function reduces to

$$[0, d] \to \mathbb{R} t \mapsto P(f_t) = \limsup_{n\to+\infty} \frac{1}{n} \log \left(\sum_{\substack{T^n x = x \\ x\in X}} \frac{1}{|(T^n)'(x)|^t} \right)$$

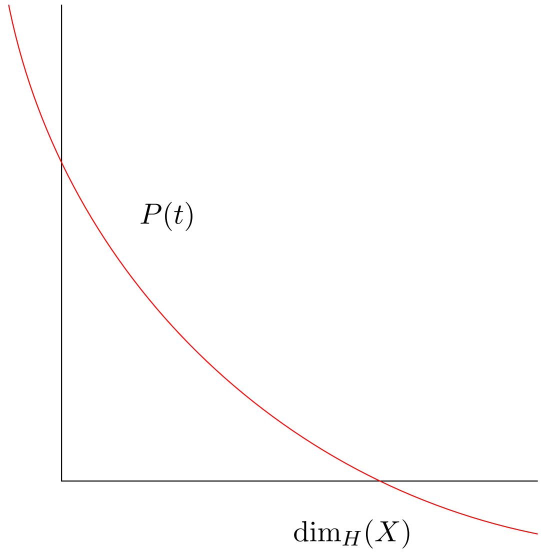
The following standard result is essentially due Bowen and Ruelle. Bowen showed the result in the context of quasi-circles and Ruelle developed the method for the case of hyperbolic Julia sets.

Lemma 36 ([Bowen-Ruelle). *Let $T : X \to X$ be a $C^{1+\alpha}$ conformal expanding map. There is a unique solution $0 \le s \le d$ to*

$$P(-s \log |T'|) = 0,$$

which occurs precisely at $s = dim_H(X)(= dim_B(X))$.

Proof. We shall explain the main ideas in the proof in the next section. □



Example 45 (Reduction to the case of linear contractions)**.** *In the case of linear iterated functions schemes this reduces to Moran's theorem. Let us assume that* $T_i = a_i x + d_i$ *then we can write*

$$\sum_{\substack{T^n x = x \\ x \in X}} \frac{1}{|(T^n)'(x)|^t} = \sum_{i_1, \cdots, i_n} \frac{1}{|a_{i_1}|^t \cdots |a_{i_k}|^t} = \left(\frac{1}{|a_1|^t} + \cdots + \frac{1}{|a_n|^t} \right)^n$$

In particular, since one readily sees that this expression is monotone decreasing as a function of t *we see from the definitions that the value* s *such that* $P(-s \log |T'|) = 0$ *is precisely the same as that for which* $1 = \frac{1}{|a_1|^s} + \cdots + \frac{1}{|a_k|^s}$*, i.e., the value given by Moran's Theorem.*

Finally, we observe that the function $t \mapsto P(f_t)$ *has the following interesting proprties*

(i) $P(0) = \log k$*;*

(ii) $t \mapsto P(f_t)$ *is strictly monotone decreasing;*

(iii) $t \mapsto P(f_t)$ *is analytic on* $[0, d]$*.*

Property (i) is immediate from the definition. We shall return to the proofs of properties (ii) and (iii) later. For the present, we can interpret analytic to mean having a convergent power series in a sufficiently small neighbourhood of each point.

One particularly nice application of the above theorem and properties of pressure is to showing the analyticity of dimension as we change the associated expanding map. More precisely:

Corollary 6. *Let T_λ, with $-\epsilon \le \lambda \le \epsilon$, be an analytic family of expanding maps. Then $\lambda \mapsto dim_H(\Lambda_\lambda)$ is analytic.*

Proof. The function $f(\lambda, t) = P(-t \log |T'_\lambda|)$ is analytic and satisfies $\frac{\partial f}{\partial \lambda}(\lambda, t) \neq 0$. Using the Implicit Function Theorem, we can often deduce that for an analytic family T_λ the dimension $\lambda \mapsto \dim(\Lambda_\lambda)$ is analytic too. □ □

This applies, in particular, to the examples of hyperbolic Julia sets and limit sets for Schottky groups.

Example 46 (Quadratic maps)**.** *The map $T_c(z) = z^2 + c$ has a hyperbolic Julia set J_c provided $|c|$ is sufficiently small. Ruelle used the above method to show that $c \mapsto dim(J_c)$ is analytic for $|c|$ sufficiently small. (He also gave the first few terms in the expansion for $dim(J_c)$, as given in the previous chapter).*

In the next section we explain the details of the proof of Theorem 2.3.2.

6.1.2 Proving the Bowen-Ruelle result

Let $T : X \to X$ be a map on $X \subset \mathbb{R}^d$. By an expanding map we mean one which locally expands distances. In the present context we can assume that there exists $C > 0$ and $\lambda > 1$ such that

$$||D_x T^n(v)|| \ge C\lambda^n ||v||, \text{ for } n \ge 1.$$

The hypothesis that T is $C^{1+\alpha}$ means that the derivative DT is α-Hölder continuous, i.e.,

$$||DT||_\alpha := \sup_{x \neq y} \frac{||D_x T - D_y T||}{||x - y||} < +\infty.$$

Here the norm in the numerator on the Right Hand Side is the norm on linear maps from $\mathbb{R}^d$ to itself (or equivalently, on $d \times d$ matrices).

Let $T : X \to X$ be a $C^{1+\alpha}$ locally expanding map on $X \subset \mathbb{R}^d$. Consider a Markov Partition $P = \{P_i\}_{i=1}^k$ for T. If we write $T_i : X \to P_i$ for the local inverses then this describes an iterated function scheme. For each $n \ge 1$ we want to consider n-tuples $\underline{i} = (i_1, \ldots, i_n) \in \{1, \ldots, k\}^n$. We shall assume that $TP_{i_r} \supset P_{i_{r-1}}$, for $r = 2, \ldots, n$. It is then an easy observation that

$$P_{\underline{i}} := T_{i_n} \cdots T_{i_2} P_{i_1}$$

is again a non-empty closed subset, and the union of such sets is equal to X.

We would like to estimate the dimension of X by making a cover using the sets $P_{\underline{i}}$, $|\underline{i}| = n$. A slight technical difficulty is that these sets are closed, rather than open. Moreover, if we try to use their interiors we see that they *might not cover X. The solution is rather easy: we simply make a cover by choosing open neighbourhoods $U_{\underline{i}} \supset P_{\underline{i}}$ which are slightly larger, and thus do form a cover for X. Let us assume that there is $0 < \theta < 1$ such that*

$$\frac{diam(U_{\underline{i}})}{diam(P_{\underline{i}})} \leq 1 + O(\theta^n), \text{ for all } \underline{i}.$$

Let us define $T_{\underline{i}} : P_{i_1} \to P_{\underline{i}}$ by $T_{\underline{i}} = T_{i_1} \circ \cdots \circ T_{i_n}$.

We can now obtain the following bounds.

Lemma 37. *We have the following bounds.*

1. *There exist $B_1, B_2 > 0$ such that for all $\underline{i}$ and all $x, y \in X$:*

$$B_1 \leq \frac{|T'_{\underline{i}}(x)|}{|T'_{\underline{i}}(y)|} \leq B_2$$

2. *There exist $C_1, C_2 > 0$ such that for all $\underline{i}$ and for all $x \in X$:*

$$C_1 \leq \frac{\mathrm{diam}(P_{\underline{i}})}{|T'_{\underline{i}}(x)|} \leq C_2.$$

In particular, for $t > 0$, there exist $C_1, C_2 > 0$ such that for any x and $n \geq 1$:

$$C_1 \leq \frac{\sum_{|\underline{i}|=n} \mathrm{diam}(U_{\underline{i}})^t}{\sum_{|\underline{i}|=n} |(T_{\underline{i}})'(x)|^t} \leq C_2$$

Proof. Part (1) is sometimes referred to as a telescope lemma. If $D = \sup_i || \log |T'_i|||_\alpha$ and $\theta = \sup_i ||T'_i||_\infty < 1$:

$$\begin{aligned} |\log |T'_{\underline{i}}(x)| - \log |T'_{\underline{i}}(y)|| &= \sum_{j=1}^{n} \left| \log |T'_{i_j}(T_{i_{j+1}} \cdots T_{i_n} x)| - \log |T'_{i_j}(T_{i_{j+1}} \cdots T_{i_n} y)| \right| \\ &\leq D \sum_{j=1}^{n} d(T_{i_{j+1}} \cdots T_{i_n} x, T_{i_{j+1}} \cdots T_{i_n} y)^\alpha \\ &\leq D \sum_{j=1}^{n} \theta^{n\alpha} d(x,y)^\alpha \leq \left(\frac{D}{1-\theta^\alpha} \right) d(x,y)^\alpha \end{aligned}$$

This uses the Chain Rule and Holder continuity. In particular, setting $C = \frac{D}{1-\theta^\alpha} > 0$ we have that for and $x, y \in X$ and all $n \geq 1$ and $|\underline{i}| = n$ with $i_1 = i$:

$$\left| \log |T'_{\underline{i}}(x)| - \log |T'_{\underline{i}}(y)| \right| \leq C d(x,y)^\alpha.$$

In particular, part (1) follows since:

$$\underbrace{e^{-C\mathrm{diam}(X)^\alpha}}_{=:B_1} \leq \frac{|T'_{\underline{i}}(x)|}{|T'_{\underline{i}}(y)|} = \exp\left(\log |T'_{\underline{i}}(x)| - \log |T'_{\underline{i}}(y)|\right) \leq \underbrace{e^{C\mathrm{diam}(X)^\alpha}}_{=:B_2}.$$

Since the contractions are conformal we can estimate

$$B_1|T'_{\underline{i}}(x)| \le \text{diam } (P_{\underline{i}}) \le B_2|T'_{\underline{i}}(x)|.$$

This suffices to deduce Part (2). $\square$

It is not surprising that the part of the approach to proving the Bowen-Ruelle result involves understanding the asymptotics of the expression $\sum_{|\underline{i}|=n} diam(U_{\underline{i}})^d$ *as* $n \to \infty$*, since this is intimately related to definition involving covers of the Hausdorff dimension of* X*. Moreover, the last Propostion tells us that it is an equivalent problem to understand the behaviour of* $\sum_{|\underline{i}|=n} |(T_{\underline{i}})'(x)|$*. Perhaps, at first sight, this doesn't seem to be an improvement. However, the key idea is to introduce a transfer operator.*

Definition 19. *Let* $C^\alpha(P)$ *be the space of Hölder continuous functions on the disjoint union of the sets in* P*. This is a Banach space with the norm* $||f|| = ||f||_\infty + ||f||_\alpha$ *where*

$$||f||_\infty = \sup_{x\in X} |f(x)| \text{ and } ||f||_\alpha = \sup_{x\neq y} \frac{|f(x) - f(y)|}{d(x, y)^\alpha}.$$

For each $t > 0$ *we define a bounded linear operator* $L_t : C^\alpha(P) \to C^\alpha(P)$ *by*

$$L_t w(x) = \sum_i |T'_i(x)|^t w(T_i x).$$

To understand the role played by the transfer operator, we need only observe that iterates of the operator applied to the constant function 1 *take the required form: for* $x \in X$

$$L^n_t 1(x) = \sum_{|\underline{i}|=n} |(T_{\underline{i}})'(x)|^t,$$

i.e., the numerator in the last line of Proposition 2.4.1 (2). In particular, to understand what happens as n *tends to infinity is now reduced to the behaviour of the operator* L_t*.*

Lemma 38 (Ruelle Operator Theorem)**.** *The operators* L_t *have the following properties.*

1. *The operator* L_t *has a simple maximal positive eigenvalue* λ_t*. Moreover the rest of the spectrum is contained in a disk of strictly smaller radius, i.e., we can choose* $0 < \theta < 1$ *and* $C > 0$ *such that* $|L^n_t 1 - \lambda^n_t| \le C\lambda^n_t\theta^n$*, for* $n \ge 1$*.*

2. *There exists a probability measure* μ *and* $D_1, D_2 > 0$ *such that for any* $n \ge 1$ *and* $|\underline{i}| = n$ *and* $x \in X$*:*

$$D_1\lambda^n_t \le \frac{\mu(P_{\underline{i}})}{|T'_{\underline{i}}(x)|^t} \le D_2\lambda^n_t.$$

3. *The map $\lambda : \mathbb{R} \to \mathbb{R}$ given by $\lambda(t) = \lambda_t$ is real analytic and $\lambda'(t) < 0$ for all $t \in \mathbb{R}$.*

We shall return to the proof of this result later. However, for the present we have an immediate corollary.

Corollary 7. *We can write $P(-t \log |T'|) = \log \lambda_t$.*

Proof. For each $|\underline{i}| = n$ we can choose a periodic point $T^n x = x$ such that By Proposition 2.4.1 (1), if we let $C_1 = B_1^t, C_2 = B_2^t > 0$ then for any $x_0 \in X$ we have $C_1|(T^n)'(x_0)|^{-t} \leq |(T^n)'(x)|^{-t} \leq C_2|(T^n)'(x_0)|^{-t}$. Summing over all possible $|\underline{i}| = n$ we have that:

$$C_1(L_t^n 1)(x_0) \leq \sum_{T^n x = x} |(T^n)'(x)|^{-t} \leq C_2(L_t^n 1)(x_0). \tag{2.2}$$

The result then follows from the definition of pressure and part (2) of Proposition 2.4.2. $\square$

In particular, properties (ii) and (iii) follow from this corollary.

By Part (2) of Proposition 2.4.1 and (2.2) we see that for some $D_1, D_2 > 0$ and $0 \leq t \leq n$:

$$D_1 \lambda_t^n \leq \sum_{|\underline{i}|=n} diam(U_{\underline{i}})^t \leq D_2 \lambda_t^n, \text{ for } n \geq 1.$$

Recalling the definition of Hausdorff dimension we can bound

$$H_\epsilon^t(X) = \inf_U \left\{ \sum_{U_i \in U} \text{diam } (U_i)^t \right\} \leq \sum_{|\underline{i}|=n} diam(U_{\underline{i}})^t \leq D_2 \lambda_t^n,$$

where the infimum is over open covers U whose elements have diameter at most $\epsilon > 0$, say, and n is chosen such that $\epsilon = \max_{|\underline{i}|=n}\{diam(U_{\underline{i}})\}$. We can therefore deduce that if $t > d$ then $\lambda_t < 1$ and thus $\lim_{\epsilon \to 0} H_\epsilon^t(X) = 0$. In particular, from the definition of Hausdorff dimension we see that $\text{diam}_H(X) \leq d$.

To obtain the lower bound for $dim_H(X)$ we can use the mass distribution principle with the measure μ. In particular, for any $|\underline{i}| = n$ and $x \in X$ we can estimate

$$\mu(P_{\underline{i}}) = \int (L_t^n \chi_{P_{\underline{i}}}) d\mu \leq D_2 \lambda_d^n |T_{\underline{i}}'(x)|^d \leq D_2 C_1^{-1} \lambda_d^n (diam(P_{\underline{i}}))^d$$

Given any $x \in X$ and any $\epsilon > 0$ we can choose n so that we can cover the ball $B(x, \epsilon)$ by a uniformly bounded number of sets $P_{\underline{i}}$ with $|\underline{i}| = n$.

In particular, since $\lambda_d = 1$ we can deduce that there exists $C > 0$ such that $\mu(B(x, \epsilon)) \leq C\epsilon^d$ for $\epsilon > 0$. Thus, by the mass distribution we dedude that $dim_H(X) \geq d$.

This completes the proof of the Bowen-Ruelle Theorem (except for the proof of Proposition 2.4.2). It remains to prove Proposition 2.4.2

Proof of Proposition 2.4.2. Fix $C > 0$. We can consider the cone of functions

$$C = \{f : C \to \mathbb{R} : 0 \leq f(x) \leq 1 \text{ and } f(x) \leq f(x)e^{C||x-y||^\alpha}, \forall x, y \in X\}.$$

It is easy to see that C is convex and closed with respect to the norm $||\cdot||_\infty$.

If $g \in C$ then for $x \neq y$ we have that

$$\begin{aligned}|g(x) - g(y)| &\leq |g(y)| \left(\exp\left(C||f||_\alpha ||x-y||^\alpha\right) - 1\right) \\ &\leq ||g||_\infty C||f||_\alpha \exp\left(C||f||_\alpha\right) ||x-y||^\alpha,\end{aligned}$$

from which we deduce that C is uniformly continuous in the $||\cdot||_\infty$ norm, and thus compact by the Arzela-Ascoli theorem.

Given $n \geq 1$ we can define $L_n(g) = L(g + 1/n)/||L(g + 1/n)||$. Since the operator L is positive, the numerator is non-zero and thus the operator L_n is well defined. Moreover, providing C is sufficiently large we have that

$$L_n f(x) \leq L_n f(x) e^{C||x-y||^\alpha}$$

from which we can easily deduce that $L_n(C) \subset C$. Using the Schauder-Tychanoff Theorem there is a fixed point $L_n g_n = g_n \in C$, i.e.,

$$L(g_n + 1/n) = ||L(g_n + 1/n)||(g_n + 1/n). \tag{2.3}$$

Finally, we can again use that C is compact in the $||\cdot||_\infty$ norm to choose a limit point $h \in C$ of $\{h\}_{n=1}^\infty$. Taking limits in (2.3) we get $L_t h = \lambda_t h$, where $\lambda_t = ||L_t h||_\infty$.

Next observe that $L_t(h + 1/n)(x) \geq \inf\{(h_n(x) + 1/n)e^{-||f||_\infty}\}$ and so $||L_t(h + 1/n)||_\infty \geq e^{-||f||_\infty}$. Taking the limit we see that $\lambda_t \geq e^{-||f||_\infty} > 0$. To show that $h > 0$, assume for a contradiction that $h(x_0) = 0$. Then since $L_t^n h(x_0) = \sum_{|\underline{i}|=n} \lambda_t^n |T'_{\underline{i}}(x_0)| h(T_{\underline{i}} x_0)$ we conclude that $h(T_{\underline{i}} x_0) = 0$ for all $|\underline{i}| = n$ and all $n \geq 1$. In particular, $h(x)$ is zero on a dense set, but then it must be identically zero contradicting $\lambda_t = ||L_t h||_\infty > 0$. To see that λ_t is a simple eigenvalue, observe that if we have a second eigenvector g with $L_t g = \lambda_t g$ and we let $t = \inf\{g(x)/h(x)\} = g(x_0)/h(x_0)$ then $g(x) - th(x) \geq 0$, but with $g(x_0) - th(x_0) = 0$. Since $g - th$ is again a positive eigenvector for L_t, the preceding argument shows that $g - th = 0$, i.e., g is a multiple of h.

Let us define a new operator $M_t w(x) = \lambda_t^{-1} w(x)^{-1} \sum_i |T'_i(x)|^t h_t(T_i x) w(T_i)$. By defintion, we have that $M_t 1 = 1$, i.e., M_t preserves the constants. Let M be the space of probability measures on X. The space M is convex and compact in the weak star topology, by Alaoglu's theorem. Since $M_t : M \to M$ we see by the Schauder-Tychanof theorem that $M_t \mu = \mu$, or equivalently, $L_t \nu = \lambda_t \nu$, where $\nu = h\mu$, i.e.,

$$\int (L_t w)(x) d\nu(x) = \lambda_t \int w(x) d\mu(x) \tag{2.4}$$

for all $w \in C(X)$. We can consider the characteristic function $\chi_{P_{\underline{i}}}$ and then

$$\mu(P_{\underline{i}}) = \int \chi_{P_{\underline{i}}} d\mu_t = \lambda_t^{-n} \int L_t^n \chi_{P_{\underline{i}}} d\mu_t = \lambda_t^{-n} \int |(T_{\underline{i}})'(y)| d\mu_t(y)$$

However, by Proposition 2.4.1 (1) we can bound

$$B_1 B_2^{-1} |(T_{\underline{i}})'(x)| \leq \int |(T_{\underline{i}})'(y)| d\mu_t(y) \leq B_2 B_1^{-1} |(T_{\underline{i}})'(x)|$$

for all $x \in X$. Thus Part (2) of Proposition 2.4.2 follows.

It is a simple calculation to show that there exists $C > 0$ such that

$$||M_t^n h||_\alpha \leq C||h||_\infty + \alpha^n ||h||_\alpha, \text{ for } n \geq 1. \tag{2.5}$$

We first claim that $M_t^n h \to \int g d\mu$ in the $|| \cdot ||_\infty$ topology. To see this we first observe from (2.5) that the family $\{M_t^n h\}_{n=1}^\infty$ is equicontinuous. We can then choose a limit point $\overline{h}$. In particular, since $M_t 1 = 1$ we see that $\sup \overline{h} \geq \sup M_t \overline{h} \geq \cdots \geq \sup M_t^n \overline{h} \to \sup \overline{h}$, from which we deduce $\sup M_t^n \overline{h} = \sup \overline{h} = \overline{h}(x)$, say, for all $n \geq 1$. In particular, $\overline{h}(T_{\underline{i}} x) = \overline{h}(x)$ for all $|\underline{i}| = n$ and $n \geq 1$ and so $\overline{h}$ is a constant function. We can denote by $\mathbb{C}^\perp$ the functions $h \in C^\alpha(X)$ which satisfy $\int h d\mu = 0$. To show that the rest of the spectrum is in a disk of smaller radius we shall apply the spectral radius theorem to $M_t : \mathbb{C}^\perp \to \mathbb{C}^\perp$ to show that its spectrum is strictly within the unit disk. (The spectra of M_t and L_t agree up to scaling by β_t). For $h \in \mathbb{C}^\perp$ the convergence result becomes $||M_t^n h||_\infty \to 0$. By applying (2.5) twice we can estimate:

$$\begin{aligned}
\|M_t^{2n} h\|_\alpha &\leq C \|M_t^n h\|_\infty + \alpha^n \|M_t^n h\|_\alpha \\
&\leq C \|M_t^n h\|_\infty + \alpha^n (C \| \|_\infty + \|h\|_\alpha) \\
&\to 0 \text{ as } n \to +\infty.
\end{aligned}$$

In particular, for n sufficiently large we see that $||M_t^{2n} h||_\alpha < 1$ and so the result on the spectrum follows.

For the final part, we observe that since λ_t is a simple isolated eigenvalue it follows by perturbation theory that it has an analytic dependence on t (as does its associated eigenfunction h_t, say). To show that λ_t is monotone decreasing we consider its derivative. Differentiating $L_t h_t = \lambda_t h_t$ we can write

$$\lambda_t' h_t + \lambda_t h_t' = L_t h_t' + L_t (\log |T'| h_t)$$

Integrating with respect to μ_t and applying (2.4) we can cancel two of the terms to get $\lambda_t' \int h_t d\mu_t = \int \log |T_t'| h_t d\mu_t$. $\square$

6.2 Julia sets and Quasi-circles

6.2.1 Julia and Mandelbrot sets

The study of Julia sets is one of the areas which has attracted most attention in recent years. We shall begin considering the general setting and specialise later to quadratic maps. Consider a map $T : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ *defined by a rational function* $T(z) = P(z)/Q(z)$*, for non-trivial relatively prime polynomials* $P, Q \in \mathbb{C}[z]$*. To avoid trivial cases, we always assume that* $d := max(deg(P), deg(Q)) \geq 2$*.*

Definition 20. *We define the Julia set* J *to be the closure of the repelling periodic points i.e.*

$$J = \mathrm{cl}\left(\left\{z \in \widehat{\mathbb{C}} : T^n(z) = z, \text{ for some } n \geq 1, \text{ and } |(T^n)'(z)| > 1\right\}\right).$$

The Julia set J *is clearly a closed* T*-invariant set (i.e.,* $T(J) = J$*). There are other alternative definitions, but we shall not require them. By contrast,* T *has at most finitely many attracting periodic points, which must be disjoint from the Julia set.*

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We choose the point $c = \frac{i}{4}$ *in the parameter space (left picture) and draw the associated Julia set for* $T(z) = z^2 + \frac{i}{4}$ *(right picture).*

Let us now restrict to polynomial maps of degree 2. We can make a change of coordinates to put these maps in a canonical form. For a fixed parameter $c \in \mathbb{C}$ *consider the map* $T_c : \mathbb{C} \mapsto \mathbb{C}$ *defined by* $T_c : z \to z^2 + c$*. Let* J_c *be the associated Julia set. To begin with, we see that when* $c = 0$ *then the Julia set is easily easily calculated.*

Example 47. $c = 0$ *For* $T_0 z = z^2$*, the repelling periodic points of period* n *are the dense set of points on the unit circle of the form* $\xi = \exp(2\pi i k/(2^n - 1))$*. The corresponding derivitive is* $|(T_0^n)'(\xi)| = 2^n$*. In particular, we have* $J_0 = \{z \in \mathbb{C} : |z| = 1\}$*, i.e., the unit circle. Thus, trivially we have that* $dim(J_0) = 1$*.*

We next consider the case of values of c *of sufficiently small modulus, where the asymptotic behaviour of the limit set is well understood through a result of Ruelle:*

Proposition 10. *For* $|c|$ *sufficiently small:*

1. *the Julia set* J_c *for* $T_c(z) = z^2 + c$ *is still a Jordan circle, but it has* $dim_B(J_c) = dim_H(J_c) > 1$*; and*

2. *the map* $c \mapsto dim_H(J_c)$ *is real analytic and we have the asymptotic*

$$dim_H(J_c) \sim 1 + \frac{|c|^2}{4 \log 2}, \qquad \text{as } |c| \to 0.$$

In a later section we shall give an outline of the proof of this result using ideas from Dynamical Systems.

At the other extreme, if c has large modulus, the asymptotic behaviour of the limit set is well understood through the following results of Falconer.

Proposition 11. *For $|c|$ sufficiently large*

1. *the Julia set for T_c is a Cantor set, with $dim_B(J_c) + dim_H(J_c) > 0$; and*

2. *the map $c \mapsto dim_H(J_c)$ is real analytic and we have the asymptotic*

$$dim_H(J_c) \sim \frac{2\log 2}{\log|c|} \quad \text{as } |c| \to +\infty \quad [?].$$

Moreover, there are also a few special cases where the Julia set (and its dimension) are well understood. For example, the case $c = -2$ is particularly simple:

Example 48. *When $c = -2$ then $J_{-2} = [-2, 2]$, i.e., a closed interval and in this case we again trivially have that $dim(J_{-2}) = 1$. For $c < -2$, the Julia set is contained in the real axis.*

Unfortunately, in general the Hausdorff dimension of the Julia set for most values of c cannot be given explicitly. However, the general nature of the Julia set is characterized by the following famous subset of the parameter space c.

Definition 21. *The Mandelbrot set $M \subset \mathbb{C}$ is defined to be the set of points c in the parameter space such that the orbit $\{T_c^n(0) : n \geq 0\}$ is bounded, i.e.,*

$$M := \{c \in \mathbb{C} : |T_c^n(0)| \not\to +\infty, \text{ as } n \to +\infty\}.$$

=3.0in Mandelbrotset.eps

The Mandelbrot set in the parameter space for c In fact, the importance of $z = 0$ in this definition is that it is a critical point for T_c, i.e., $T_c'(0) = 0$. The significance of the Mandelbrot set is that it actually characterizes the type of Julia set J_c one gets for T_c.

Proposition 12. *If $c \notin M$ then J_c is a Cantor set. If $c \in M$ then J_c is a connected set.*

For more specific choices for the parameter c we have to resort to numerical computation if we want to know the Hausdorff dimension of J_c. We shall study this problem in detail in a latter chapter. However, for the moment, we shall illustrate this by examples of each type of behaviour.

example

(i) *Let us consider two points in the Mandelbrot set. For $c = i/4$,say, we can estimate*

$$\dim_H(J_{i/4}) = 1.02321992890309691\ldots$$

For $c = 1/100$, say, we can estimate

$$\dim_H(J_{1/100}) = 1.00003662\cdots$$

(ii) *Let us consider two points outside of the Mandelbrot set. For $c = -3/2 + 2i/3$, say, we can estimate*

$$dim_H(J_{-3/2+2i/3}) = 0.9038745968111...$$

For $c = -5$,say, we can estimate

$$dim_H(J_{-5}) = 0.4847982944381604305383984 7...$$

However, an important ingredient in the method of computation of these values is that the Julia set should satisfy an additional property which is particularly useful in our analysis our analysis. More precisely, we need to assume that T_c is hyperbolic in the following sense.

Definition We say that the rational map is *hyperbolic* if there exist $\beta > 1$ and $C > 0$ such that for any $z \in \mathbb{C}$ we have $(T^n)'(z)| \geq C\beta^n$, for all $n \geq 1$.

Hyperbolicity, in various guises, is something that underpins a lot of our analysis in different settings. For the particular setting of rational maps, hyperbolicity can be shown to be equivalent to the Julia set J being disjoint from the orbit of the critical points $C = \{z : T'(z) = 0\}$ (i.e. $J \cap (\cup_{n=0}^{\infty} T^n(C)) = \emptyset$). However, we shall not require this observation in the sequel.

Proposition 13. *If T_c is hyperbolic then $dim_H(J_c) = dim_B(J_c)$.*

Proof. Actually, in the case of hyperbolic maps we can think of the Julia set as being the limit set of an iterated function scheme with respect to the two inverse branches for T_c. In this case, the result is just a special case of more general results (which we return to in a later chapter). □

As a cautionary tale, we should note that once one takes c outside of the region in the parameter space corresponding to hyperbolic maps, then the situation becomes more complicated. For example, the dimension of the Julia set may no longer be even continuous in c, in contrast to the hyperbolic case where there is actually a real analytic dependence. This is illustrated by the following.

Remark 10 (Parabolic Explosions)**.** *Of course, as c crosses the boundary of the Mandelbrot set the Julia set J_c (and its Hausdorff dimension) can change more dramatically. Douady studied the case as $c \to \frac{1}{4}$ (along the real axis). As c increases the dimension dim(J_c) increases monotonically, with derivative tending to infinity. However, as c increases past $\frac{1}{4}$ there is a discontinuity where the dimension suddenly stops.*

Let us return to studying the Mandelbrot set. Although the Mandelbrot set is primarily a set in the parameter space for the quadratic maps, it has a particularly interesting structure in its own right. Some of its main features are described in the following proposition.

1. The set M lies within the ball of radius 2 given by $\{c \in \mathbb{C} : |c| \leq 2\}$;
2. The set M is closed, connected and simply connected;
3. The interior $\mathrm{int}(M)$ is a union of simply connected components;
4. The largest component of $\mathrm{int}(M)$ is the *main cardioid* defined by
$$M_1 = \{w \in \mathbb{C} : |1 - \sqrt{1-4w}| < 1\}$$
and for any $c \in M_1$ the map T_c is hyperbolic;
5. For $c \notin M$, the map T_c is hyperbolic.

Proof. For part (1), suffices to show that if $|c| \geq 2$ then the sequence $\{T_c^n(0) : n \geq 0\}$ is unbounded. If $|z| > 2$, then $|z^2+c| \geq |z^2|-|c| > 2|z|-|c|$. If $|z| \geq |c|$, then $2|z|-|c| > |z|$. So, if $|z| > 2$ and $|z| \geq c$, $|z^2+c| > |z|$, so the sequence is increasing. (It takes a bit more work to prove it is unbounded and diverges.) If $|c| > 2$, the sequence diverges.

The Mandelbrot set is known to be a simply connected set in the plane from a theorem of Douady and Hubbard that there is a conformal isomorphism from the complement of the Mandelbrot set to the complement of the unit disk.

For the other properties we refer the reader to any book on rational maps (e.g., [**?**]). □

Although we don't have a comprehensive knowledge of which parameter values c lead to T_c being hyperbolic, we do have some partial information. For example, it is known that a component H of $\mathrm{int}(M)$ contains a parameter c for which T_c is hyperbolic if and only if $T_{c'}$ is hyperbolic for *every* $c' \in H$. In particular, any c in the central cartoid M_1 the map T_c has the attracting fixed point $\frac{1}{2}(1 - \sqrt{1-4w})$, and thus is hyperbolic because of another equivalent condition for hyperbolicity is: Either $c \notin M$ or T_c has an attracting cycle. We call H a *hyperbolic component.*

At first sight, one might imagine that there is little direction between the metric properties of the Mandelbrot set and the associated Julia sets. However, there are are a number of surprising connections. We mention only the following.

Theorem 10 (Shishikura)**.** *The boundary of M has Hausdorff dimension 2. For generic points c in the boundary the associated Julia set for T_c has Hausdorff dimension 2.*

Although considerable work has been in recent years done on understanding the structure of the Mandelbrot set, and enormous progress has been made, there remain a number of major outstanding questions. The solution to these would give fundamental insights into the nature of the Mandelbrot set.

Major Open Problems However, it is a major conjecture that the boundary ∂M is locally connected (i.e., if every neighbourhood of $\partial M \cap B(x, \epsilon)$ contains a connected open neighbourhood). Another important question is whether there exist any examples of Julia sets which can have positive measure. Finally, it is apparently unknown whether every component of $\mathrm{int}(M)$ is hyperbolic.

6.2.2 Random iterated function schemes

6.2.3 β-expansions and Fat Sierpinski triangles

Chapter 7

Projections and slices

In this chapter we discuss two important results due to Marstrand.[1]

7.1 The Projection Theorem

We begin with one of the classical projection theorems. Let $A \subseteq \mathbb{R}^2$ and $p_\theta : \mathbb{R}^2 \to \mathbb{R}$ correspond to the linear projection onto the line at an angle θ to the x axis. More precisely, let $\underline{\theta} = (\cos\theta, \sin\theta)$ and for $\underline{x} = (x, y)$ we write $\underline{x} \cdot \underline{\theta} = (x\cos\theta + y\sin\theta)$ and then the projection is given by

$$\begin{aligned} &p_\theta : \mathbb{R}^2 \to \mathbb{R} \\ &p_\theta(x, y) = \underline{x} \cdot \underline{\theta} \end{aligned}$$

Example 49. *Consider the middle third Cantor set $C \subset [0, 1]$ and let $X = C \times \{0\} \subset [0, 1] \times [0, 1]$ then $\dim_H(X) = \frac{\log 2}{\log 3}$. For $\theta \neq \pm\pi$ then $p_\theta(X) \subset \mathbb{R}$ is a scaled version of C which again has $\dim_H(p_\theta X) = \frac{\log 2}{\log 3}$. On the other hand for $\theta = \pm\pi$ we have that $p_\theta(X) = \{0\} \subset \mathbb{R}$ which therefore has $\dim_H(p_\theta X) = 0$.*

Let λ denote one dimensional Lebesgue measure on the real line.

Theorem 11 (Marstrand Projection Theorem)**.** *Let $A \subset \mathbb{R}^2$ and $\dim_H A = s$.*

1. *If $s \leq 1$ then for almost all θ, $\dim_H p_\theta(A) = \dim_H A$.*

2. *If $s > 1$ then for almost all θ, $\lambda(p_\theta(A)) > 0$.*

[1] John Marstrand was essentially supervised by Besicovich and his famous results from 1954 came from his PhD thesis. Marstrand wasn't very prolific, but he had many outside interests including becoming the British over 50 Fell-racing champion. Falconer attributes to him the insightful comment "There is only one idea in mathematical analysis: you integrate a function in two ways and apply Fubini's theorem. The difficulty is finding the right function."

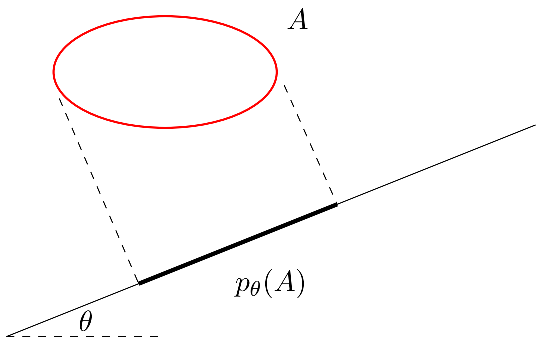

Figure 7.1: The result shows that *if the set is small enough there is no drop in the Hausdorff dimension for typical directions.*

Although this result was first proved in 1954, Kaufmann introduced an alternative method, which we will follow. We begin with a preliminary lemma, which is a version of Frostman's lemma.

Lemma 39. *Assume that $H^t(X) > 0$. We require the following fact: There exists a compact set $K \subset X$ with $0 < H^t(K) < +\infty$ and $b > 0$ such that $H^t(K \cap B(x,r)) \leq br^t$*

We omit the proof.

Lemma 40. *Let $0 < s < t$.*

1. *If $H^t(X) > 0$ there there exists a measure on μ on X such that for all $t' > t$,*
$$\int_X \int_X \frac{d\mu(x)d\mu(y)}{|x-y|^{t'}} < +\infty$$
2. *If μ is a probability measure such that*
$$\int_X \int_X \frac{d\mu(x)d\mu(y)}{|x-y|^{s}} < +\infty$$
then $\dim_H(X) \geq s$.

Proof. By the previous lemma we can choose a compact set K. Let $\mu = H^t|K$ be the restriction to K. [2]

We begin with part (1).We can define $\phi : K \to \mathbb{R}$ by

$$\phi(x) = \int_K \frac{d\mu(y)}{|x-y|^{t'}} \text{ for each } x \in K.$$

[2]Here we are using that $H^t(\cdot)$ gives rise to a measure. This requires a proof, which we have omitted

We can then bound

$$\begin{aligned}\phi(x) &= \int_{|x-y|\le 1} \frac{d\mu(y)}{|x-y|^{t'}} + \int_{|x-y|\ge 1} \frac{d\mu(y)}{|x-y|^{t}} \\ &= \sum_{n=1}^{\infty} \int_{\frac{1}{2^n}\le |x-y|\le \frac{1}{2^{n-1}}} \frac{d\mu(y)}{|x-y|^{t'}} + \int_{|x-y|\ge 1} \frac{d\mu(y)}{|x-y|^{t'}} \\ &\le \sum_{n=1}^{\infty} \frac{\mu(B(x,\frac{1}{2^n}))}{2^{nt'}} + \mu(\mathbb{R}^n) \\ &\le \sum_{n=1}^{\infty} \frac{b}{2^{n(t'-t)}} + \mu(\mathbb{R}^n) \le C\end{aligned}$$

for some constant $C > 0$. Thus we have that

$$\int_X \int_X \frac{d\mu(x)d\mu(y)}{|x-y|^{t'}} = \int_X \phi(x)d\mu(x) \le C.$$

This completes the proof of Part (1).

To prove part (2), let us now define $\psi : K \to \mathbb{R}$ by

$$\psi(y) = \int_K \frac{d\mu(x)}{|x-y|^s} \in L^1(K,\mu).$$

In particular, by choosing $M > 0$ sufficiently large the set

$$A_M = \{y \in K : \psi(y) \le M\}$$

satisfies $\mu(A_M) > 0$. Let $\nu = \mu|A_M$ be the (further) restriction to A_M. Then for all $x \in A$ and $r > 0$ we have

$$M \ge \int_{A_M} \frac{d\nu(y)}{|x-y|^s} \ge \int_{B(x,r)\cap A_M} \frac{d\nu(y)}{|x-y|^s} \ge \frac{1}{r^s}\nu(B(x,r))$$

In particular, $\nu(B(x,r)) \le Mr^s$ for all $r > 0$. Thus by the Mass Distribution Principle we have that $\dim_H(A) \ge s$. This completes the proof. □

After this preparation, we now come to the proof of the theorem.

Proof of Marstrand Projection Theorem. For part (1), let $A \subset \mathbb{R}^2$ where $\dim_H(A) < 1$. We begin by observing that for any θ we have that $p_\theta : X \to \mathbb{R}$ is Lipschitz and thus $\dim_H(p_\theta(X)) \le \dim_H(X)$. It remains to show that for almost every θ we have an equality.

Fix any $t < \dim_H(A)$ then from the definition of Hausdorff dimension we know that $H^t(A) > 0$. Thus by the first part of the second lemma there exists a probability measure μ on A such that

$$\int_A \int_A \frac{\mathrm{d}\mu(x)\mathrm{d}\mu(y)}{|x-y|^t} < \infty. \tag{6.0}$$

We denote by $\mu_\theta = p_\theta \mu$ the projection of the measure μ onto the line $\mathbb{R}$ (i.e., $\mu_\theta(I) = \mu(p_\theta^{-1} I)$ for any interval $I \subset \mathbb{R}$) then

$$\mu_\theta([a,b]) = \mu(A \cap p_\theta^{-1}([a,b])) = \mu\{\underline{x} \in A : a \leq \underline{x} \cdot \underline{\theta} \leq b\}.$$

For any particular value of θ, to show that we have that $\dim_H(p_\theta A) > t$ it sufficesto show that

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\mathrm{d}\mu_\theta(u)\mathrm{d}\mu_\theta(v)}{|u-v|^t} < \infty \tag{6.1}$$

and apply part (2) of the second Lemma to μ_θ. Therefore, if we can show that

$$I := \int_0^\pi \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\mathrm{d}\mu_\theta(u)\mathrm{d}\mu_\theta(v)}{|u-v|^t} \right) \mathrm{d}\theta < \infty \tag{6.2}$$

then by Fubini's Theorem we have for almost all θ the inner integral (6.2) is finite, i.e., (6.1) holds for a.e. θ as required.

It now remains to establish (6.2). From the definition of μ_θ we can rewrite this as

$$I = \int_0^\pi \int_A \int_A \frac{\mathrm{d}\mu(x)\mathrm{d}\mu(y)\mathrm{d}\theta}{|x \cdot \underline{\theta} - y \cdot \underline{\theta}|^t} = \left(\int_0^\pi \frac{\mathrm{d}\theta}{|\underline{\theta} \cdot \underline{\tau}|^t} \right) \int_A \int_A \frac{\mathrm{d}\mu(x)\mathrm{d}\mu(y)}{|x-y|^t}$$

and we know by (6.0) that the second part of this last term is finite. Thus it only remains to show that,

$$\int_0^\pi \frac{\mathrm{d}\theta}{|\underline{\theta} \cdot \underline{\tau}|^t} < \infty. \tag{6.3}$$

We can rewrite this last integral as

$$\int_0^\pi \frac{\mathrm{d}\theta}{|\underline{\tau} \cdot \underline{\theta}|^t} = \int_0^\pi \frac{\mathrm{d}\theta}{|\cos(\tau - \theta)|^t}.$$

Moreover, the derivative of $\cos(\tau - \theta)$ is bounded away from 0 when $\cos(\tau - \theta)$ is equal to 0 so when $|\cos(\tau - \theta)|^t$ is close to 0 it can be bounded below by Cx^t for some $C > 0$. Since $t < 1$ this means

$$\int_0^\pi \frac{\mathrm{d}\theta}{|\cos(\tau - \theta)|^t} < \infty.$$

Thus $I < \infty$ for any $t < s$ and so the proof is complete.

We turn to the proof of part (2). Let $d = \dim(K) > 1$. Let us first assume that $0 < m_d(K) < +\infty$ and there exists $C > 0$ such that

$$m_d(K \cap B_r(x)) \leq Cr^d$$

for $x \in K$ and $0 < r \leq r$. We can then define a measure μ on $\mathbb{R}^2$ by $\mu(A) = \mu_d(K \cap A)$, where A is a Borel set.

Let μ_θ be the projection of the measure onto the real line $\mathbb{R}$ such that $\int f d\mu_\theta = \int (f \circ \pi_\theta) d\mu$. It suffices to show that for almost all $\theta \in (-\pi/2, \pi/2)$ the support of μ_θ has positive measure.

Lemma 41 (Riemann-Lebesgue)**.** *Let η be a finite measure on $\mathbb{R}$ with compact support. Let*

$$\widehat{\eta}(p) = \int_{-\infty}^{+\infty} e^{ixp} d\eta(x)$$

be the Fourier transform of the measure. If $0 < \int_{-\infty}^{+\infty} |\widehat{\eta}(p)|^2 dp < +\infty$ then the support of η has positive Lebesgue measure.

Proof of Riemann-Lebesgue Theorem. Since $0 < \int_{-\infty}^{+\infty} |\widehat{\eta}(p)|^2 dp < +\infty$ we have by Plancherel's theorem that $\phi(x) = \int_{-\infty}^{+\infty} e^{ixp}\widehat{\eta}(p)dp$ is well defined, square integrable and $d\eta\phi dx$ and

$$\int_{-\infty}^{+\infty} |\phi(x)|^2 = \int_{-\infty}^{+\infty} |\widehat{\eta}(p)|^2| > 0.$$

The support of ϕ, and thus support of η, cannot have zero Lebesgue measure. $\square$

We return to the proof of Part (2) of the Marstrand Theorem. We want to show that for for almost all $\theta \in (-\pi/2, \pi/2)$ we have that the Fourier transform satisfies

$$|\widehat{\mu}_\theta(p)|^2 = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{i(y-x)p} d\mu_\theta(y) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{i(v-u).v_\theta} d\mu(u)d\mu(v).$$

Since

$$|\widehat{\mu}_\theta(p)|^2 + |\widehat{\mu}_{\theta(p)+\pi}|^2 = \frac{1}{\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \cos(pi(v-u).v_\theta)d\mu(u)d\mu(v)$$

we can integrate over θ to write

$$\int_0^{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{i(v-u).v_\theta} d\mu(u)d\mu(v)\, d\theta = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} . \int_0^{2\pi} \cos(pi(v-u).v_\theta)d\theta d\mu(u)d\mu(v)$$

by Fubini's theorem. Let $J(z) = \frac{1}{2\pi}\int_0^{2\pi} \cos(\cos\theta)d\theta$ and then we can write

$$\int_0^{2\pi} |\widehat{\mu}_\theta(p)|^2 d\theta \int\int J(p\|v-u\|))d\mu(u)d\mu(v).$$

Thus we can write

$$\begin{aligned}
\int_{-a}^{a}\int_0^{2\pi} |\widehat{\mu}_\theta(p)|^2 d\theta dp &\leq \int\int J(p\|v-u\|))d\mu(u)d\mu(v) \\
&= \int\int\int_{-a}^{a} J(p\|v-u\|)d\mu(u)d\mu(v) \\
&= \int\int \left(\int_{a\|u-v\|}^{-a\|u-v\|} J_0(z)dz\right)\frac{1}{\|v-u\|} d\mu(u)d\mu(v).
\end{aligned}$$

Since $\int_{-\infty}^{+\infty} J_0(z)dz < +\infty$ we can write $\int_{-a}^{a} \int_0^{2\pi} |\widehat{\mu}_\theta(p)|^2 d\theta dp \leq C \int\int \frac{1}{\|v-u\|} d\mu(u)d\mu(v)$ uniformly in $a > 0$. It is easy to see the last integral is finite. Let $0 < \alpha < 1$,

$$\begin{aligned}\int \frac{1}{\|v-u\|} d\mu(v) &= \int_{\|u-v\|\geq 1} \frac{1}{\|v-u\|} d\mu(v) + \sum_{n=1}^{\infty} \int_{\alpha^n \leq \|u-v\| \leq \alpha^{n-1}} \frac{1}{\|v-u\|} d\mu(v) \\ &\leq \mu(\mathbb{R}^2) + \sum_{n=1}^{\infty} \alpha^{-n} \mu(B_{\alpha^{n-1}}(u)) \\ &\leq \mu(\mathbb{R}^2) + \frac{C}{\alpha - \alpha^d}\end{aligned}$$

for all $u \in \mathbb{R}^2$. Thus

$$\int\int \frac{1}{\|v-u\|} d\mu(v)d\mu(u) \leq \mu(\mathbb{R}^2) \left(\mu(\mathbb{R}^2) + + \frac{C}{\alpha - \alpha^d} \right) < +\infty.$$

Letting $a \to +\infty$ and using Fubini's theorem we get

$$\int_0^{2\pi} \int_{-\infty}^{\infty} |\widehat{\mu}_\theta(p)|^2 dp d\theta \leq C \int\int \frac{1}{\|v-u\|} d\mu(v)d\mu(u) < +\infty$$

Thus $\int_{-\infty}^{\infty} |\widehat{\mu}_\theta(p)|^2 dp < +\infty$ for almost all $\theta \in (-\pi/2, \pi/2)$.

We claim that $\int_{-\infty}^{\infty} |\widehat{\mu}_\theta(p)|^2 dp > 0$ for all $\theta \in (-\pi/2, \pi/2)$. Otherwise $\int_{-\infty}^{\infty} |\phi(x)|^2 dx = 0$ and then $\phi = 0$ almost everywhere. Since $d\mu_\theta = \phi dx$. But this would imply $\mu_\theta(\mathbb{R}) = int_{-\infty}^{\infty} \phi(x)dx = 0$ and so $\mu(\mathbb{R}^2) = 0$, contradicting the assumption that $m_d(X) > 0$.

In the general case, we can choose $d' < d$ (with $m_{d'}(X) = +\infty$) and then choose $X' \subset X$ with $0 < m_{d'}(X) < +\infty$ (see [**?**]). The above approach applied to X' shows that for almost every θ we have $\pi_\theta(X')$. $\subset \pi_\theta(X)$ has positive Lebesgue measure.

□

Example 50 (Example)**.** *Consider the iterated function scheme in $\mathbb{R}^2$ given by contractions T_1, T_2, T_3 of the form*

$$\begin{aligned}T_1(x,y) &= (x/3, y/3) \\ T_2(x,y) &= (x/3, y/3) + (0,1) \\ T_3(x,y) &= (x/3, y/3) + (1,0)\end{aligned}$$

and let $\Lambda \subset \mathbb{R}^2$ be the associated Limit set. Since the iterated function scheme theorem holds we know that this set has Hausdorff dimension $\dim_H(\Lambda) = 1$.

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For the iterated function scheme T_1, T_2, T_3 we know the Hausdorff Dimension of the limit set (since Moran's Theorem applies). Thus for "typical" λ be know the Hausdorff Dimension of the limit set for S_1, S_2, S_3.

Consider the projection $p_\theta : \mathbb{R}^2 \to \mathbb{R}$ onto the line at an angle θ. The image limit set $p_\theta(\Lambda) \subset \mathbb{R}$ is the limit set for the iterated function scheme on $\mathbb{R}$ given by contractions T_1, T_2, T_3 of the form

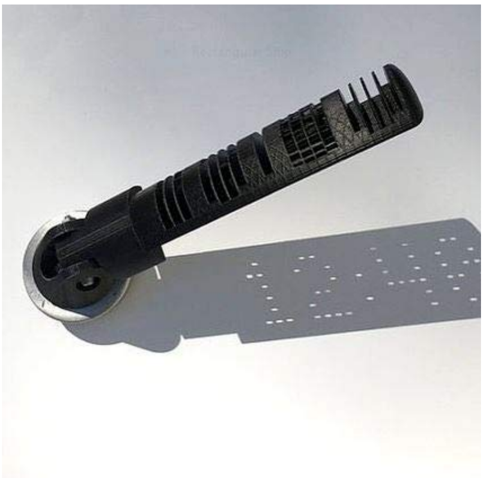
$$
\begin{aligned}
S_1(x) &= x/3 \\
S_2(x) &= x/3 + 1 \\
S_3(x) &= x/3 + \lambda
\end{aligned}
$$

(up to scaling the line by $\cos\theta$) where $\lambda = \tan\theta$ on the real line. Let us denote $\Lambda_\lambda = p_\theta(\Lambda)$.

The open set condition does not apply in this case. However, from Theorem 6.1 we can deduce that for a.e. λ (or equivalently for a.e. θ) we have that $\dim_H(\Lambda) = 1$. Clearly, this cannot be true for all λ. For example, when $\lambda = 0$ then $S_1 = S_2$ and the iterated function scheme has a limit set consisting only of a Cantor set (the middle $(1 - 2\lambda)$ Cantor set) with Hausdorff Dimension $-\log 2/\log\lambda$.

There is a natural generalization to projections $p : \mathbb{R}^n \to \mathbb{R}^m$.

Remark 11 (Fractal Sundial)**.** *Falconer proposed that it would be possible to construct a (more complicated) Fractal set X with the property that the projection in different directions could be prescribed sets. For example, given a three dimension set X one could consider the different projections as shadows from sunlight. As the position of the sun moves during the day so the projection changes. Therefore, a judicious construction of X might lead to shadows which actually display the time, i.e., a digital "fractal" sundial. In*



particular, the mathematical principle here is that given sets $Y_\theta \subset \mathbb{R}$ there exists $X \subset \mathbb{R}^2$ such that $p_\theta(X) = Y_\theta \subset \mathbb{R}$ (up to a set of zero Lebesgue measure).

7.2 The Slice Theorem

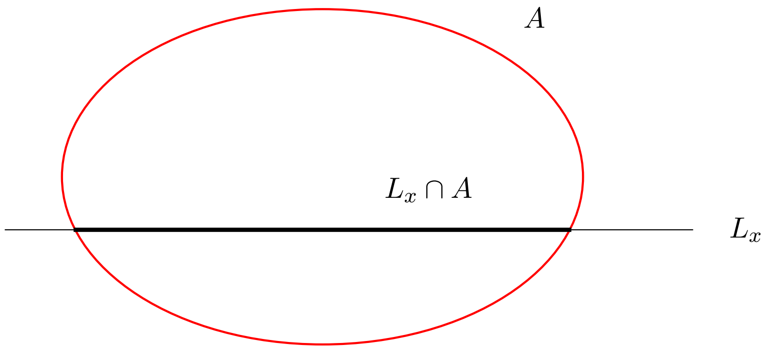
Assume that $A \subset \mathbb{R}^2$ has dimension $\dim_H(A)$. Let

$$L_x = \{(x, y) : y \in \mathbb{R}\}$$

be a vertical line. We can make the following assertion about the dimension of a typical intersection $A \cap L_x$.

The next theorem shows that *if the set is large enough then typical slices have dimensions that drop by at least* 1.

Theorem 12 (Marstrand's Slice Theorem)**.** *Assume that* $\dim_H(A) \geq 1$, *then for almost every* $x \in \mathbb{R}$ *we have that* $\dim_H(A \cap L_x) \leq \dim_H(A) - 1$.



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For a typical vertical slice through a large set A the dimension of the slice drops by at least 1.

We begin with a preliminary result

Lemma 42. *For* $1 \leq \alpha \leq 2$ *we can write*

$$H^\alpha(A) \geq \int H^{\alpha-1}(A \cap L_x)dx$$

Proof. Given $\epsilon, \delta > 0$, let $\{U_i\}$ be an open cover of A with $\operatorname{diam}(U_i) < \epsilon$ and such that

$$\sum_i \operatorname{diam}(U_i) \leq H^\alpha_\epsilon(A) + \delta.$$

We can cover each U_i by a square $I_i \times J_i$ aligned with the axes (whose sides are of length l_i at most the diameter of U_i, i.e., $\operatorname{diam}(U_i) < \epsilon$).

Consider a function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x, y) = \sum_i \chi_{I_i \times J_i}(x, y) l_i^{\alpha-2},$$

where

$$\chi_{I_i \times J_i}(x,y) = \begin{cases} 1 & \text{if } x \in I_i, y \in J_i \\ 0 & \text{otherwise.} \end{cases}$$

The sets $\{L_x \cap (I_i \times J_i)\}$ form a cover for $L_x \cap A$ of diameter $\epsilon > 0$. Thus using this cover we have that

$$H_\epsilon^{\alpha-1}(A \cap L_x) \leq \sum_{\{i \,:\, x \in I_i\}} l_i^{\alpha-1}. \tag{6.3}$$

For a fixed x we have

$$\int_{-\infty}^{\infty} f(x,y)dy = \int_{-\infty}^{\infty} \left(\sum_i \chi_{I_i \times J_i}(x,y) l_i^{\alpha-2} \right) dy = \epsilon \sum_{\{i \,:\, x \in I_i\}} l_i^{\alpha-1}$$

Thus we have that

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y)dxdy = \epsilon \int_{-\infty}^{\infty} \left(\sum_{i \,:\, x \in I_i} l_i^{\alpha-1} \right) dx$$

In particular, by (6.3) we have that

$$\begin{aligned} \int H^{\alpha-1}(A \cap L_x)dx &\leq \int_{-\infty}^{\infty} \left(\sum_{i \,:\, x \in I_i} l_i^{\alpha-1} \right) dx = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y)dxdy \\ &\leq \sum_i l_i^{\alpha-2} l_i^2 = \sum_i l_i^\alpha \\ &\leq H^\alpha(A) + \delta \end{aligned}$$

using that $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y)dxdy = \sum_i \text{Area}(I_i \times J_i) l_i^{\alpha-2}$.

Letting $\delta \to 0$ gives

$$\int H_\epsilon^{\alpha-1}(A \cap L_x)dx \leq H_\epsilon^\alpha(A)$$

Letting $\epsilon \to 0$ gives that $H_\epsilon^{\alpha-1}(A) \nearrow H_\epsilon^{\alpha-1}(A)$ and so

$$\int H^{\alpha-1}(A \cap L_x)dx \leq H^\alpha(A)$$

This completes the proof of the lemma. □

After this preparation, we now have a short proof of the Slice theorem.

Proof of Theorem 12. Let $\alpha > \dim_H(A)$ then by Lemma 42

$$0 = H^\alpha(A) = \int_{-\infty}^{\infty} H^{\alpha-1}(A \cap L_x)dx.$$

Thus, by Fubini's Theorem $H^{\alpha-1}(A \cap L_x) = 0$ for a.e. x. In particular, $\dim_H(A \cap L_x) \leq \alpha - 1$ for such x, as required. □

Example 51. *Fix $\frac{1}{3} < \lambda < \frac{1}{2}$. Consider the iterated function scheme in $\mathbb{R}^2$ given by contractions T_1, T_2, T_3 of the form*

$$\begin{aligned} T_1(x, y) &= (\lambda x, \lambda y) \\ T_2(x, y) &= (\lambda x, \lambda y) + (0, 1) \\ T_3(x, y) &= (\lambda x, y) + (1, 0) \end{aligned}$$

and let $\Lambda \subset \mathbb{R}^2$ be the associated Limit set. Since $\lambda < \frac{1}{2}$ the Open Set Condition holds and by Moran's Theorem we know that the Limit set Λ has Hausdorff dimension $\dim_H(\Lambda) = -\frac{\log 3}{\log \lambda} > 1$. Let us take the vertical slices $L_x \cap \Lambda$ through this limit set.

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The dimension drop on typical slices is strictly greater than 1. The projection onto the x-axis is a middle $(1 - 2\lambda)$ Cantor set X. For $x \in X$ the Haudorff Dimension $\dim_H(L_x \cap \Lambda)$ is in the range $[0, -\frac{\log 2}{\log \lambda}]$. However, X has zero measure. On the complement $\mathbb{R} - X$ we have that $L_x \cap \Lambda = \emptyset$. In particular, $\dim_H(L_x \cap \Lambda) = 0 < \dim_H(\Lambda) - 1$ (a strict inequality).

7.3 A generalized slice theorem

Assume that $A \subset \mathbb{R}^2$ has dimension $\dim_H(A)$. Again, let $L_x = \{(x, y) : y \in \mathbb{R}\}$ be a vertical line. The following relates $\dim_H(A)$ to typical values $\dim_H(A \cap L_x)$ for a typical x, with respect to a more general measure μ.

Theorem 13 (Generalized Marstrand's Slice Theorem))**.** *Let $B \subset \mathbb{R}$. Assume that μ is a probability measure on B and $C > 0$ with $\mu(I) \leq C(\mathrm{diam}(I))^\alpha$, for intervals $I \subset \mathbb{R}$. If $A \subset \mathbb{R}^2$ then*

$$\dim_H(A) \geq \alpha + \dim_H(A \cap L_x)$$

the for almost every $x \in B$ with respect to μ.

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For a typical vertical slice through a large set A (relative to a measure μ on B) the dimension of the slice drops by at least the value α (depending on the measure μ).

Proof. The proof is similar to that of Theorem 6.3. Fix $\gamma > \dim_H(A)$. If we can show that

$$\int H^{\gamma-\alpha}(A \cap L_x) d\mu(x) < +\infty$$

then by Fubini's Theorem $H^{\gamma-\alpha}(A \cap L_x) < +\infty$ for a.e. (μ) x. In particular, $\dim_H(A \cap L_x) \leq \gamma - \alpha$ for a.e. (μ) x, by definition.

We can cover B by squares $I_i \times J_i$ aligned with the axes whose side lengths l_i satisfy $\sum_i l_i^\gamma < \epsilon$. If we define

$$f(x,y) = \sum_i \chi_{I_i \times J_i}(x,y) l_i^{\gamma-\alpha-1}$$

then we can write

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dyd\mu(x) = \sum_i l_i^{\gamma-\alpha-1}\mathrm{diam}(A_i)\mu(B_i) \le C\sum_i l_i^\alpha \le C\epsilon \tag{6.4}$$

We can denote

$$Q_i^x = \begin{cases} J_i & \text{if } x \in I_i \\ \emptyset & \text{otherwise} \end{cases}$$

then these sets form cover of $F \cap L_x$. By Fubini's theorem we can interchange integrals and write

$$\begin{aligned} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dyd\mu(x) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dyd\mu(x) \\ &= \int \left(\sum_i \mathrm{diam}\ (Q_i^x)^{\gamma-\alpha}\right) d\mu(x) \qquad (6.5) \\ &\ge \int H_\epsilon^{\gamma-\alpha}(L_x \cap F)d\mu(x). \end{aligned}$$

Thus by (6.4) and (6.5):

$$0 \le \int H_\epsilon^{\gamma-\alpha}(L_x \cap F)d\mu(x) \le C\epsilon$$

Finally, letting $\delta \to 0$ gives

$$\int H_\epsilon^{\gamma-\alpha}(F \cap L_x)dx \le H_\epsilon^{\gamma-\alpha}(F),$$

and then letting $\epsilon \to 0$ gives

$$\int H^{\gamma-\alpha}(F \cap L_x)dx = 0.$$

Thus Fubini's Theorem gives that the integrand is finite almost everywhere, i.e., $H^{\gamma-\alpha}(F \cap L_x) = 0$ for a.e. (μ) x. In particular, $\dim_H(A \cap L_x) \le \gamma - \alpha$ for a.e. (μ) x. Since γ can be chosen arbitrarily close to $dim_H(A)$ this completes the proof. □ □

The slicing theorems generalize to k-dimensional slices of sets in R^n.

A popular way to get one fractal from another is to drop down to a lower dimension, either by projecting or slicing. In the interests of clarity of exposition we will concentrate on the case of two dimensions and one dimension.

7.4 Application

Example 52. *Let X be the gasket (with 3 squares in a 2×2 grid). Almost every vertical slice X_x has dimension $\dim(X_x) = \frac{1}{2} < \dim(X) - 1 = \frac{\log 8}{\log 3} - 1$.*

For almost every $x \in [0,1]$ we can consider the binary expansion $x = \sum_{n=1}^{\infty} \frac{x_n}{2^n}$ Then for almost all x the frequency with which $x_n = 1$ equals $\frac{1}{2}$. Moreover, such points ????????????????????????

Example 53. *Let X be the gasket (with 3 squares in a 2×3 grid). The vertical sets X_x consist of a single point.*

On the other hand, the horizonal projection corresponding to $y = \sum_{n=1}^{\infty} \frac{y_n}{3^n}$ with $y_n \in \{0,1,2\}$ The dimension of the intersection is

$$(\log_3 2) \liminf_N \frac{1}{N} \sum_{n=1}^{N} (1 - y_n)$$

which is $\frac{1}{2}$ for almost every y.

For almost every $x \in [0,1]$ we can consider the binary expansion $x = \sum_{n=1}^{\infty} \frac{x_n}{2^n}$ $0 < m < d$. Then for almost all x the frequency with which $x_n = 1$ equals $\frac{1}{2}$. Moreover, such points ????????????????????????

A higher dimensional generalization is the following

Theorem 14. *Let $0 < m < d$. Let $A \subset \mathbb{R}^d$ such that $\operatorname{diam}(A) > d - m$ and let E be an m-dimensional subspace. Then for almost every $x \in E^{\perp}$ we have that*

$$\dim(A \cap (E + x)) \leq \dim(A) - (d - m).$$

Exercise 9. *Show that the proof of ??? generalizaes to prove Theorem ???.*

Example 54. *Fix $\frac{1}{3} < \lambda < \frac{1}{2}$. Consider the iterated function scheme in $\mathbb{R}^2$ given by contractions T_1, T_2, T_3 of the form*

$$\begin{aligned} T_1(x,y) &= (\lambda x, \lambda y) \\ T_2(x,y) &= (\lambda x, \lambda y) + (0,1) \\ T_3(x,y) &= (\lambda x, y) + (1,0) \end{aligned}$$

and let $\Lambda \subset \mathbb{R}^2$ be the associated Limit set. Since $\lambda < \frac{1}{2}$ the Open Set Condition holds and by Moran's Theorem we know that the Limit set Λ has Hausdorff dimension $\dim_H(\Lambda) = -\frac{\log 3}{\log \lambda} > 1$. Let us take the vertical slices $L_x \cap \Lambda$ through this limit set.

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The dimension drop on typical slices is strictly greater than 1. The projection onto the x-axis is a middle $(1 - 2\lambda)$ Cantor set X. For $x \in X$ the Haudorff Dimension $\dim_H(L_x \cap \Lambda)$ is in the range $[0, -\frac{\log 2}{\log \lambda}]$. However, X has zero measure. On the complement $\mathbb{R} - X$ we have that $L_x \cap \Lambda = \emptyset$. In particular, $\dim_H(L_x \cap \Lambda) = 0 < \dim_H(\Lambda) - 1$ (a strict inequality).

Assume that $A \subset \mathbb{R}^2$ has dimension $\dim_H(A)$. Again, let $L_x = \{(x,y) : y \in \mathbb{R}\}$ be a vertical line. The following relates $\dim_H(A)$ to typical values $\dim_H(A \cap L_x)$ for a typical x, with respect to a more general measure μ.

Theorem 6.5 (Generalized Marstrand's Slice Theorem) Let $B \subset \mathbb{R}$. Assume that μ is a probability measure on B and $C > 0$ with $\mu(I) \leq C(\mathrm{diam}(I))^\alpha$, for intervals $I \subset \mathbb{R}$. If $A \subset \mathbb{R}^2$ then

$$\dim_H(A) \geq \alpha + \dim_H(A \cap L_x)$$

the for almost every $x \in B$ with respect to μ.

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For a typical vertical slice through a large set A (relative to a measure μ on B) the dimension of the slice drops by at least the value α (depending on the measure μ).

Proof. The proof is similar to that of Theorem 6.3. Fix $\gamma > \dim_H(A)$. If we can show that

$$\int H^{\gamma-\alpha}(A \cap L_x) d\mu(x) < +\infty$$

then by Fubini's Theorem $H^{\gamma-\alpha}(A \cap L_x) < +\infty$ for a.e. (μ) x. In particular, $\dim_H(A \cap L_x) \leq \gamma - \alpha$ for a.e. (μ) x, by definition.

We can cover B by squares $I_i \times J_i$ aligned with the axes whose side lengths l_i satisfy $\sum_i l_i^\gamma < \epsilon$. If we define

$$f(x,y) = \sum_i \chi_{I_i \times J_i}(x,y) l_i^{\gamma-\alpha-1}$$

then we can write

$$\begin{aligned} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y) dy d\mu(x) &= \sum_i l_i^{\gamma-\alpha-1} \mathrm{diam}(A_i) \mu(B_i) \\ &\leq C \sum_i l_i^\alpha \leq C\epsilon \end{aligned} \tag{6.4}$$

We can denote

$$Q_i^x = \begin{cases} J_i & \text{if } x \in I_i \\ \emptyset & \text{otherwise} \end{cases}$$

then these sets form cover of $F \cap L_x$. By Fubini's theorem we can interchange integrals and write

$$\begin{aligned} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y) dy d\mu(x) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y) dy d\mu(x) \\ &= \int \left(\sum_i \mathrm{diam}\ (Q_i^x)^{\gamma-\alpha} \right) d\mu(x) \\ &\geq \int H_\epsilon^{\gamma-\alpha}(L_x \cap F) d\mu(x). \end{aligned} \tag{6.5}$$

Thus by (6.4) and (6.5):

$$0 \le \int H_\epsilon^{\gamma-\alpha}(L_x \cap F)d\mu(x) \le C\epsilon$$

Finally, letting $\delta \to 0$ gives

$$\int H_\epsilon^{\gamma-\alpha}(F \cap L_x)dx \le H_\epsilon^{\gamma-\alpha}(F),$$

and then letting $\epsilon \to 0$ gives

$$\int H^{\gamma-\alpha}(F \cap L_x)dx = 0.$$

Thus Fubini's Theorem gives that the integrand is finite almost everywhere, i.e., $H^{\gamma-\alpha}(F \cap L_x) = 0$ for a.e. (μ) x. In particular, $\dim_H(A \cap L_x) \le \gamma - \alpha$ for a.e. (μ) x. Since γ can be chosen arbitrarily close to $dim_H(A)$ this completes the proof. □ □

The slicing theorems generalize to k-dimensional slices of sets in R^n.

7.5 Slices

Let $A \subset \mathbb{R}^2$ be a Borel set. We can consider the one dimensional vertical slices

$$A_x = \{h : (x, y) \in A\} \text{ for } x \in A$$

We can formulate the following classic slice theorem.

Theorem 15. *Asume that $A \subset \mathbb{R}^2$ has $\dim_H(A) \ge 1$. Then $\dim_H(A_x) \le \dim_H(A) - 1$ for almost every $x \in \mathbb{R}$ with respect to Lebesgue measure.*

If $\dim_H(A) < 1$ then $A_x = \emptyset$ for almost all x (in fact, except on a set of dimension at most $\dim(A)$)

Proof. We begin with the following

Claim. For $1 \le \alpha \le 2$

$$H^\alpha(A) \ge \int H^{\alpha-1}(A_x)dx$$

Assuming this claim, we can choose $\alpha > \dim_H(A)$ and then by the claim

$$0 = H^\alpha(A) \ge \int H^{\alpha-1}(A_x)dx$$

and the result follows since $H^{\alpha-1}(A_x) = 0$ for almost every x, and thus $\dim_H(A_x) \le \alpha - 1$, for almost every x.

Proof of claim. Fix $\epsilon, \delta > 0$ and let $\{U_j\}$ be a cover for A with $\mathrm{diam}(U_j) < \epsilon$ and such that

$$\sum_j (\mathrm{diam}(U_j))^\alpha \leq H_\epsilon^\alpha(A) + \delta.R$$

For each open set U_j we can choose a square $U_j \subset S_j$ with sides of length $\mathrm{diam}(U_j)$, aligned with the axes of $\mathbb{R}^2$.

Let $I_j \subset \mathbb{R}^2$ be the vertical projection onto the horizontal axis and define a function $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x,y) = \sum_j \chi_{S_j}(x,y)(\mathrm{diam}(U_j))^{\alpha-2}.$$

For each x the slices $S_{j,x} := S_j \cap \{(x,y) : y \in \mathbb{R}\}$ form a cover for the slices A_x and have length

$$\lambda(S_{j,x}) = \begin{cases} \mathrm{diam}(U_j) & x \in I_j \\ 0 & x \notin I_j \end{cases}$$

Using this cover for A_x we can bound

$$H_\epsilon^{\alpha-1}(A_x) \leq \sum_j \lambda(S_{j,x})^{\alpha-1} \leq \sum_{j\,:\,x\in I_j} (\mathrm{diam}(U_j))^{\alpha-1}.$$

If we fix x then

$$\int_{\mathbb{R}} \sum_{j\,:\,x\in I_j} \chi_{S_j}(x,y)(\mathrm{diam}(U_j))^{\alpha-2} dxdy = \sum_{j\,:\,x\in I_j} (\mathrm{diam}(U_j))^{\alpha-2}$$

which implies

$$\int\int f(x,y)dxdy = \int_{\mathbb{R}} \sum_j (\mathrm{diam}(U_j))^{\alpha-1}.$$

Therefore,

$$\begin{aligned} \inf H_\epsilon^{\alpha-1}(A+x)dx &\leq \int_{\mathbb{R}} \left(\sum_j \mathrm{diam}(U_j))^{\alpha-1} \right) dx \\ &= \int\int f(x,y)dxdy \\ &= \sum_j \mathrm{diam}(U_j))^{\alpha-2}\mathrm{diam}(U_j))^2 \\ &= \sum_j \mathrm{diam}(U_j))^\alpha \leq H_\epsilon^\alpha(A_x) + \delta. \end{aligned}$$

Letting $\delta \to 0$ gives

$$\int H_\epsilon^{\alpha-1}(A_x)dx \leq H_\epsilon^\alpha(A).$$

Finally, as $\epsilon \to 0$ we have that $H_\epsilon^{\alpha-1}(A_x) \to H^{\alpha-1}(A_x)$ and so [3] we have that

$$\int H^{\alpha-1}(A_x)dx \leq H^\alpha(A).$$

as required.

□

7.6 Differences of Cantor sets: Hausdorff Dimension and positive measure

Let $X, Y \subset \mathbb{R}$ then we define the difference

$$X - Y = \{t \in \mathbb{R} : \exists x \in X, y \in Y \text{ such that } x - y = t\}.$$

As a corollary to the projection theorem we have the following result on the difference of Cantor sets.

Theorem 16. *Let $X, Y \subset \mathbb{R}$ be Cantor sets.*

1. *If $\dim_H(X) + \dim_H(Y) < 1$ then for almost all $\lambda > 0$ the set*

$$X - \lambda Y = \{x - \lambda y : x, y \in X, Y\} \subset \mathbb{R}$$

has Hausdorff Dimension $\dim_H X + \dim_H Y$.

2. *If $\dim_H(X) + \dim_H(Y) > 1$ then for almost all $\lambda > 0$ the set $X - \lambda Y$ has positive Lebesgue measure.*

Proof. We can consider the product space $X \times Y$ which has Hausdorff dimension $\dim_H(X) + dim_H(Y)$. We can now consider the projection $\pi_\theta : X \times Y \to \mathbb{R}$ in the direction with angle θ with $\tan(\theta) = \lambda$. In particular, $\pi_\theta(x, y) = x\cos\theta - y\sin\theta$. In particular, assuming $\cos\theta \neq 0$ we can divide by $\cos\theta$ to get $x - \lambda y$.

By the projection theorem we have that if $\dim_H(X) + \dim_H(Y) < 1$ then $\dim_H(X - \lambda Y) = \dim_H(X \times Y) = \dim_H(X) + \dim_H(Y)$ for almost all directions θ, which is equivalent to almost all λ. On the other hand, if $\dim_H(X) + \dim_H(Y) > 1$ then $X - \lambda Y$ and $X \times Y$ has positive Lebesgue measure for almost all λ. □

Proposition 14. *If $dim_B(X) + dim_B(Y) < 1$ then $dim_B(X - Y) < 1$ and $X - Y$ has zero Lebesgue measure.*

[3] By the Monotone Convergence Theorem

Proof. We can choose $\dim_B(X) < d_1$ and $\dim_B(Y) < d_2$ with $d_1 + d_2 < 1$. There exists $\epsilon_0 > 0$ such that for any $0 < \epsilon < \epsilon_0$ we can cover X by $[\epsilon^{-d_1}]$ intervals of length ϵ and we can cover Y by $[\epsilon^{-d_2}]$ intervals of length ϵ. If I and J are intervals of length ϵ then $X - Y$ has length 2ϵ. Thus $X - Y$ is covered by $\epsilon^{-(d_1+d_2)}$ thus $N_{2\epsilon}(X - Y) \leq \epsilon^{-(d_1+d_2)}$. □

Lemma 43. *Let $X, Y \subset \mathbb{R}$ be Cantor sets with thickness τ_1 and τ_2 with $\tau_1.\tau_2 > 1$ then one of the following occurs:*

1. *X is contained in a gap of Y*
2. *Y is contained in a gap of X*
3. *$X \cap Y \neq \emptyset$*

We have the following version of the Marstrand projection theorem.

Theorem 17. *Let $X, Y \subset \mathbb{R}$ with $dim_B(X) + dim_B(Y) > 1$ then $X - \lambda Y$ has positive Lebesgue measure for almost every $\lambda \in \mathbb{R}$.*

It is possible to show that for dynamically defined limit sets:

Theorem 18. *For a dynamically defined Cantor set X we have* $\dim_H(X) = \dim_B(X)$.

Theorem 19. *Let X be a limit set and $d = \dim_H(X)$ then $0 < m_d(X)$. Moreover, for all $x \in X$ and $0 < r \leq 1$ and*

$$\frac{1}{c} \leq \frac{m_d(B_r(x))}{r^d} \leq c$$

This also holds in two dimensions.

Lemma 44. *For dynamically defined Cantor sets the Hausdorff dimension and thickness depend continuously on the contractions.*

7.7 Sums of Continued fraction cantor sets

We can consider dynamically defined Cantor sets given by finitely many branches of the Gauss maps, i.e., let $S \subset \mathbb{N}$ be a finite set and then let $T_i : [0, 1] \to [0, 1]$ by $T_i(x) = \frac{1}{x+i}$ for $i \in S$. Let X_S be the limit set for this iterated function scheme. In the next result we can take $S = \{1, 2, 3, 4\}$.

Theorem 20. *For $S = \{1, 2, 3, 4\}$ we have that $X_S + X_S = (\sqrt{2}-1, 4\sqrt{2}-4)$. In other words, every number in the interval $(\sqrt{2} - 1, 4\sqrt{2} - 4)$ is the sum of two continued fractions whose coefficients do not exceed 4.*

As it is explained in Cusick-Flahive book (cf. the first two lines of the proof of Theorem 1 in Chapter 6), $M \cap [\sqrt{5}, \sqrt{10}) \subset U + U$, where U is the set of continued fractions with 1 and 2 in which 121 never occur.

Lemma 45. $\dim(U+U) \leq 2\dim(U) < 0.93$

Cusick and Flahive explain (still in the proof of Theorem 1 of Chapter 6) Hall's theorem from 1971 using an explicit description of the structure of U ultimately leading him to the fact that $U+U$ has zero Lebesgue measure. In fact, this gives an (implicit) upper bound on the dimension of $U+U$ (and, a fortiori, on the dimension of $M\cap[\sqrt{5},\sqrt{10})$) along the following lines. Among several estimates, Cusick and Flahive mention that Hall noticed that U is a Cantor set obtained by a subdivision process where each interval I of a given stage is decomposed into four intervals $I(11)$, $I(12)$, $I(21)$, $I(22)$ such that either

a) $\|II(11)\| < 0.15$, $\|I(12)\| < 0.015$, $\|I(21)\| < 0.013$, $\|I(22)\| < 0.007$; or

b) $\|I(11)\| < 0.131$, $\|I(12)\| < 0.013$, $\|I(21)\| < 0.059$, $\|I(22)\| < 0.003$

From this fact, Hall showed that $U+U$ has zero Lebesgue measure, but Hall morally got an upper bound on dimension because it is not hard to see that $\dim(U) < s$ for any s such that there is $A < 1$ with

$$\|I(11)\|^s + \|I(12)\|^s + \|I(21)\|^s + \|I(22)\|^s < A\|^s$$

for all I. Since

$(0.15)^(0.465) + (0.015)^(0.465) + (0.013)^(0.465) + (0.007)^(0.465) < 0.79$; and

$(0.131)^(0.465) + (0.013)^(0.465) + (0.059)^(0.465) + (0.003)^(0.465) < 0.986$

we derive that $\dim(U) < 0.465$ and, a fortiori, $\dim(U+U) \leq 2dim(U) < 0.93$.

Chapter 8

Falconer's Theorem

The situation of estimating the dimension of non-conformal maps, or maps whose images have overlaps, can be quite challenging. However, there are some approaches to this which work for "typical points"

8.1 Affine contractions of the line with overlaps

Assume that we have a finite set of affine contractions $T_i : \mathbb{R} \to \mathbb{R}$ of the intervals of the form $T_i(x) = a_i x + b_i$ $(i = 1, \cdots, k)$ for $0 \leq x \leq 1$. *However, we will not necessarily assume the open set condition.*

Let us fix $0 < a_1, \cdots, a_d < 1$ but consider the d-tuple $\underline{b} = (b_1, \cdots, b_d)$ within $\mathbb{R}^d$. Let $X_{\underline{b}}$ be the associated attractor, i.e., the smallest closed non-empty set such that $\cup_{i=1}^n T_i X = X$. The following theorem looks a little like Moran's theorem, except that we don't assume the Open Set condition, but instead we have a conclusion that only holds for typical maps (corresponding to almost all allowed $\underline{b}$ with respect to d-dimensional Lebesgue measure).

Theorem 21 (after Falconer)**.** *For almost all $(b_1, \cdots, b_k) \in \mathbb{R}^k$ the Hausdorff dimension and Box dimension of X coincide (i.e., $\dim_H(X) = \dim_B(X)$). Moreover, their common value d is then the unique solution to*

$$a_1^d + \cdots + a_k^d = 1.$$

Proof. The upper bound follows from the part of the proof of Moran's theorem that $\overline{\dim}_B(X) \leq d$. It remains to show that $\dim_H(X) \geq d$ to complete the present proof.

Fix $\epsilon > 0$. We recall that in order to show that $\dim_H(X) \geq d - \epsilon$, say, it suffices to show that there exists a probability measure d on X such that

$$\int\int \frac{d\mu(x)d\mu(y)}{|x-y|^{d-\epsilon}} < +\infty.$$

This implies that

$$\dim_H(X_{\underline{b}}) \geq d - \epsilon$$

To achieve this we can associate the space of sequences $\Sigma = \{1, \cdots, d\}^{\mathbb{N}}$. There is then a natural well defined map $\pi : \Sigma \to B_{\underline{b}}$ given by

$$\pi_{\underline{b}}(\underline{x}) = \lim_{n \to +\infty} T_{x_1} T_{x_2} \cdots T_{x_n}(0). = \sum_{n=0}^{\infty} b_{x_n} \lambda_{x_1} \lambda_{x_2} \cdots \lambda_{x_n}$$

which is easily seen to converge since

$$0 \leq a_{x_1} a_{x_2} \cdots a_{x_n} \leq c^n.$$

We can then define a Bernoulli measure ν on Σ associated to the probability vector $(p_1, \cdots, p_d)$ where we make the choices $p_i = a_i^d$ for $1 \leq i \leq k$. We can then push the measure down to $X_{\underline{b}}$ to the probability measure $\mu_{\underline{b}} = \nu \pi^{-1}$ on $X_{\underline{b}}$, i.e., $\mu_{\underline{b}}(B) = \nu(\pi_{\underline{b}}^{-1} B)$ for any Borel set $B \subset X_{\underline{b}}$.

For any $R > 0$ we can consider the integral over the box $[-R, R]^k$ and then hope to show that

$$\int_{\underline{b} \in [-R,R]^k} \left(\int \int \frac{d\mu_{\underline{b}}(x) d\mu_{\underline{b}}(y)}{|x - y|^{d-\epsilon}} \right) d\underline{b} < +\infty.$$

In particular, this implies that for almost all $\underline{b} \in [-R, R]^k$ (with respect to the usual Lebesgue measure)

$$\int \int \frac{d\mu_{\underline{b}}(x) d\mu_{\underline{b}}(y)}{|x - y|^{d-\epsilon}} < +\infty.$$

This would imply that $\dim_H(X_{\underline{b}}) \geq d - \epsilon$ for almost all $\underline{b} \in [-R, R]^k$ (with respect to the usual Lebesgue measure).

Returning to the double integral, we can use the definition of $\mu_{\underline{b}}$ to write the inner integral as

$$\int \int \frac{d\mu_{\underline{b}}(x) d\mu_{\underline{b}}(y)}{|x - y|^{d-\epsilon}} = \int_{\Sigma} \int_{\Sigma} \frac{d\nu(\underline{i}) d\nu(\underline{j})}{|\pi_b(\underline{i}) - \pi_b(\underline{j})|^{d-\epsilon}}$$

where $\underline{i}, \underline{j} \in \Sigma$. We can substitute this expression into the double integral and then switch the order of integration (formally using Fubini's lemma) to write

$$\begin{aligned}\int_{\underline{b} \in [-R,R]^k} \left(\int \int \frac{d\mu_{\underline{b}}(x) d\mu_{\underline{b}}(y)}{|x - y|^{d-\epsilon}} \right) d\underline{b} &= \int_{\underline{b} \in [-R,R]^k} \left(\int_{\Sigma} \int_{\Sigma} \frac{d\nu(\underline{i}) d\nu(\underline{j})}{|\pi_b(\underline{i}) - \pi_b(\underline{j})|^{d-\epsilon}} \right) d\underline{b} \\ &= \int_{\Sigma} \int_{\Sigma} \left(\int_{\underline{b} \in [-R,R]^k} \frac{d\underline{b}}{|\pi_b(\underline{i}) - \pi_b(\underline{j})|^{d-\epsilon}} \right) d\nu(\underline{i}) d\nu(\underline{j}).\end{aligned}$$

In summary, since $R > 0$ and $\epsilon > 0$ can be chosen arbirarily it only remains to show that this final double integral is finite to complete the proof. To this end, given $\underline{j} \in \Sigma$ and $m \geq 1$ we can partition

$$\Sigma = \cup_{m=0}^{\infty} \Delta_m(\underline{j}) \cup \{j\}$$

where

$$\Delta_m(\underline{j}) = \{\underline{i} \in \Sigma : i_r = j_r \text{ for } 1 \leq r \leq m \text{ but } i_{m+1} \neq j_{m+1}\}$$

for which we have by definition $\mu(\Delta_m(\underline{j})) = (a_{j_1} \cdots a_{j_n})^d$

We can now observe that for $\underline{i} \in \Delta_m(\underline{j})$ that

$$\begin{aligned}\pi_b(\underline{i}) - \pi_b(\underline{j}) &= \sum_{n=m}^{\infty} (b_{i_n} a_{i_1} a_{i_2} \cdots a_{i_n} - b_{j_n} a_{j_1} a_{j_2} \cdots a_{j_n}) \\ &= a_{i_1} a_{i_2} \cdots a_{i_n} \left(b_{i_{n+1}} - b_{j_{n+1}} + E(a_1, \cdots, a_d) \right)\end{aligned}$$

where for $|c| < \frac{1}{3}$ we have that the linear map $E : \mathbb{R}^d \to \mathbb{R}$ has norm $\|E\| \leq \frac{2c}{1-c} < 1$. [1] This is the crucial observation in the proof. It has echos later in the idea of "transversality"

Changing variables to $y = b_{i_{n+1}} - b_{j_{n+1}} + E(a_1, \cdots, a_d)$ for fixed $\underline{i} \in \Delta_m(\underline{j})$ we can bound

$$\left(\int_{\underline{b} \in [-R,R]^k} \frac{d\underline{b}}{|\pi_b(\underline{i}) - \pi_b(\underline{j})|^{d-\epsilon}} \right) \leq \frac{C}{(a_{j_1} a_{j_2} \cdots a_{j_n})^{d-\epsilon}}$$

for some constant $C > 0$.

We can now bound the double integral as

$$\begin{aligned}&\int_\Sigma \int_\Sigma \left(\int_{\underline{b} \in [-R,R]^k} \frac{d\underline{b}}{|\pi_b(\underline{i}) - \pi_b(\underline{j})|^{d-\epsilon}} \right) d\nu(\underline{i}) d\nu(\underline{j}) \\ &\leq \int_\Sigma \left(\sum_{n=0}^{\infty} \int_{\Delta_n(\underline{j})} \frac{C}{(a_{j_1} a_{j_2} \cdots a_{j_n})^{d-\epsilon}} \right) d\nu(\underline{j}) \\ &\leq \int_\Sigma \left(\sum_{n=0}^{\infty} \mu(\Delta_n(\underline{j})) \frac{C}{(a_{j_1} a_{j_2} \cdots a_{j_n})^{d-\epsilon}} d\nu(\underline{i}) \right) d\nu(\underline{j}) \\ &\leq \sum_{n=0}^{\infty} (a_{j_1} a_{j_2} \cdots a_{j_n})^d \frac{C}{(a_{j_1} a_{j_2} \cdots a_{j_n})^{d-\epsilon}} \\ &\leq C \sum_{n=0}^{\infty} c^{\epsilon n} = \frac{C}{1 - c^\epsilon} < +\infty\end{aligned}$$

as required.

□

[1] As observed by Peres, this condition can be relaxed to $c\frac{1}{2}$.

Chapter 9

Transversality

9.1 The $\{0, 1, 3\}$-Problem

A similar technique to the one outlined in the previous section can be used to help calculate the dimension of certain self-similar sets where the similarities overlap. Let $F = \{f_0, f_1, f_2\}$ be an IFS on where,

$$\begin{aligned} f_0(x) &= \lambda x \\ f_1(x) &= \lambda x + 1 \\ f_2(x) &= \lambda x + 3. \end{aligned}$$

For $\lambda \leq \frac{1}{4}$ the Open Set Condition applies and the Hausdorff dimension of the attractor $\Lambda(\lambda)$ is thus $-\frac{log 3}{\log \lambda}$. When $\lambda \in (\frac{1}{4}, \frac{1}{3})$ the OSC can not be applied and the problem of whether $\dim \Lambda(\lambda) = -\frac{\log 3}{\log \lambda}$ is still not fully solved. A generic solution was found by Pollicott and Simon in [**?**].

Theorem 22 (Pollicott-Simon, 1994)**.** *For almost all* $\lambda \in (\frac{1}{4}, \frac{1}{3}]$,

$$\dim \Lambda(\lambda) = -\frac{\log 3}{\lambda}.$$

The method of proof is extremely similar to that of the projection theorem. Let μ be $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$-Bernoulli measure on Σ_3. A projection $\Pi_\lambda : \Sigma_3 \to$ can be defined by,

$$\Pi_\lambda(\underline{i}) = \sum_{k=0}^{\infty} i_k \lambda^k.$$

Thus on each possible attractor $\Lambda(\lambda)$ a self-similar measure ν_λ can be defined by $\nu_\lambda = \mu \circ \Pi_\lambda^{-1}$. Let $\epsilon > 0$ and $s_\epsilon(\lambda) = -\frac{\log 3}{\log(\lambda+\epsilon)}$. Note that the proof is complete, using Lemma **??** if it can be shown that,

$$I = \int_{\frac{1}{4}}^{\frac{1}{3}} \iint \frac{\mathrm{d}\nu_\lambda(x)\mathrm{d}\nu_\lambda(y)\mathrm{d}\lambda}{|x-y|^{s_\epsilon(\lambda)}} < \infty$$

for all $\epsilon > 0$. Using the projection Π_λ the inner two integrals can be transferred to Σ_3.

$$I = \int_{\frac{1}{4}}^{\frac{1}{3}} \iint \frac{\mathrm{d}\mu(\underline{i})\mathrm{d}\mu(\underline{j})\mathrm{d}\lambda}{|\Pi_\lambda(\underline{i}) - \Pi_\lambda(\underline{j})|^{s_\epsilon(\lambda)}}.$$

We then turn I into a product of two expressions using Fubini's theorem. We let $t = \max_\lambda s_\epsilon(\lambda)$ and note that $t < 1$. It can be seen that,

$$\begin{aligned} |\Pi_\lambda(\underline{i}) - \Pi_\lambda(\underline{j})|^{s_\epsilon(\lambda)} &= \lambda^{|\underline{i}\wedge\underline{j}|s_\epsilon(\lambda)} \left(\sum_{k=0}^{\infty} a_k\lambda^k\right)^{s_\epsilon(\lambda)} \\ &\geq \left(\frac{1}{3}+\epsilon\right)^{s_\epsilon(\lambda)|\underline{i}\wedge\underline{j}} \left(\sum_{k=0}^{\infty} a_k\lambda^k\right)^{t} \end{aligned}$$

where $\{a_k\}_{k\in_0}$ is a sequence such that $a_k \in \{0, \pm1, \pm2, \pm3\}$ and $a_0 \neq 0$. Substituting this back into I and using Fubini's Theorem we get

$$I \leq \int \frac{\mathrm{d}\lambda}{\left(\sum_{k=0}^{\infty} a_k\lambda^k\right)^t} \int_{\Sigma_3}\int_{\Sigma_3} \frac{\mathrm{d}\mu(\underline{i})\mathrm{d}\mu(\underline{j})}{\left(\frac{1}{3}+\epsilon\right)^{|\underline{i}\wedge\underline{j}|}}.$$

By simple integration on Σ_3 it can be seen that,

$$\begin{aligned} \int\int \frac{\mathrm{d}\mu(\underline{i})\mathrm{d}\mu(\underline{j})}{\left(\frac{1}{3}+\epsilon\right)^{|\underline{i}\wedge\underline{j}|}} &\leq \sum_{k=0}^{\infty} \sum_{[i_0,i_1,\ldots,i_{k-1}]} \frac{\mu([i_0,i_1,\ldots,i_{k-1}])^2}{\left(\frac{1}{3}+\epsilon\right)^k} \\ &= \sum_{k=0}^{\infty} \frac{\frac{1}{3}^{k+1}}{\left(\frac{1}{3}+\epsilon\right)^k} < \infty \end{aligned}$$

Thus to show that $I < \infty$ it remains to show that,

$$\int \frac{\mathrm{d}\lambda}{\left(\sum_{k=0}^{\infty} a_k\lambda^k\right)^t} < \infty$$

for any sequence $\{a_k\}_{k\in_0}$ where each $a_k \in \{0, \pm1, \pm2, \pm3\}$ and $a_0 = 0$. Let $g(\lambda)$ be a power series of that form. In Lemma 1 of [**?**] it is shown that whenever such a power series $g(\lambda)$ is close to 0 its derivative is bounded away from 0. Thus a transversality condition is satisfied. The integral can be shown to be finite by splitting it into two parts, one part where $g(\lambda)$ is bounded away from 0 and one where $g(\lambda)$ is close to zero but the derivative is bounded away from 0. The first part is clearly finite and the second part is finite because $t < 1$ and $g(\lambda)$ can be bounded below by linear functions in this region. This method will be used again in chapters 4 and 5.

In [**?**] a general result about when specific power series satisfy a transversality condition is given. Let

$$F_b = \{f(\lambda) = \sum_{k=0}^{\infty} f_k\lambda^K : g_k \in [-b, b]\}.$$

As in [?] we now define,

$$y(b)=\min\{x>0:\exists f\in F_b \text{ where } f(x)=f'(x)\}.$$

Theorem 23 (Peres-Solomyak,1996)**.** *The function* $y(b):[1,\infty)\to[0,1]$ *is strictly decreasing, continuous and piecewise algebraic.* $y(1)\approx 0.649$, $y(2)=0.5$ *and* $y(b)\geq(\sqrt{b}+1)^{-1}$ *with equality when* $b\geq 3+\sqrt{8}$.

A proof can be found in [?]. The following corollary is crucial when trying to use the transversality technique to calculate the dimension or measure of self-similar sets.

Corollary 8. *Let* $f\in F_b$. *We have that,*

1. *for any* $s<1$ *there exists* $K(s)>0$,

$$\int_0^{y(b)}\frac{d\lambda}{|f(\lambda)|^s}<K(s).$$

2. *There exists* $C>0$ *such that,*

$$\mathcal{L}\{\lambda\in(0,b(k-1)):|f(\lambda)|\leq\epsilon\}\leq C\epsilon.$$

The first part is extremely useful when proving theorems of a similar type to Theorem 25. The second part is useful in the case when we wish to show that a class of self-similar sets have positive Lebesgue measure for almost all parameter values. We will now look at the $\{0,1,3\}$ problem in the region $\lambda\in[\frac{1}{3},y(3)]$ to outline how this method works. Let μ and ν_λ be defined exactly as in the proof of Thereom 25.

Theorem 24. *For a.e.* $\lambda\in[\frac{1}{3},y(3)]$ ν_λ *is absolutely continuous and hence* $\mathcal{L}(\Lambda(\lambda))>0$.

This result was proved in [?]. The method of proof relies that because of Lemma **??** to show that a measure ν_λ is absolutely continuous it suffices to show that,

$$\int\liminf_{r\to 0}\frac{\nu_\lambda(B(x,r))}{2r}\mathrm{d}\nu_\lambda(x)<\infty.$$

Thus to show that ν_λ is absolutely continuous for a.e. $\lambda\in(\frac{1}{3},y(3))$ it is sufficient to show for any $\epsilon>0$

$$I=\int_{\frac{1}{3}+\epsilon}^{y(3)}\int\liminf_{r\to 0}\frac{\nu_\lambda(B(x,r))}{2r}\mathrm{d}\nu_\lambda(x)\mathrm{d}\lambda<\infty.$$

The first step is to apply Fatou's Lemma and lift to the shift space. Thus

$$\begin{aligned}I &\leq \liminf_{r\to 0}\frac{1}{2r}\int_{\frac{1}{3}+\epsilon}^{y(3)}\int\nu_\lambda(B(x,r))\mathrm{d}\nu_\lambda(x)\mathrm{d}\lambda\\ &\leq \liminf_{r\to 0}\frac{1}{2r}\int_{\frac{1}{3}+\epsilon}\int_{\Sigma_3}\int_{\Sigma_3}{}_{\{\omega,\tau:|\Pi_\lambda(\omega)-\Pi_\lambda(\tau)|\leq r\}}\mathrm{d}\mu(\omega)\mathrm{d}\mu(\tau)\mathrm{d}\lambda.\end{aligned}$$

Applying Fubini's Theorem bounds I by an expression which allows part (ii) of Corollary 9 to be used. This gives

$$I \leq \liminf_{r\to 0} \frac{1}{2r} \int_{\Sigma_3} \int_{\Sigma_3} \mathcal{L}\{\lambda \in \left(\frac{1}{3}+\epsilon, y(3)\right) : |\Pi_\lambda(\omega) - \Pi_\lambda(\tau)| \leq r\} \mathrm{d}\mu(\omega)\mathrm{d}\mu(\tau).$$

It can be seen that,

$$|\Pi_\lambda(\omega) - \Pi_\lambda(\tau)| = \lambda^{|\omega\wedge\tau|} g(\lambda)$$

where $g(\lambda) \in F_\lambda$ for all $\omega, \tau \in \Sigma_3$. Thus (ii) of Corollary 9 gives that,

$$\mathcal{L}\{\lambda \in \left(\frac{1}{3}+\epsilon, y(3)\right) : |\Pi_\lambda(\omega) - \Pi_\lambda(\tau)| \leq r\} \leq 2C\left(\frac{1}{3}+\epsilon\right)^{|\omega\wedge\tau|} r$$

for some $C > 0$. This gives,

$$I \leq C \int_{\Sigma_3} \int_{\Sigma_3} \left(\frac{1}{3}+\epsilon\right)^{-|\omega\wedge\tau|} \mathrm{d}\mu(\omega)\mathrm{d}(\tau)$$

which can be seen to be finite by simply integrating on the shift space as in Theorem 25. This is the standard method for using transversality that for a.e parameter a family of measures are absolutely continuous. This method has been successfully used in many contexts. These include self-affine sets ([**?**]), Bernoulli convolutions ([**?**],[**?**],[**?**]), non linear hyperbolic IFS ([**?**]), Parabolic IFS and random continued fraction expansions ([**?**]).

The $\{0, 1, 3\}$-Problem A similar technique to the one outlined in the previous section can be used to help calculate the dimension of certain self-similar sets where the similarities overlap. Let $F = \{f_0, f_1, f_2\}$ be an IFS on where,

$$f_0(x) = \lambda x f_1(x) = \lambda x + 1 f_2(x) = \lambda x + 3.$$

For $\lambda \leq \frac{1}{4}$ the Open Set Condition applies and the Hausdorff dimension of the attractor $\Lambda(\lambda)$ is thus $-\frac{log3}{\log\lambda}$. When $\lambda \in (\frac{1}{4}, \frac{1}{3})$ the OSC can not be applied and the problem of whether $\dim\Lambda(\lambda) = -\frac{\log 3}{\log\lambda}$ is still not fully solved. A generic solution was found by Pollicott and Simon in [**?**].

Theorem 25 (Pollicott-Simon, 1994)**.** *For almost all* $\lambda \in (\frac{1}{4}, \frac{1}{3}]$,

$$\dim\Lambda(\lambda) = -\frac{\log 3}{\lambda}.$$

The method of proof is extremely similar to that of the projection theorem. Let μ be $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$-Bernoulli measure on Σ_3. A projection $\Pi_\lambda : \Sigma_3 \to$ can be defined by,

$$\Pi_\lambda(\underline{i}) = \sum_{k=0}^{\infty} i_k \lambda^k.$$

Thus on each possible attractor $\Lambda(\lambda)$ a self-similar measure ν_λ can be defined by $\nu_\lambda = \mu \circ \Pi_\lambda^{-1}$. Let $\epsilon > 0$ and $s_\epsilon(\lambda) = -\frac{\log 3}{\log(\lambda+\epsilon)}$. Note that the proof is complete, using Lemma **??** if it can be shown that,

$$I = \int_{\frac{1}{4}}^{\frac{1}{3}} \iint \frac{\mathrm{d}\nu_\lambda(x)\mathrm{d}\nu_\lambda(y)\mathrm{d}\lambda}{|x-y|^{s_\epsilon(\lambda)}} < \infty$$

for all $\epsilon > 0$. Using the projection Π_λ the inner two integrals can be transferred to Σ_3.

$$I = \int_{\frac{1}{4}}^{\frac{1}{3}} \iint \frac{\mathrm{d}\mu(\underline{i})\mathrm{d}\mu(\underline{j})\mathrm{d}\lambda}{|\Pi_\lambda(\underline{i}) - \Pi_\lambda(\underline{j})|^{s_\epsilon(\lambda)}}.$$

We then turn I into a product of two expressions using Fubini's theorem. We let $t = \max_\lambda s_\epsilon(\lambda)$ and note that $t < 1$. It can be seen that,

$$\begin{aligned}
|\Pi_\lambda(\underline{i}) - \Pi_\lambda(\underline{j})|^{s_\epsilon(\lambda)} &= \lambda^{|\underline{i}\wedge\underline{j}|s_\epsilon(\lambda)} \left(\sum_{k=0}^{\infty} a_k \lambda^k\right)^{s_\epsilon(\lambda)} \\
&\geq \left(\frac{1}{3}+\epsilon\right)^{s_\epsilon(\lambda)|\underline{i}\wedge\underline{j}} \left(\sum_{k=0}^{\infty} a_k \lambda^k\right)^{t}
\end{aligned}$$

where $\{a_k\}_{k\in 0}$ is a sequence such that $a_k \in \{0, \pm1, \pm2, \pm3\}$ and $a_0 \neq 0$. Substituting this back into I and using Fubini's Theorem we get

$$I \leq \int \frac{\mathrm{d}\lambda}{\left(\sum_{k=0}^{\infty} a_k\lambda^k\right)^t} \int_{\Sigma_3}\int_{\Sigma_3} \frac{\mathrm{d}\mu(\underline{i})\mathrm{d}\mu(\underline{j})}{\left(\frac{1}{3}+\epsilon\right)^{|\underline{i}\wedge\underline{j}|}}.$$

By simple integration on Σ_3 it can be seen that,

$$\begin{aligned}
\int\int \frac{\mathrm{d}\mu(\underline{i})\mathrm{d}\mu(\underline{j})}{\left(\frac{1}{3}+\epsilon\right)^{|\underline{i}\wedge\underline{j}|}} &\leq \sum_{k=0}^{\infty} \sum_{[i_0,i_1,\ldots,i_{k-1}]} \frac{\mu([i_0,i_1,\ldots,i_{k-1}])^2}{\left(\frac{1}{3}+\epsilon\right)^k} \\
&= \sum_{k=0}^{\infty} \frac{\frac{1}{3}^{k+1}}{\left(\frac{1}{3}+\epsilon\right)^k} < \infty
\end{aligned}$$

Thus to show that $I < \infty$ it remains to show that,

$$\int \frac{\mathrm{d}\lambda}{\left(\sum_{k=0}^{\infty} a_k\lambda^k\right)^t} < \infty$$

for any sequence $\{a_k\}_{k\in 0}$ where each $a_k \in \{0, \pm1, \pm2, \pm3\}$ and $a_0 = 0$. Let $g(\lambda)$ be a power series of that form. In Lemma 1 of [?] it is shown that whenever such a power series $g(\lambda)$ is close to 0 its derivative is bounded away from 0. Thus a transversality condition is satisfied. The integral can be shown to be finite by splitting it into two parts, one part where $g(\lambda)$ is bounded away from 0 and one where $g(\lambda)$ is close to zero but the derivative is bounded away from 0. The first part is clearly finite and the second part is finite because $t < 1$ and $g(\lambda)$ can be bounded below by linear functions in this region. This method will be used again in chapters 4 and 5.

In [**?**] a general result about when specific power series satisfy a transversality condition is given. Let

$$F_b = \{f(\lambda) = \sum_{k=0}^{\infty} f_k \lambda^K : g_k \in [-b, b]\}.$$

As in [**?**] we now define,

$$y(b) = \min\{x > 0 : \exists f \in F_b \text{ where } f(x) = f'(x)\}.$$

Theorem 26 (Peres-Solomyak,1996)**.** *The function $y(b) : [1, \infty) \to [0, 1]$ is strictly decreasing, continuous and piecewise algebraic. $y(1) \approx 0.649$, $y(2) = 0.5$ and $y(b) \geq (\sqrt{b} + 1)^{-1}$ with equality when $b \geq 3 + \sqrt{8}$.*

A proof can be found in [**?**]. The following corollary is crucial when trying to use the transversality technique to calculate the dimension or measure of self-similar sets.

Corollary 9. *Let $f \in F_b$. We have that,*

1. *for any $s < 1$ there exists $K(s) > 0$,*
$$\int_0^{y(b)} \frac{d\lambda}{|f(\lambda)|^s} < K(s).$$
2. *There exists $C > 0$ such that,*
$$\mathcal{L}\{\lambda \in (0, b(k-1)) : |f(\lambda)| \leq \epsilon\} \leq C\epsilon.$$

The first part is extremely useful when proving theorems of a similar type to Theorem 25. The second part is useful in the case when we wish to show that a class of self-similar sets have positive Lebesgue measure for almost all parameter values. We will now look at the $\{0, 1, 3\}$ problem in the region $\lambda \in [\frac{1}{3}, y(3)]$ to outline how this method works. Let μ and ν_λ be defined exactly as in the proof of Thereom 25.

Theorem 27. *For a.e. $\lambda \in [\frac{1}{3}, y(3)]$ ν_λ is absolutely continuous and hence $\mathcal{L}(\Lambda(\lambda)) > 0$.*

This result was proved in [**?**]. The method of proof relies that because of Lemma **??** to show that a measure ν_λ is absolutely continuous it suffices to show that,

$$\int \liminf_{r \to 0} \frac{\nu_\lambda(B(x, r))}{2r} \mathrm{d}\nu_\lambda(x) < \infty.$$

Thus to show that ν_λ is absolutely continuous for a.e. $\lambda \in (\frac{1}{3}, y(3))$ it is sufficient to show for any $\epsilon > 0$

$$I = \int_{\frac{1}{3}+\epsilon}^{y(3)} \int \liminf_{r\to 0} \frac{\nu_\lambda(B(x,r))}{2r} \mathrm{d}\nu_\lambda(x)\mathrm{d}\lambda < \infty.$$

The first step is to apply Fatou's Lemma and lift to the shift space. Thus

$$\begin{aligned} I &\leq \liminf_{r\to 0} \frac{1}{2r} \int_{\frac{1}{3}+\epsilon}^{y(3)} \int \nu_\lambda(B(x,r))\mathrm{d}\nu_\lambda(x)\mathrm{d}\lambda \\ &\leq \liminf_{r\to 0} \frac{1}{2r} \int_{\frac{1}{3}+\epsilon} \int_{\Sigma_3} \int_{\Sigma_3} {}_{\{\omega,\tau:|\Pi_\lambda(\omega)-\Pi_\lambda(\tau)|\leq r\}} \mathrm{d}\mu(\omega)\mathrm{d}\mu(\tau)\mathrm{d}\lambda. \end{aligned}$$

Applying Fubini's Theorem bounds I by an expression which allows part (ii) of Corollary 9 to be used. This gives

$$I \leq \liminf_{r\to 0} \frac{1}{2r} \int_{\Sigma_3} \int_{\Sigma_3} \mathcal{L}\{\lambda \in \left(\frac{1}{3}+\epsilon, y(3)\right) : |\Pi_\lambda(\omega) - \Pi_\lambda(\tau)| \leq r\}\mathrm{d}\mu(\omega)\mathrm{d}\mu(\tau).$$

It can be seen that,

$$|\Pi_\lambda(\omega) - \Pi_\lambda(\tau)| = \lambda^{|\omega\wedge\tau|} g(\lambda)$$

where $g(\lambda) \in F_\lambda$ for all $\omega, \tau \in \Sigma_3$. Thus (ii) of Corollary 9 gives that,

$$\mathcal{L}\{\lambda \in \left(\frac{1}{3}+\epsilon, y(3)\right) : |\Pi_\lambda(\omega) - \Pi_\lambda(\tau)| \leq r\} \leq 2C\left(\frac{1}{3}+\epsilon\right)^{|\omega\wedge\tau|} r$$

for some $C > 0$. This gives,

$$I \leq C \int_{\Sigma_3} \int_{\Sigma_3} \left(\frac{1}{3}+\epsilon\right)^{-|\omega\wedge\tau|} \mathrm{d}\mu(\omega)\mathrm{d}(\tau)$$

which can be seen to be finite by simply integrating on the shift space as in Theorem 25. This is the standard method for using transversality that for a.e parameter a family of measures are absolutely continuous. This method has been successfully used in many contexts. These include self-affine sets ([**?**]), Bernoulli convolutions ([**?**],[**?**],[**?**]), non linear hyperbolic IFS ([**?**]), Parabolic IFS and random continued fraction expansions ([**?**]).

We shall formulate a simple version of this result in one dimension, although a version is valid in arbitrary dimensions.

Let us fix $0 < \lambda < \frac{1}{2}$. We want to consider affine maps $T_i : \mathbb{R} \to \mathbb{R}$ $(i = 1, \ldots, k)$ of the real line $\mathbb{R}$ defined by $T_i x = \lambda x + b_i$, for $i = 1, \ldots, k$, where $b_1, \ldots, b_k \in \mathbb{R}$. Let us use the notation $\underline{b} = (b_1, \ldots, b_k) \in \mathbb{R}^k$ and then let us denote by $\Lambda_{\underline{b}}$ the associated limit set.

Theorem 28 (Theorem 6.6 (Falconer's Theorem))**.** *For almost all* $\underline{b} = (b_1, \ldots, b_k) \in \mathbb{R}^k$ *we have that* $\dim_H \Lambda_{\underline{b}} = -\log k / \log \lambda$.

Of course, this if $T_1, \ldots, T_k$ satisfy the Open Set Condition then the formula for Hausdorff Dimension automatically holds by Moran's Theorem.

We begin with a preliminary result.

Lemma 46. *Consider a power series* $f_{\underline{a}}(z) = a_0 + \sum_{n=1}^{\infty} a_n z^n$ *where* $a_n \subset \{b_i - b_j : 1 \leq i, j \leq k\}$ *and* $a_0 \neq 0$. *Then* $\int_{|\underline{b}|} |f_{\underline{a}}(z)|^s d\underline{b} < +\infty$

Proof of Theorem 6.6. Let $U \subset \mathbb{R}$ be an open set chosen such that $T_i U \subset U$ for all $1 \leq i \leq k$. Given $\delta > 0$ we can choose n sufficiently large that $\lambda^n \mathrm{diam}(U) \leq \delta$. Let us cover $\Lambda_{\underline{b}}$ by open sets $\{T_{\underline{i}}(U) : |\underline{i}| = n\}$. Given $s > 0$ can estimate

$$H_\delta^s(\Lambda_{\underline{b}}) \leq \sum_{|\underline{i}|=n} \mathrm{diam}(U_{\underline{i}}) \leq (k\lambda^s)^n$$

In particular, for any $s > -\log k / \log \lambda$ we have that $(k\lambda^s) < 1$ and so we deduce that $\dim_H \Lambda_{\underline{b}} \leq s|$. In particular, $\dim_H \Lambda_{\underline{b}} \leq -\log k / \log \lambda$.

On the other hand, let us consider the Bernoulli measure $\nu = (\frac{1}{k}, \cdots, \frac{1}{k})^{\mathbb{Z}^+}$ on the associate sequence space $\Sigma = \{1, \ldots, k\}^{\mathbb{Z}^+}$. Let $\pi_{\underline{b}} : \Sigma \to \Lambda_{\underline{b}}$ be the natural coding given by $\pi_{\underline{b}}(\underline{i}) = \lim_{n \to +\infty} T_{i_0} \cdots T_{i_n}(0)$. We can consider the associated measure $\mu_{\underline{b}} = \pi_{\underline{b}} \nu$ (i.e., $\mu_{\underline{b}}(I) = (\pi_{\underline{i}}^{-1} I)$). Let us fix $s > -\log k / \log \lambda$ For any $R > 0$ we can write

$$\int_{|\underline{b}| \leq R} \left(\int_{\Lambda_{\underline{b}}} \int_{|\underline{b}| \leq R} \frac{d\mu_(\underline{b})(x) d\mu(\underline{b})}{|x - y|^s} \right) d\underline{b} = \int_{|\underline{b}| \leq R} \left(\int_\Sigma \int_\Sigma \frac{d\nu(\underline{i}) d\nu(\underline{j})}{|\pi_{\underline{b}}(\underline{i}) - \pi_{\underline{b}}(\underline{j})|^s} \right) d\underline{b},$$

where we integrate over the ball of radius R with respect to lebesgue measure. Moreover, using Fubini's theorem we can reverse the order of the integrals in the last expression to get

$$\int_\Sigma \int_\Sigma \left(\int_{|\underline{b}| \leq R} \frac{d\underline{b}}{|\pi_{\underline{b}}(\underline{i}) - \pi_{\underline{b}}(\underline{j})|^s} \right) d\nu(\underline{i}) d\nu(\underline{j}) \tag{6.6}$$

If the sequences $\underline{i}$, $\underline{j}$ agree in the first n spaces (but differ in the $(n+1)$st place) then we can write

$$\pi_{\underline{b}}(\underline{i}) - \pi_{\underline{b}}(\underline{i}) = \lambda^{n+1} \left((b_{i_{n+1}} - b_{j_{n+1}}) + \sum_{m=1}^{\infty} \lambda^m (b_{i_{n+m+1}} - b_{j_{n+m+1}}) \right)$$

where $b_{i_{n+1}} \neq b_{j_{n+1}}$ are distinct elements from $\{b_1, \ldots, b_k\}$. In particular, differentiating in the direction corresponding to $b_{i_{n+1}}$ (whilst fixing the other directions) we see that

$$\begin{aligned}\left|\frac{\partial(\pi_{\underline{b}}(\underline{i})-\pi_{\underline{b}}(\underline{i}))}{\partial b_{i_{n+1}}}\right| &= \lambda^{n+1}\left|\left(1+\sum_{m=1}^{\infty}\lambda^m\frac{\partial(b_{i_{n+m+1}}-b_{j_{n+m+1}})}{\partial b_{i_{n+1}}}\right)\right| \\ &\geq \lambda^{n+1}\left(1-\sum_{m=1}^{\infty}\lambda^m\right)\geq C\lambda^{n+1}\end{aligned}$$

for some $C>0$. We can then write

$$\int_{|\underline{b}|\leq R}\frac{d\underline{b}}{|\pi_{\underline{b}}(\underline{i})-\pi_{\underline{b}}(\underline{i})|^s}\leq D\lambda^{-s(n+1)} \tag{6.7}$$

for some $D>0$. Substituting (6.7) into (6.6) we have that

$$\begin{aligned}\int_{\underline{b}}\left(\int_{\Lambda_{\underline{b}}}\int_{\Lambda_{\underline{b}}}\frac{d\mu_{\underline{b}}(x)d\mu_{\underline{b}}(y)}{|x-y|^s}\right)d\underline{b} &\leq C^s\int_{\Sigma}\left(\sum_{n=1}^{\infty}\sum_{i_0,\ldots,i_n}\underbrace{\mu[i_0,\ldots,i_n]}_{=(\frac{1}{k})^{n+1}}\lambda^{-s(n+1)}\right)d\mu(\underline{i}) \\ &\leq C^s\sum_{n=1}^{\infty}\left(\frac{\lambda^{-s}}{k}\right)^{n+1}<+\infty\end{aligned}$$

By Fubini's Theorem we deduce that for almost every $\underline{b}$ we have that the integrand is finite almost everywhere, i.e.,

$$\int_{\Lambda_{\underline{b}}}\int_{\Lambda_{\underline{b}}}\frac{d\mu_{\underline{b}}(x)d\mu_{\underline{b}}(y)}{|x-y|^s}<+\infty$$

provided $s<-\log k/\log\lambda$. In particular, we deduce from lemma 6.2 that for such $\underline{b}$ we have $\dim_H(\Lambda_{\underline{b}})>s$. Since s can be chosen arbitrarily close to $-\log k/\log\lambda$ the result follows. $\square$

Chapter 10

Measure and dimension

10.1 Hausdorff dimension of measures

Let μ denote a probability measure on a set X. We can define the Hausdorff dimension μ in terms of the Hausdorff dimension of subsets of Λ.

Definition 22. *For a given probability measure μ we define the Hausdorff dimension of the measure by*

$$\dim_H(\mu) = \inf\{\dim_H(X) : \mu(X) = 1\}.$$

We next want to define a local notion of dimension for a measure μ at a typical point $x \in X$.

Definition 23. *The upper and lower pointwise dimensions of a measure μ are measurable functions $\overline{d}_\mu, \underline{d}_\mu : X \to \mathbb{R} \cup \{\infty\}$ defined by*

$$\overline{d}_\mu(x) = \limsup_{r \to 0} \frac{\log \mu(B(x,r))}{\log r} \text{ and } \underline{d}_\mu(x) = \liminf_{r \to 0} \frac{\log \mu(B(x,r))}{\log r}$$

where $B(x,r)$ is a ball of radius $r > 0$ about x.

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The pointwise dimensions describe how the measure μ is distributed. We compare the measure of a ball about x to its radius r, as r tends to zero. There are interesting connections between these different notions of dimension for measures.

Theorem 29. *If $\underline{d}_\mu(x) \geq d$ for a.e. (μ) $x \in X$ then $\dim_H(\mu) \geq d$.*

Proof. We can choose a set of full μ measure $X_0 \subset X$ (i.e., $\mu(X_0) = 1$) such that $\underline{d}_\mu(x) \geq d$ for all $x \in X_0$. In particular, for any $\epsilon > 0$ and $x \in X$

we have $\limsup_{r\to 0} \mu(B(x,r))/r^{d-\epsilon} = 0$. Fix $C > 0$ and $\delta > 0$, and let us denote

$$X_\delta = \{x \in X) : \mu(B(x,r)) \leq Cr^{d-\epsilon}, \quad \forall 0 < r \leq \delta\}.$$

Let $\{U_i\}$ be any δ-cover for X. Then if $x \in U_i$, $\mu(U_i) \leq C\mathrm{diam}(U_i)^{d-\epsilon}$. In particular,

$$\mu(X_\delta) \leq \sum_{U_i \cap X_\delta} \mu(U_i) \leq C \sum_i \mathrm{diam}(U_i)^{d-\epsilon}.$$

Thus, taking the infimum over all such cover we have $\mu(X_\delta) \leq CH_\delta^{d-\epsilon}(X_\delta) \leq CH^{d-\epsilon}(X)$. Now letting $\delta \to 0$ we have that $1 = \mu(X_0) \leq CH^{d-\epsilon}(X)$. Since $C > 0$ can be chosen arbitrarily large we deduce that $H^{d-\epsilon}(X) = +\infty$. In particular, $\dim_H(X) \geq d - \epsilon$ for all $\epsilon > 0$. Since $\epsilon > 0$ is arbitrary, we conclude that $\dim_H(X) \geq d$. □

We have the following simple corollary, which is immediate from the definition of $\dim_H(\mu)$.

Corollary 10. *Given a set $X \subset \mathbb{R}^d$, assume that there is a probability measure μ with $\mu(X) = 1$ and $\underline{d}_\mu(x) \geq d$ for a.e. (μ) $x \in X$. Then $\dim_H(X) \geq d$.*

In the opposite direction we have that a uniform bound on pointwise dimensions leads to an upper bound on the Hausdorff Dimension.

Theorem 30. *If $\overline{d}_\mu(x) \leq d$ for a.e. (μ) $x \in X$ then $\dim_H(\mu) \leq d$.*

Moreover, if there is a probability measure μ with $\mu(X) = 1$ and $\overline{d}_\mu(x) \leq d$ for **every** *$x \in X$ then $\dim_H(X) \leq d$.*

Proof. We begin with the second statement. For any $\epsilon > 0$ and $x \in X$ we have $\limsup_{r\to 0} \mu(B(x,r))/r^{d+\epsilon} = \infty$. Fix $C > 0$. Given $\delta > 0$, consider the cover U for X by the balls

$$\{B(x,r) : 0 < r \leq \delta \text{ and } \mu(B(x,r)) > Cr^{d+\epsilon}\}.$$

We recall the following classical result.

Besicovitch covering lemma. There exists $N = N(d) \geq 1$ such that for any cover by balls we can choose a sub-cover $\{U_i\}$ such that any point x lies in at most N balls.

Thus we can bound

$$H_\delta^{d+\epsilon}(X) \leq \sum_i \mathrm{diam}\ (U_i)^{d+\epsilon} \leq \frac{1}{C} \sum_i \mu(B_i) \leq \frac{N}{C}.$$

Letting $\delta \to 0$ we have that $H^{d+\epsilon}(X) \leq \frac{N}{C}$. Since $C > 0$ can be chosen arbitrarily large we deduce that $H^{d+\epsilon}(X) = 0$. In particular, $\dim_H(X) \leq d + \epsilon$ for all $\epsilon > 0$. Since $\epsilon > 0$ is arbitrary, we deduce that $\dim_H(X) \leq d$.

The proof of the first statement is similar, except that we replace X by a set of full measure for which $\overline{d}_\mu(x) \leq d$. □

Let us consider the particular case of iterated function schemes.

Example 55 (Iterated Function Schemes and Bernoulli measures)). *For an iterated function scheme $T_1, \cdots, T_k : U \to U$ we can denote as before*

$$\Sigma = \{\underline{x} = (x_m)_{m=0}^{\infty} : x_m \in \{1, \cdots, k\}\}$$

with the Tychonoff product topology. The shift map $\sigma : \Sigma \to \Sigma$ is a local homeomorphism defined by $(\sigma x)_m = x_{m+1}$. The kth level cylinder is defined by,

$$[x_0, \ldots, x_{k-1}] = \{(i_m)_{m=0}^{\infty} \in \Sigma : i_m = x_m \text{ for } 0 \le m \le k-1\},$$

(i.e., all sequences which begin with $x_0, \ldots, x_{k-1}$). We denote by $W_k = \{[x_0, \ldots, x_{k-1}]\}$ the set of all kth level cylinders (of which there are precisely k^n).

Notation For a sequence $\underline{i} \in \Sigma$ and a symbol $r \in \{1, \ldots, k\}$ we denote by $k_r(\underline{i}) = \text{card}\{0 \le m \le k-1 : i_m = r\}$ the number of occurrences of r in the first k terms of $\underline{i}$.

Consider a probability vector $\underline{p} = (p_0, \ldots, p_{n-1})$ and define the Bernoulli measure of any kth level cylinder to be,

$$\mu([i_0, \ldots, i_{k-1}]) = {p_0}^{k_0(\underline{i})} {p_1}^{k_1(\underline{i})} \cdots {p_{n-1}}^{k_{n-1}(\underline{i})}.$$

A probability measure μ on σ is said to be invariant under the shift map if for any Borel set $B \subset X$, $\mu(B) = \mu(\sigma^{-1}(B))$. We say that μ is ergodic if any Borel set $B \subseteq \Sigma$ such that $\sigma^{-1}(X) = X$ satisfies $\mu(X) = 0$ or $\mu(X) = 1$. A Bernoulli measure is both invariant and ergodic.

We now introduce the concept of entropy. Entropy We start by defining entropy for general ergodic systems before going back to shift spaces. All of the details given here can be found in Chapter 4 in [**?**]. Let (X, B, μ) be a probability space and let $T : X \to X$ be a measure persevering transformation. A partition of (X, B, μ) is a finite or countable collection of elements in B whose union is X. For example if we take $X = \Sigma_n$, B to be the standard sigma algebra for Σ_n and μ to be evenly weighted Bernoulli measure the cylinders $\{[0], \ldots, [n-1]\}$ form a partition of Σ_n. For a finite partition $A = \{A_1, \ldots, A_m\}$ we define

$$H_\mu(A) = -\sum_{i=1}^{m} \mu(A_i) \log(\mu(A_i)).$$

For two partitions $A = \{A_1, A_2, \ldots, A_m\}$ and $C = \{C_1, \ldots, C_l\}$ we define the join to be,

$$A \vee C = \{A_i \cap C_j : 1 \le i \le m, 1 \le j \le l\}.$$

This definition also holds for countable partitions. The partition $T^{-k}(A)$ for $k \in$ is defined as,

$$T^{-k}(A) = \{T^{-k}(A_1), T^{-k}(A_2), \ldots, T^{-k}(A_m).$$

We define the entropy of T with respect to A to be,

$$h_\mu(T, A) = \lim_{k \to \infty} \frac{1}{k} H\left(\vee_{i=0}^{k-1} T^{-1} A\right).$$

This limit is shown to exist in [**?**]. The entropy of the transformation can now be defined as,

$$h_\mu(T) = \sup h(T, A).$$

To directly calculate the entropy of a transformation using this definition is usually extremely difficult. The idea of a generating partition often makes the calculation much easier. A countable partition A is said to be a generating partition if

$$\vee_{k=-\infty}^{\infty} T^n(A) = B$$

where B is the Borel sigma algebra for the measure space. If A is a generator and $H_\mu(A) < \infty$ then $h_\mu(T) = h_\mu(T, A)$.

We now return to Σ_n. In this case the set of cylinders $\{[0], \ldots, [n-1]\}$ is a generating partition.

Definition 24. *For any ergodic and invariant measure μ on Σ the entropy of μ is defined to be the value*

$$h_\mu(\sigma) = \lim_{k \to \infty} -\frac{1}{k} \sum_{\omega_k \in W_k} \mu(\omega_k) \log(\mu(\omega_k)).$$

In particular, for a Bernoulli measure μ associated to a probability vector $\underline{p} = (p_0, \ldots, p_{n-1})$ the entropy can easily seen to be simply

$$h_\mu(\sigma) = -\sum_{i=0}^{n-1} p_i \log p_i.$$

An important classical result for entropy is the following.

Theorem 31 (Shannon-McMillan-Brieman Theorem)**.** *Let μ be an ergodic σ-invariant measure on Σ. For μ almost all $\underline{i} \in \Sigma$,*

$$\lim_{k \to \infty} -\frac{1}{k} \log \mu([i_0, \ldots, i_{n-1}]) = h_\mu(\sigma).$$

We can define a continuous map $\Pi : \Sigma \to \Lambda$ by $\Pi(\underline{i}) = \lim_{k \to \infty} T_{i_0} \cdots T_{i_k}(0)$. We can associated to a probability measure μ on Σ a measure ν on Λ defined by $\nu = \mu \circ \Pi_\lambda^{-1}$. In particular, when μ is a $\underline{p}$-Bernoulli measure the measure ν satisfies,

$$\nu(A) = \sum_{i=0}^{n-1} p_i \nu(f_i^{-1}(A)).$$

In the case where all the contractions $T_1, \ldots, T_k$ are similarities it is possible to use the Shannon-Mcmillan-Brieman Theorem to get an upper bound on the Hausdorff dimension of ν. Let T_i have contraction ratio $|T_i'| = r_i < 1$, say, and let

$$\chi = \sum_{i=0}^{n-1} p_i \log r_i < 0$$

be the Lyapunov exponent of ν.

Proposition 15. *Consider a conformal linear iterated function scheme $T_1, \cdots, T_k$ satisfying the open set condition. Let ν be the image of a Bernoulli measure. Then*

$$\dim_H(\nu) = \frac{\sum_{i=0}^{n-1} p_i \log p_i}{\sum_{i=0}^{n-1} p_i \log r_i} \left(= \frac{h_\mu(\sigma)}{|\chi|} \right)$$

Without the open set condition we still get an inequality $\leq$.

Proof. The idea is to apply Theorem 5.1 and Theorem 5.2.

For two distinct sequences $\omega, \tau \in \Sigma$ we denote by $|\omega \wedge \tau| = \min\{k : \omega_k \neq \tau_k\}$ the first term in which the two sequences differ. For two sequences $\omega, \tau \in \Sigma$ we denote by $|\omega \wedge \tau| = \min\{k : \omega_k \neq \tau_k\}$ the first term in which the two sequences differ. Given $\omega, \tau \in \Sigma$ let $m = |\omega \wedge \tau|$, then we define a metric by

$$d(\omega, \tau) = \prod_{i=0}^{k-1} r_i^{m_i(\omega)} \left(= \prod_{i=0}^{k-1} r_i^{m_i(\tau)} \right).$$

We can apply Theorem 5.1 (1). To show $\dim \nu \geq s$ for some s it is sufficient to show that

$$\liminf_{r \to 0} \frac{\log \nu(B(x, r))}{\log r} \leq s$$

for a.e. (ν) $x \in \Lambda$. Since $\nu = \mu \circ \Pi^{-1}$, it is sufficient to show that for μ-almost all $\tau \in \Sigma$,

$$\liminf_{r \to 0} \frac{\log \nu(B(\Pi^{-1}\tau, r))}{\log r} \leq s.$$

A useful property of this metric d is that the diameter of any cylinder in the shift space is the same as the diameter of the projection of the cylinder in $\mathbb{R}^n$. Fix $\tau \in \Sigma_n$ and let $x = \Pi^{-1}\tau$. For $r > 0$ there exists $k(r)$ such that,

$$[i_1, \ldots, i_{k(r)}, i_{k(r)+1}] \leq 2r \leq [i_1, \ldots, i_{k(r)}]$$

and $k(r) \to \infty$ as $r \to 0$. Hence

$$\lim_{r\to 0} \frac{\log(\nu(B(x,r)))}{\log r} = \lim_{k\to\infty} \frac{\log(\mu([\tau_0,\ldots,\tau_{k-1}]))}{\log(\mathrm{diam}([\tau_0,\ldots,\tau_{k-1}]))}.$$

(Without the open set condition $\nu(B(x,r))$ can be much bigger than $\mu([\tau_1,\ldots,\tau_{k(r)-1}])$.)

By the Shannon-McMillan-Brieman Theorem we have that,

$$\lim_{n\to\infty} \frac{1}{n} \log(\mu([\tau_0,\ldots,\tau_{n-1}])) \to \sum_{i=0}^{n-1} p_i \log p_i = h_\mu(\sigma)$$

for μ almost all τ and by the Birkhoff Ergodic theorem we have that

$$\lim_{n\to\infty} \frac{1}{n} \log \mathrm{diam}[\tau_0,\ldots,\tau_{n-1}] \to \sum_{i=0}^{n-1} p_i \log r_i = \chi$$

for μ almost all τ. Hence for μ almost all τ where $x = \Pi\tau$ (or equivalently, ν almost all x)

$$\lim_{r\to 0} \frac{\log(\nu(B(x,r)))}{\log r} = \frac{h_\mu(\sigma)}{\chi}.$$

Thus by Theorem 5.1 and Theorem 5.2 the result follows □. □

It is follows from the proof that we still get an upper bound $\dim_H(\nu)$ if we replace μ by any other ergodic σ-invariant measure on Σ or if we don't assume the Open Set Condition.

A more general statement is the following:

Proposition 16. *Let $T : X \to X$ be a conformal expanding map on a compact metric space. If μ is an ergodic invariant measure then the pointwise dimension $d_\mu(x)$ exists for μ-almost every x. Moreover*

$$d_\mu(x) = \frac{h_\mu(T)}{\int_X \log |T'|\, d\mu}$$

for μ-almost every x.

Proof. The proof follows the same general lines as above. Let $P = \{P_1,\ldots,P_k\}$ be an Markov partition for T and let $C_n(x) = \cap_{i=0}^{n-1} T^{-i} P_{x_i}$ be a cylinder set containing a point x. By the Shannon-McMillan Brieman theorem $-\frac{1}{n}\log\mu(C_n) \to h(\mu)$, a.e. (μ). By the Birkhoff Ergodic Theorem we expect $\frac{1}{n}\log|\mathrm{diam}(C_n)| \sim -\frac{1}{n}\log|(T^n)'(x)| \to \int \log|T'|d\mu$, a.e. (μ) □ □

10.1.1 Multifractal Analysis

For a measure μ on a set X we can ask about the set of points x for which the limit

$$d_\mu(x) = \lim_{r\to 0} \frac{\log \mu(B(x,r))}{\log r}$$

exists. Let $X_\alpha = \{x : \text{ the limit } d_\mu(x) = \alpha\}$ be the set for which the limit exists, and equals α. There is a natural decomposition of the set X by "level sets":

$$X = \bigcup -\infty < \alpha < \infty X_\alpha \cup \{x \in X \mid d_\mu(x) \text{ does not exist}\}.$$

To study this decomposition one defines the following:

The *dimension spectrum* is a function $f_\mu : \mathbb{R} \to [0,d]$ given by $f_\mu(\alpha) = \dim_H(X_\alpha)$, i.e., the Hausdorff dimension of the set X_α.

The "multifractal analysis" of the measure μ describes the size of the sets X_α through the behaviour of the function f_μ.

Example 56. *Let us consider an iterated function scheme $T_1, \ldots, T_k$ with similarities satisfying the open set condition. Consider the Bernoulli measure μ associated with the vector $(p_1, \ldots, p_k)$. We have already seen that:*

1. *$d_\mu(x)$ exists for a.e. (μ) x and is equal to $\dim_H(\mu)$. (In this particular case, this limit is equal to $\frac{\sum_{i=1}^k p_i \log p_i}{\sum_{i=1}^k p_i \log r_i}$).*

We claim that the following is also true.

"(2)" Except in the very special case $p_i = r_i^{\dim_H(\Lambda)}$, for $i = 1, \ldots, k$, there is an interval (a,b) containing $\dim_H(\Lambda)$ such that $f_\mu : (a,b) \to \mathbb{R}$ is analytic.

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Multifractal analysis describes the size of sets X_α for which the pointwise dimension is exactly α.

Sketch proof of (2). For each α, we can write

$$X_\alpha = \Pi \left\{ \underline{x} \in \Sigma : \lim_{n\to+\infty} \frac{\sum_{j=1}^n \log p_{x_j}}{\sum_{j=1}^n \log r_{x_j}} = \alpha \right\}.$$

For each $q \in \mathbb{R}$, we can choose $T(q) \in \mathbb{R}$ such that $P(-T(q)\log|r_{x_0}| + q \log p_{x_0}) = 0$. There exists an associated Bernoulli measure ν_q and constants $C_1, C_2 > 0$ such that

$$C_1 \le \frac{\nu_q([i_1, \cdots, i_n])}{\prod_{i=0}^{n-1} \exp\left(-T(q)\log r_{x_i} + q \log p_{x_i}\right)} \le C_2. \tag{5.1}$$

Furthermore, we associate to q the particular value

$$\alpha(q) = \frac{\int \log p_{x_0} d\nu_q}{\int \log r_{x_0} d\nu_q}.$$

For a.e. (ν_q) $x \in X_{\alpha(q)}$ we have that $d_{\nu_q}(x) = \alpha(q)$ by the Birkhoff ergodic theorem and the definition of X_α. In particular, $\nu_q(X_\alpha) = 1$. [1] If $(r_1, \ldots, r_k) \neq (p_1, \ldots, p_k)$ then $f_\nu(\alpha)$ and $T(q)$ are strictly convex (and are Legendre transforms of each other).

We then claim that:

"(a)" $\alpha(q)$ is analytic

"(b)" $f_\nu(\alpha(q)) = (dim_H X_{\alpha(q)}) = T(q) + q\alpha(q)$. and then (2) follows.

For part (a) observe that since $P(\cdot)$ is analytic, we deduce from the Implicit Function Theorem that the function $T(q)$ is analytic as a function of q. Observe that $T(0) = \dim_H X$. We can check by direct computation that $T'(q) \leq 0$ and $T''(q) \geq 0$.

Part (b) follows from the observation that $d_{\nu_q}(x) = T(q) + q\alpha(q)$ for a.e. $x \in K_\alpha$ and $\bar{d}_{\nu_q}(x) = T(q) + q\alpha(q)$ for all $x \in K_\alpha$ by (5.1). We then apply Theorem 5.1 and Theorem 5.2. □ □

Example 57 (Expanding maps). *Let $T\ I \to I$ be an expanding transformation on the unit interval I. Let μ be a T-invariant ergodic probability measure. We say that μ is a Gibbs measure if $\phi(x) = \log \frac{d\mu T}{d\mu}$ is piecewise C^1 (or merely Hölder continuous would suffice. The most familiar example of a Gibbs measure is given by the following.*

Proposition 17 ('Folklore Lemma'). *There is a unique absolutely continuous invariant probability measure ν (i.e., we can write $d\nu(x) = \rho(x)dx$).*

The main result is the following.

Proposition 18. *Assume that μ is a Gibbs measure (but not ν):*

1. *The pointwise dimension $d_\mu(x)$ exists for μ-almost every $x \in I$. Moreover, $d_\mu(x) = d_\mu \equiv h_\mu(T)/\int_X \log |T'|\, d\mu$ for μ-almost every $x \in I$.*
2. *The function $f_\mu(\alpha)$ is smooth and strictly convex on some interval $(\alpha_{min}, \alpha_{max})$ containing d_μ.*

Let ψ be a positive function defined by $\log \psi = \phi - P(\phi)$, where $P(\phi)$ denotes the pressure of ϕ. Clearly ψ is a Hölder continuous function on I such that $P(\log \psi) = 0$ and μ is also the equilibrium state for $\log \psi$. We define the two parameter family of Hölder continuous functions $\phi_{q,t} = -t \log |T'| + q \log \psi$. Define the function $t(q)$ by requiring that $P(\phi_{q,t(q)}) = 0$ and let μ_q be the equilibrium state for $\phi_{q,t(q)}$

[1]We can also identify $\alpha(q) = -T'(q)$, then it has a range $[\alpha_1, \alpha_2] \subset \mathbb{R}^+$.

10.2 Computing Lyapunov exponents

In many examples, the Lyapunov exponents $\int \log |T'(x)| d\mu(x)$ can be computed in much the same way that Hausdorff dimension was. More precisely, this integral can be approximated by periodic orbit estimates. In the interests of definiteness, consider the absolutely continuous T-invariant measure ν.

Lemma 47. *Let*

$$m_n = \frac{\sum_{x \in Fix(T^n)} \delta_x / |(T^n)'(x)|}{\sum_{x \in Fix(T^n)} 1/|(T^n)'(x)|}, \quad n \geq 1,$$

where $Fix(T^n) = \{x \in [0,1] : T^n x = x\}$ and δ_x is the Dirac measure supported on x. Then $m_n \to \mu$ in the weak-star topology.

However, for any $f \in C^\omega([0,1])$ we have exponential convergence, i.e., $\exists 0 < \theta < 1$, $C > 0$ such that $|\int f dm_n - \int f d\mu| \leq C\theta^n$.

Aim. *We will present a different family of invariant measures μ_M with super-exponential convergence for $f \in C^\omega([0.1])$, i.e., $\exists 0 < \theta < 1$, $C > 0$ such that $|\int f d\mu_n - \int f d\mu| \leq C\theta^{n^2}$.*

For example, taking $f(x) = \log |T'(x)|$ gives approximations to the metric entropy $h(\mu)$. Similarly, taking $f(x) = e^{2\pi i n x}$, $n \in \mathbb{Z}$ gives approximations to

the Fourier coefficients $\hat{\mu}(n) = \int_0^1 e^{2\pi i n x} d\mu(x)$ of μ.

For definiteness, let us consider the case of the absolutely continuous invariant measure ν. We construct the family of approximating measures by a more elaborate regrouping of the periodic points to define new invariant probability measures. Let λ_n be the sequence of numbers given by

$$\lambda_n = \frac{\sum_{\underline{k}=(k_1,\ldots,k_m), k_1+\ldots+k_m \leq n} \frac{(-1)^m r(\underline{k})}{m!} \left(\sum_{i=1,\ldots,m\, x \in Fix(T^{k_i})} k_i \log |T'(x)| \right)}{\sum_{\underline{k}=(k_1,\ldots,k_m), k_1+\ldots+k_m \leq n} \frac{(-1)^m r(\underline{k})}{m!} \left(\sum_{i=1,\ldots,m\, x \in \mathrm{Fix}(T^{k_i})} k_i \right)}$$

where we write

$$r(\underline{k}) = \prod_{j=1}^{m} \sum_{z \in \mathrm{Fix}(T^{k_j})} \frac{1}{k_j |(T^{k_j})'(z) - 1|}.$$

and $\mathrm{Fix}(T^n) = \{x \in [0,1] : T^n x = x\}$.
We have the following superexponentially converging estimate.

Theorem 32. *If $T : [0,1] \to [0,1]$ is a C^ω piecewise expanding Markov map with absolutely continuous invariant measure μ then there exists $C > 0$ and $0 < \theta < 1$ with $|\lambda_n - \int \log |T'| \, d\nu| \leq C\theta^{n^2}$*

Example 58. *Consider the family $T_{\frac{1}{4\pi}} : [0,1] \to [0,1]$ defined by*

$$T_{\frac{1}{4\pi}}(x) = 2x + \varepsilon \sin 2\pi x \ (mod\ 1),$$

for $-\frac{1}{2\pi} < \varepsilon < \frac{1}{2\pi}$.

=2.0in graphexponent.eps

A plot of the non-linear analytic expanding map of the interval $T_{\frac{1}{4\pi}}(x) = 2x + \varepsilon \sin 2\pi x \ (mod\ 1)$

We can estimate the Lyapunov exponent $\int \log |T'_{1/4\pi}| \, d\nu$ in terms of the estimates

$$\lambda_n \to \int \log |T'_{1/4\pi}| \, d\nu \ \textit{[super-exponential rate]}$$

n	*using* λ_n
6	0.6837719
7	0.68377196
8	0.68377196024
9	0.6837719602421451
10	0.6837719602421451396
11	0.683771960242145139619160
12	0.6837719602421451396191607 1

Chapter 11

Besicovich and Multifractal Analysis

Besicovich studied the dimension of the set of points in the unit interval for which the frequency of the digits takes given values. For the purposes of illustration, we will consider the dyadic expansions, to base 2. Given

$$x = \sum_{n=1}^{\infty} \frac{x_n}{2^n}$$

where $x_n \in \{0, 1\}$ we can ask what the frequency of the digits x_n.

Definition 25. *Given $\alpha \in \mathbb{R}$ we let*

$$A_\alpha = \left\{ x = \sum_{n=1}^{\infty} \frac{x_n}{2^n} \ : \ \lim_{N \to +\infty} \frac{1}{N} \sum_{n=1}^{N} x_n = \alpha \right\}.$$

For a full measure set of x we can show using the Birkhoff Ergodic Theorem that for $\alpha = \frac{1}{2}$ the set $A_{1/2}$ has full Lebesgue measure.

Theorem 33 (Besicovich)**.** *For any $0 < \alpha < 1$,*

$$\dim(A_\alpha) = -\alpha \log \alpha - (1 - \alpha) \log(1 - \alpha).$$

The proof uses the following result.

Lemma 48. *Let ν be a probablity measure and let $f_n \in L^2(X, \nu)$ be an othogonal family of functions, i.e., $\int f_i j_j d\nu(x) = 0$ for $i \neq j$, with $\int |f_i(x)| d\nu(x) \leq 1$. Then*

$$\frac{1}{N} \sum_{n=1}^{N} f_n(x) \to 0$$

for almost all x (with respect to ν.)

We can define a measure ν which on the dyadic intervals

$$\mu\left(\left[\sum_{n=1}^{N}\frac{x_n}{2^n},\sum_{n=1}^{N}\frac{x_n}{2^n}+\frac{1}{2^N}\right]\right)=\alpha^{\sum_{n=1}^{N}x_n}(1-\alpha)^{N-\sum_{n=1}^{N}x_n}.$$

We begin with then following observation which illustrates why this measure is useful.

Claim 1. $\nu(A_\alpha)=1$

Proof of claim 1. Let us define $f_n(x)=x_n-\alpha$. Then A_α is the set of points x for which

$$\frac{1}{N}\sum_{n=1}^{N}f_n(x)\to 0.$$

To show this is a set of full ν measure we want to apply the previous lemma. This requires proving two properties:

1. $\int f_n^s d\nu \le \|f_n\|_\infty \le 1$.
2. To show orthogonality, we can write

$$\begin{aligned}\int f_n f_m d\nu &= \int (x_n-\alpha)(x_n-\alpha)d\nu(x)\\ &= (-\alpha)^2\nu([0,1/4])-\alpha(1-\alpha)(\nu[1/4,3/4])+\alpha^2\nu([3/4,1])\end{aligned}$$

   but then $\nu([0,1/4])=\alpha^2$, $\nu([3/4],1)=(1-\alpha)^2$ and $\nu[1/4,3/4]=2\alpha(1-\alpha)$. Thus we can see $\int f_n f_m d\nu=0$ for $n\neq m$.

We next need Billingsley's lemma. Let $I_n(x)$ be the nth level dyadic interval containing x.

Lemma 49. *Billingsley] Let μ be a finite measure on $[0,1]$. Let $A\subset[0,1]$ with $\nu(A)>0$. Let $0\le\alpha_1\le\alpha_2$ with*

$$\alpha_1\le\liminf_{n\to+\infty}\frac{\log\mu(I_n(x))}{\log(1/2^n)}\le\alpha_2$$

for all $x\in A$ then $\alpha_1\le dim_H(A)\le\alpha_2$.'

Proof. The upper bound implies that

$$\limsup_{n\to+\infty}\frac{\mu(I_n(x))}{1/2^{n\alpha_2}}\ge 1$$

We can fix $\epsilon>0$ and choose an open set $V\supset A$ with $\mu(V)$ close to $\mu(A)$. For every $x\in A$ we can choose n sufficiently large that

$$\frac{\mu(I_n(x))}{1/2^{n\alpha_2}}>C.$$

covers A and let $\{J_k\}$ be a disjoint subcover. Let $n(x)$ be the smallest n satisfying this inequality and als $2^n < \epsilon$ and $I_n(x) \subset V$.

By assumption $\{I_{n(x)}(x)\}$ covers A and let $\{J_k\}$ be a disjoint subcover. Clearly $\mathrm{diam}(J_k) \leq \epsilon$ for each k, and

$$\sum_k |J_k|^\alpha \leq \sum_k \mu J_k) \leq \mu(V)$$

In particular, we can deduce

$$H^\alpha_\epsilon(A) \leq \mu(A)/C \tag{3}$$

and since the right hand side of (3) is independent of ϵ we have that

$$H^\alpha(A) \leq \mu(A)/C \tag{3}$$

The lower bound implies

$$\limsup_{n\to+\infty} \frac{\mu(I_n(x))}{1/2^{n\alpha_1}} \leq 1.$$

Let

$$A_m = \{x \in A : \mu(I_n(x)) < C\mathrm{diam}(|I_n(x)|) \text{ for all } n > m\}$$

Since $A = \cup_m A_m$ and $A_{m+1} \supset A_m$ we have that $\mu(A) = \lim_{m\to+\infty} \mu(A_m)$ and thus it suffices to prove the result for A_m.

Fix $\epsilon < 2^{-m}$ and consider an cover of A by dyadic intervals. Then

$$\sum_k |J_k|^\alpha \geq \sum_k \mu(J_k) \geq \frac{\mu(A_m)}{C}$$

Thus $\mathcal{H}^\alpha_\epsilon(X) \geq \frac{\mu(A_m)}{C}$. Letting $\epsilon \to 0$ and $m \to +\infty$ gives the result. $\square$

Chapter 12

IFS and overlaps

12.1 One dimensional Iterated Function Schemes with overlaps

In this chapter we shall consider one dimensional iterated function schemes with over laps (i.e., such that the Open set condition fails). In this context we will concentrate on two particular examples. We will be interested in: the Hausdorff dimension of the limit set; and the properties of naturally associated measures (absolute continuity, dimension, etc.), The key tool in our study here is the application of the so called "transversality method" which helps in showing certain integrals are finite. We have already seen this in another guise, in the proofs in the previous chapter.

7.1 Transversality: Properties of Power Series A general result about when specific power series satisfy a transversality condition is given. Let F_b be a family of analytic functions such that $f(0) = 1$ and whose coefficients are real numbers that lie all in an interval $[-b, b]$, for some $b > 0$, i.e.,

$$F_b = \left\{ f(t) = 1 + \sum_{k=1}^{\infty} c_k t^k \, : \, c_k \in [-b, b] \right\}.$$

In practise, we shall only need to consider the case where $b \in \mathbb{N}$. Of course, every function $f \in F_b$ converges on the interval $(-1, 1)$. [1] We now define,

$$y(b) = \min\{x > 0 : \exists f \in F_b \text{ where } f(x) = f'(x) = 0\},$$

i.e., the first occurrence of a double zero for any function F_b.

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[1] Of course, the power series converges on the unit disk D on the complex plane. As an aside, we recall that any analytic function $F : D \to \mathbb{C}$ which is simple (i.e., it is one-one onto its image) must necessarily have a bound on its coefficients of the form $|c_k| \leq k$ (Bieberbach Conjecture)

The dotted line shows the function which has the first double zero (at $y(b)$). Any other function which gets δ-close to the horizontal axis before $y(b) - \epsilon$ must have slope at least δ (in modulus).

The basic idea is that we can deal with real valued functions $f \in F_b$ on an interval $[0, y(b) - \epsilon]$, for any $\delta > 0$, which have the property that when they cross the x-axis their slope has to be bounded away from zero. For example, when $\delta > 0$ a function is said to be δ-transversal if whenever its graph comes within δ of t-axis then its slope is at most $-\delta$ or at least δ (i.e, $|f(t)| \leq \delta$ implies $|f'(t)| \geq \delta$). In particular, given $\epsilon > 0$ we can find $\delta = \delta(\epsilon)$ such that every $f \in F_b$ is δ-transversal on $[0, y(b) - \epsilon]$.

Claim It is possible to numerically compute $y(1) \approx 0.649\ldots$ and also to show that $y(2) = 0.5$.

Example Consider the series $f(t) = 1 - \sum_{k=1}^{\infty} t^k = 1 - \frac{t}{1-t} \in F_1$ (with $b = 1$). The first zero is at $t = \frac{1}{2} < y(1)$ but the derivative $f'(t) = -\frac{1}{(1-t)^2}$ takes the value $f'(\frac{1}{2}) = -4 < 0$.

Approach to Claim To illustrate the method consider the case $b = 1$. The basic idea is to consider functions $h \in F_1$ of the special form

$$h(x) = 1 - \underbrace{\sum_{i=1}^{k-1} x^i}_{\frac{x - x^{k+1}}{1-x}} + a_k x^k + \underbrace{\sum_{i=k+1}^{\infty} x^i}_{\frac{x^{k+1}}{1-x}} \tag{7.1}$$

with $|a_k| \leq 1$. We claim that if we can find any such function, a value $0 < x_0 < 1$ and $0 < \delta < 1$ such that $h(x_0) > \delta$ and $h'(x_0) < -\delta$ then $y(1) \geq x_0$. More precisely, for $f \in F_b$ we have that if $g(x) < \delta$ then $g'(x) < -\delta$.

Observation: By construction $h''(x)$ is a power series with at most one sign change, and thus has at most one zero on $(0, 1)$. In particular, $h(x) > \delta$ and $h'(x) < -\delta$ for all $0 \leq x \leq x_0$.

There are two cases to consider:

If $k = 1$ then $h'(0) = a_1$. In particular, $h'(0) < h'(x_0) < -\delta$ (by the observation above); and

If $k \neq 1$ Then $h'(0) = -1 < -\delta$.

Let $g \in F_b$ and let

$$f(t) := g(t) - h(t) = 1 + \sum_{i=1}^{k-1} \underbrace{(b_n - 1)}_{c_i \geq 0} t^i - + \underbrace{(a_k - b_k)}_{c_k} t^k - \sum_{i=l+1}^{\infty} \underbrace{(1 - b_i)}_{c_i \geq 0} t^i. \tag{7.2}$$

Since for $0 \leq x \leq x_0$ we have $h(x) > \delta$ then if $g(x) < \delta$ we have that $f(x) = g(x) - h(x) < 0$. However, because of the particular form of $f(x)$ in (7.2), with positive coefficients followed by negative coefficients, one easily sees that $f(x) < 0$ implies $f'(x) = g'(x) - h'(x) < 0$. Finally, since by the observation $h'(x) < -\delta$ we deduce that $g'(x) < -\delta$, as required.

In particular, if let

$$h(x) = 1 - x - x^2 - x^3 + \frac{1}{2}x^4 + \sum_{i=5}^{\infty} x^i$$

then one can check that $h(2^{-2/3}) > 0.07$ and $h'(2^{-2/3}) < -0.09$ and so $y(1) \geq 2^{-2/3}$ A more sophisticated choice of $h(x)$ leads to the better bounds described above. □

A general result shows the following.

Proposition 7.1 The function $y : [1, \infty) \to [0, 1]$ is strictly decreasing, continuous and piecewise algebraic function. Moreover,

$y(b) \geq (\sqrt{b} + 1)^{-1}$ for $1 \leq b < 3 + \sqrt{8}$; and

$y(b) = (\sqrt{b} + 1)^{-1}$ for $b \geq 3 + \sqrt{8}$ The proof uses a variation on the proof of the claim above.

The following technical corollary is crucial when trying to use the transversality technique to calculate the dimension or measure of the limit sets for self-similar sets.

Proposition 7.2 ("Transversality Lemma") Let $b > 0$.

Given $0 < s < 1$ there exists $K > 0$ such that

$$\int_0^{y(b)} \frac{\mathrm{d}\lambda}{|f(\lambda)|^s} \leq K,$$

for all $f \in F_b$;

There exists $C > 0$ such that,

$$\mathcal{L}\rceil\lfloor\{0 \leq \lambda \leq y(b) : |f(\lambda)| \leq \epsilon\} \leq C\epsilon.$$

for all $f \in F_b$ and all sufficiently small $\epsilon > 0$. Proof To see part (1), we can write

$$[0, y(b)] = \underbrace{\{x \in [0, y(b)] : |f(x)| > \delta\}}_{=:S_1} \cup \underbrace{\{x \in [0, y(b)] : |f'(x)| > \delta\}}_{=:S_2}.$$

In particular, we can bound

$$\int_0^{y(b)} \frac{\mathrm{d}\lambda}{|f(\lambda)|^s} \leq \int_{S_1} \frac{\mathrm{d}\lambda}{|f(\lambda)|^s} + \int_{S_2} \frac{\mathrm{d}\lambda}{|f(\lambda)|^s} \leq \frac{1}{\delta^s} + \frac{1}{\delta^s}$$

For part (2) we need only observe that if $|f(x)| \leq \epsilon \leq \delta$ then x is contained in an interval I upon which $-\epsilon \leq f(t) \leq \epsilon$ is monotone and, by δ-transversality, we have that $|f'(t)| \geq \delta$. In particular, the length of I is at most $(2/\delta)\epsilon$ and I contains a zero. The result easily follows form the observation that the number of zeros of f is uniformly bounded. (For example, by Jenson's formula from complex analysis the number $n(x_0)$ of zeros $z_1, \cdots, z_{n(x_0)}$ (ordered by modulus) of $f(z)$ with $|z_i| < x_0$ satisfies

$$\prod_{i=1}^{n(x_0)} \frac{x_0}{|z_i|} = \exp\left(\int_0^{2\pi} \log|f(re^{i\theta})| d\theta\right) \leq 1 + \frac{bx_0}{1-x_0}$$

and we also have

$$\prod_{i=1}^{n(x_0)} \frac{x_0}{|z_i|} \geq \prod_{i=1}^{n(x_0-\epsilon)} \frac{x_0}{|z_i|} \geq \left(\frac{x_0-\epsilon}{x_0}\right)^{n(x_0-\epsilon)}.$$

Comparing these two expressions gives a uniform bound. □The first part is extremely useful when proving theorems involving generic conclusions. The second part is useful in the case when we wish to show that a class of self-similar sets have positive Lebesgue measure for almost all parameter values.

7.2 The $\{0,1,3\}$-Problem We want to describe the dimension of certain self-similar sets where the images of the similarities overlap. Given $0 < \lambda < 1$, let $\{T_0, T_1, T_2\}$ be an iterated function scheme on $\mathbb{R}$ where,

$$T_0(x) = \lambda x T_1(x) = \lambda x + 1 T_2(x) = \lambda x + 3.$$

Observe that:

"(i)" For $\lambda \in (0, \frac{1}{4})$ the Open Set Condition holds (since $T_i([0,1]) \cap T_j([0,1]) = \emptyset$, for $i \neq j$) and the dimension of the associated limit set $\Lambda(\lambda)$ is $\dim_H \Lambda(\lambda) = \dim_B \Lambda(\lambda) = -\frac{\log 3}{\log \lambda}$, by Moran's Theorem.

"(ii)" When $\lambda \in (\frac{1}{4}, \frac{1}{3})$ the Open Set Condition does not hold, and we only know that $\dim_H \Lambda(\lambda) \leq \dim_B \Lambda(\lambda) \leq -\frac{\log 3}{\log \lambda}$. The problem of whether $\dim_H \Lambda(\lambda) = -\frac{\log 3}{\log \lambda}$ holds for a specific value of λ is far from well understood, in general. This class of problems was studied by Keane, Smorodinsky and Solomyak. In particular they showed:

"(iii)" For $\frac{2}{5} < \lambda < 1$ we have that $\Lambda(\lambda)$ is an open interval.

A generic description of the behaviour of $\dim_H(\Lambda(\lambda))$ in the region $(\frac{1}{4}, \frac{1}{3})$ is given by the following result.

Theorem 7.3

"(a)" For almost all $\lambda \in (\frac{1}{4}, \frac{1}{3}]$,

$$\dim_H \Lambda(\lambda) = \dim_B \Lambda(\lambda) = -\frac{\log 3}{\log \lambda};$$

and

"(b)" There is a dense set of values $D \subset (\frac{1}{4}, \frac{1}{3}]$ such that for $\lambda \in D$ we have that $\dim_H \Lambda(\lambda) \leq \dim_B \Lambda(\lambda) < -\frac{\log 3}{\log \lambda}$

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In the range $0 < \lambda \leq \frac{1}{4}$ we always have $\dim_H \Lambda(\lambda) = -\log 3/\log \lambda$; but for $\frac{1}{4} < \lambda \leq \frac{1}{3}$ we only know the result for a.e. λ; for $\frac{2}{5} < \Lambda < 1$ we always have $\dim_H \Lambda(\lambda) = 1$.

Proof To prove part (a), it is first easy to see from the definitions that $\dim_H \Lambda(\lambda) \leq \dim_B \Lambda(\lambda) \leq -\frac{\log 3}{\log \lambda}$. We now consider the opposite inequality. Let $\mu = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})^{\mathbb{Z}^+}$ be the usual $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$-Bernoulli measure on the space of sequences $\Sigma = \{0, 1, 2\}^{\mathbb{Z}^+}$. For any $0 < \lambda < 1$ we can define the map $\Pi_\lambda : \Sigma \to \mathbb{R}$ by

$$\Pi_\lambda(\underline{i}) = \sum_{k=0}^{\infty} i_k \lambda^k.$$

Thus on each possible attractor $\Lambda(\lambda)$ a self-similar measure ν_λ can be defined by $\nu_\lambda = \mu \circ \Pi_\lambda^{-1}$. Given $\epsilon > 0$ let $s_\epsilon(\lambda) = -\frac{\log 3}{\log(\lambda + \epsilon)}$. Note that the proof can be completed (as in the proofs in the previous chapter) if it can be shown that,

$$I = \int_{\frac{1}{4}}^{\frac{1}{3}} \left(\int\int \frac{d\nu_\lambda(x) d\nu_\lambda(y)}{|x - y|^{s_\epsilon(\lambda)}} \right) d\lambda < \infty$$

for all $\epsilon > 0$. In particular, the finiteness of the integrand, for almost all λ, allows us to deduce that for those values $\dim_H \Lambda(\lambda) \geq s_\epsilon(\lambda)$. Since the value of $\epsilon > 0$ is arbitrary, we get the lower bound $\dim_H \Lambda(\lambda) \geq -\frac{\log 3}{\log \lambda}$.

Using the map Π_λ the inner two integrals can be rewritten in terms of the measure μ on Σ and we can rewrite the last expressions as

$$I = \int_{\frac{1}{4}}^{\frac{1}{3}} \left(\int\int \frac{d\mu(\underline{i}) d\mu(\underline{j})}{|\Pi_\lambda(\underline{i}) - \Pi_\lambda(\underline{j})|^{s_\epsilon(\lambda)}} \right) d\lambda.$$

We then turn I into a product of two expressions. More precisely, let $t = \max_{\frac{1}{4} \leq \lambda \frac{1}{3}} s_\epsilon(\lambda)$ and note that $t < 1$. In particular, if $\underline{i} \neq \underline{j}$ then they agree until the $|\underline{i} \wedge \underline{j}|$-th term and we can write

$$|\Pi_\lambda(\underline{i}) - \Pi_\lambda(\underline{j})|^{s_\epsilon(\lambda)} = \lambda^{|\underline{i} \wedge \underline{j}| s_\epsilon(\lambda)} \left(\sum_{k=0}^{\infty} a_k \lambda^k \right)^{s_\epsilon(\lambda)} \geq \left(\frac{1}{3} + \epsilon \right)^{s_\epsilon(\lambda) |\underline{i} \wedge \underline{j}|} \left(\sum_{k=0}^{\infty} a_k \lambda^k \right)^t,$$

where $\{a_k\}_{k \in \mathbb{Z}^+}$ is the sequence $a_k := i_{k+|\underline{i} \wedge \underline{j}|} - j_{k+|\underline{i} \wedge \underline{j}|} \in \{0, \pm 1, \pm 2, \pm 3\}$ and $a_0 \neq 0$. Substituting this back into the integrand in I and using Fubini's Theorem we get

$$I \leq \int_\Sigma \int_\Sigma \frac{d\mu(\underline{i}) d\mu(\underline{j})}{\left(\frac{1}{3} + \epsilon\right)^{|\underline{i} \wedge \underline{j}|}} \left(\int_{\frac{1}{4}}^{\frac{1}{3}} \frac{d\lambda}{\left(\sum_{k=0}^{\infty} a_k \lambda^k\right)^t} \right). \tag{7.3}$$

We can estimate the first integral in (7.3) by

$$\int\int \frac{\mathrm{d}\mu(\underline{i})\mathrm{d}\mu(\underline{j})}{\left(\frac{1}{3}+\epsilon\right)^{|\underline{i}\wedge\underline{j}|}} \leq \sum_{k=0}^{\infty} \sum_{[i_0,i_1,\ldots,i_{k-1}]} \frac{\mu([i_0,i_1,\ldots,i_{k-1}])}{\left(\frac{1}{3}+\epsilon\right)^k} = \sum_{k=0}^{\infty} \frac{\frac{1}{3}^k}{\left(\frac{1}{3}+\epsilon\right)^k} < \infty.$$

Thus to show that $I < \infty$ it remains to bound the second integral in (7.3) by

$$\int \frac{\mathrm{d}\lambda}{\left(\sum_{k=0}^{\infty} a_k\lambda^k\right)^t} < \infty$$

for any sequence $\{a_k\}_{k\in\mathbb{Z}^+}$, where $a_k \in \{0, \pm 1, \pm 2\}$ and $a_0 \neq 0$. Let $f(\lambda) = 1 + \sum_{k=0}^{\infty} \left(\frac{a_k}{a_0}\right) \lambda^k$ then we can apply part (1) of Proposition 7.1 to deduce that the integral is finite, since $y(2) \geq \frac{1}{3}$.

To prove part (b), we need only observe that if for some n we can find distinct $(i_1,\ldots,i_n), (j_1,\ldots,j_n) \in \{0,1\}^n$ such that

$$\sum_{k=1}^{n} i_k\lambda^k = \sum_{k=1}^{n} j_k\lambda^k$$

then at the n-th level of the construction at least two of the 2^n intervals of length λ^n coincide. In particular, it is easy to see that

$$\dim_B(\Lambda(\lambda)) \leq -\frac{n-1}{n}\frac{\log 3}{\log\lambda}.$$

It is then an easy to matter to show that the set D of such λ is dense in $(\frac{1}{4}, \frac{1}{3})$. □

Remark It is also possible to show a corresponding result where generic λ is understood in a topological sense: for λ is a dense G_δ set (i.e., a countable intersection of open dense sets).

Remark Of course one can prove somewhat similar results where $\{0, 1, 3\}$ is replaced by some other finite set of numbers. These are usually called *deleted digit expansions.*

7.3 The Erdös-Solomyak Theorem We recall some results about the properties of self-similar measures. Let $\lambda \in (0, 1)$. We let,

$$T_0(x) = \lambda x T_1(x) = \lambda x + 1.$$

Let $\nu = \nu_\lambda$ be a measure such that for all $J \subset \left[0, \frac{1}{1-\lambda}\right]$,

$$\nu(J) = \frac{1}{2}\nu(T_0^{-1}(J)) + \frac{1}{2}\nu(T_1^{-1}(J)). \tag{7.4}$$

In fact, is unique probability measure satisfying this identity called the *self-similar measure.* Equivalently, we say this is a Bernoulli convolution with respect to $\underline{p} = (\frac{1}{2}, \frac{1}{2})$.

In particular, we wish to know whether the measures ν_λ are absolutely continuous or not (i.e., whenever B is a Borel set with $\mathrm{Leb}(B) = 0$ then $\nu_\lambda(B) = 0$). To begin with, it is an easy exercise to see that if $0 < \lambda < \frac{1}{2}$ then the Iterated Function Scheme $\{T_0, T_1\}$ satisfies the Open Set Condition, thus $\Lambda(\lambda)$ is a Cantor set with

$$\dim_H(\Lambda(\lambda)) = -\frac{\log 3}{\log \lambda},$$

by Moran's Theorem and, in particular, has zero Lebesgue measure. Thus ν_λ is singular with respect to Lebesgue measure.

Jessen-Wintner Theorem The measure ν_λ is either absolutely continuous or singular with respect to Lebesgue measure Leb (i.e, either every set B with $Leb(B) = 0$ satisfies $\nu_\lambda(B) = 0$, or there exists a set B with $Leb(B) = 0$ and $\nu_\lambda(B) = 1$).

Proof Every measure ν_λ can be written in the form $\nu_\lambda = \nu^{abs} + \nu^{sing}$, where $\nu^{abs} << Leb$ and $\nu^{sing} \perp Leb$ (This is the Lebesgue decomposition theorem). However, substituting into (7.4) we see that both ν^{abs} and ν^{sing} satisfy the identity. By uniqueness we have that one of them must be zero. $\square$

Next we recall one of the classical theorems in Harmonic Analysis. Let us define the Fourier transform $\widehat{\nu} : \mathbb{R} \to \mathbb{R}$ by

$$\widehat{\nu}(u) = \int e^{iut} d\nu(t), \text{ for } u \in \mathbb{R}.$$

The following result describes the behaviour of $\widehat{\nu}(u)$ as $|u| \to +\infty$.

Riemann-Lebesgue Theorem If the measure ν is absolutely continuous then $\widehat{\nu}(u) \to 0$ as $|u| \to +\infty$.

We can use the Riemann-Lebesgue Theorem to show that for some value of $\lambda \in [\frac{1}{2}, 1]$ the measure ν_λ is singular.

Pisot Numbers We recall that $\theta > 1$ is an algebraic integer if it is a zero of a polynomial $P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ with $a_{n-1}, \ldots, a_0 \in \mathbb{Z}$. Let $\theta_1, \ldots, \theta_{n-1} \in \mathbb{C}$ be the other roots of $P(x)$. We call λ a *Pisot Number* if $|\theta_1|, \cdots, |\theta_{n-1}| < 1$.

Clearly, there are at most countably many Pisot numbers (since there are at most countably many such polynomials $P(x)$). The smallest Pisot numbers are $\theta = 1.3247\cdots$ (which is a root for $x^3 - x - 1$) and $\theta = 1.3802\cdots$ (which is a root for $x^3 - x - 1$). However, perhaps the most important feature of these numbers is the following:

$$\min_{k \in \mathbb{N}} |\theta^n - k| = O(\Theta^n) \text{ as } n \to +\infty$$

where $\Theta = \max\{|\theta_1|, \ldots, |\theta_{n-1}|\} < 1$.

The following highly influential Theorem was published by Erdös in 1939.

Erdös's Theorem If $\theta := 1/\lambda$ is a Pisot number then the measure ν_λ is singular.

Proof This is based on the study of the Fourier Transform of the measure ν_λ. In fact, if we let $\delta(x)$ be the Dirac measure on $x \in \mathbb{R}$ then

$$\frac{1}{2^n} \sum_{i_1 \ldots i_n \in \{0,1\}} \delta\left(\sum_{j=1}^n i_j \lambda^j\right) \to \nu_\lambda$$

(where convergence is in the weak star topology) as $n \to +\infty$, and so we can write

$$\widehat{\nu_\lambda}(u) := \int_{-\infty}^{\infty} e^{itx} d\nu_\lambda(x) = \lim_{n\to\infty} \prod_{k=0}^n \left(\frac{e^{-iu\lambda^k} + e^{iu\lambda^k}}{2}\right)$$

For a Pisot number θ we can choose for each $n \geq 1$ a natural number $k_n \in \mathbb{N}$ such that $|\theta^n - k_n| = O(\Theta^{-n})$. In particular, if we let $u \in \mathbb{N}$ then we can show that there exists $c > 0$ such that

$$\prod_{k=0}^n \left(\frac{e^{-iu\lambda^k} + e^{iu\lambda^k}}{2}\right) > c \text{ for all } n \geq 0$$

In particular, we can bound $\inf_{m\in\mathbb{N}} \nu_\lambda(m) > 0$. Thus $\nu_\lambda(u) \not\to 0$ as $u \to +\infty$. By the Riemann-Lebesgue Lemma ν_λ is not absolutely continuous. Thus, by the Jessen-Wintner theorem, we deduce that ν_λ is singular. □

Erdös also showed the following:

"(i)" If $\lambda = 2^{-1/k}$, for some $k \geq 1$, then ν_λ is absolutely continuous; and

"(ii)"There exists $\epsilon > 0$ such that for almost all $\lambda \in [1-\epsilon, 1]$ the measure ν_λ is absolutely continuous. He went onto conjecture that for almost all $\lambda \in [\frac{1}{2}, 1]$ the measure is absolutely continuous. This was eventually proved in 1995 by Solomyak:

Erdös-Solomyak Theorem For almost all $\lambda \in [\frac{1}{2}, 1]$ the measure ν_λ is absolutely continuous.

There is a useful criteria for the measure ν_λ to be absolutely continuous.

Absolute Continuity Lemma The measure ν_λ is absolutely continuous if

$$\int \left(\liminf_{r\to 0} \frac{\nu_\lambda(B(x,r))}{2r}\right) \mathrm{d}\nu_\lambda(x) < \infty.$$

Proof of the Absolute Continuity theorem From the hypotheses we see that for a.e. (ν_λ) x we have that $\underline{D}(x) := \left(\liminf_{r\to 0} \frac{\nu_\lambda(B(x,r))}{2r}\right) < +\infty$. It therefore suffices to show that if $\mathrm{leb}(A) = 0$ and $u > 0$, then the set $X_u := \{x \in A : \underline{\mathrm{D}}(\mathrm{x}) \leq u\}$ satisfies $\nu_\lambda(X_u) = 0$.

Let us fix $\epsilon > 0$. For each $x \in X_u$ we can choose a sequence $r_i \searrow 0$ with $\mu(B(x, r_i))/2r_i \leq u + \epsilon$. Let us denote $A = X_u$. By the Besicovitch covering lemma, we can choose a cover $\{B_i\}$ with is a union of two families $\{B_i^{(0)}\} \cup \{B_i^{(1)}\}$ (each of which consists of balls which are pairwise disjoint). In particular, let us assume that $\mu(\cup_i B_i^{(0)}) > \frac{1}{2}$. In particular, we can bound

$$\mu(A - \cup_i B_i^{(0)}) \leq \mu(A) - \mu(\cup_i B_i^{(0)}) \leq \frac{1}{2}\mu(A),$$

for $\eta > 0$. We can proceed inductively, replacing A by $A - \cup_i B_i^{(0)}$. Finally, taking the union of the families of balls at each step we arrive at a countable family of balls $\{B_i\}$ such that:

$\mu(X_u - \cup_i B_i) = 0$; and

$\mu(B_i) \leq (u+\epsilon)\lambda(B_i) = (u+\epsilon)2r_i$ In particular,

$$\mu(X_u) \leq \sum_i \mu(B_i) \leq (u+\epsilon)\sum_i \lambda(B_i) \leq (u+\epsilon)(\mathrm{leb}(X_u)+\epsilon).$$

In particular, since $\epsilon > 0$ is arbitrary we have that $\mu(X_u) \leq u\mathrm{leb}(X_u) = 0$. □

We follow a variation on Solomyak's original proof (due to Peres and Soloymak) which makes use of this lemma.

Proof of the Erdös-Solomyak Theorem We will also let $\mu = (\frac{1}{2}, \frac{1}{2})^{\mathbb{Z}^+}$ be the usual $(\frac{1}{2}, \frac{1}{2})$-Bernoulli measure defined on the sequence space, $\Sigma = \{0,1\}^{BbbZ^+}$. As usual, we let $\Pi_\lambda : \{0,1\}^{\mathbb{N}} \to \mathbb{R}$ be defined by,

$$\Pi_\lambda(\underline{i}) = \sum_{n=0}^{\infty} i_n \lambda^n.$$

We can also write $\nu_\lambda = \Pi_\Lambda \mu$ (i.e., $\nu_\lambda(B) = \mu(\Pi_\Lambda^{-1}B)$ for all intervals $B \subset \mathbb{R}$).

To begin with, we want to show that ν_λ is absolutely continuous for a.e. $\lambda \in \left(\frac{1}{2}, y(2)\right)$, where $y(2) = 0.68\cdots$. In this case, it is sufficient to show for any $\epsilon > 0$

$$I = \int_{\frac{1}{2}+\epsilon}^{y(2)} \left(\int \liminf_{r\to 0} \frac{\nu_\lambda(B(x,r))}{2r} \mathrm{d}\nu_\lambda(x) \right) \mathrm{d}\lambda < \infty.$$

In particular, since $\epsilon > 0$ is arbitrary we can then deduce that for almost every $\lambda \in \left(\frac{1}{2}, y(2)\right)$ we have that the integrand is finite. Thus for such λ we can apply the previous lemma to deduce that ν_λ is absolutely continuous, as required.

The first step is to apply Fatou's Lemma (to move the lim inf outside of the integral) and then reformulate the integral in terms of integrals on the sequence space Σ. Thus

$$I \leq \liminf_{r\to 0} \frac{1}{2r} \int_{\frac{1}{2}+\epsilon}^{y(2)} \left(\int \nu_\lambda(B(x,r))\mathrm{d}\nu_\lambda(x) \right) \mathrm{d}\lambda \leq \liminf_{r\to 0} \frac{1}{2r} \int_{\frac{1}{2}+\epsilon} \left(\int_\Sigma^{y(2)} \int_\Sigma {}_{\{\omega,\tau:|\Pi_\lambda(\omega)-\Pi_\lambda(\tau)|\leq r\}} \mathrm{d}\mu(\omega)\mathrm{d}\mu(\tau$$

Applying Fubini's Theorem bounds I (to switch the oder of the integrals) gives

$$I \leq \liminf_{r\to 0} \frac{1}{2r} \int_\Sigma \int_\Sigma \mathcal{L}\rceil\lfloor \left\{ \lambda \in \left(\frac{1}{2}+\epsilon, y(2)\right) \,:\, |\Pi_\lambda(\omega) - \Pi_\lambda(\tau)| \leq r \right\} \mathrm{d}\mu(\omega)\mathrm{d}\mu(\tau).$$

To simplify this bound observe that

$$|\Pi_\lambda(\omega) - \Pi_\lambda(\tau)| = \lambda^{|\omega\wedge\tau|} g(\lambda)$$

where $g(\lambda) \in F_\lambda$ for all $\omega, \tau \in \Sigma$. Thus by definition of $y(2)$ and Proposition 7.2 we have that

$$\mathcal{L}\rceil\lfloor \left\{ \lambda \in \left(\frac{1}{2}+\epsilon, y(2)\right) : \, |\Pi_\lambda(\omega) - \Pi_\lambda(\tau)| \leq r \right\} \leq 2C \left(\frac{1}{3}+\epsilon\right)^{|\omega\wedge\tau|} r$$

for some $C > 0$. This allows us to bound:

$$I \leq C \int_\Sigma \int_\Sigma \left(\frac{1}{2}+\epsilon\right)^{-|\omega\wedge\tau|} \mathrm{d}\mu(\omega)\mathrm{d}(\tau) \leq C \int_\Sigma \left(\sum_{n=0}^{\infty} \frac{1}{2^n} \left(\frac{1}{2}+\epsilon\right)^{-n} \right) \mathrm{d}(\tau) < +\infty$$

which can be seen to be finite by simply integrating on the shift space. Since $\epsilon > 0$ is arbitrary, this shows that ν_λ is absolutely continuous for a.e. $\lambda \in [\frac{1}{2}, y(2)]$.

We shall just sketch how to extend this result to the larger interval $[\frac{1}{2}, 1]$. Recall from the proof of Erdös's theorem that the Fourier transform of the measure ν_λ takes the form

$$\widehat{\nu}_\lambda(u) = \prod_{k=0}^{\infty} \left(\frac{e^{-iu\lambda^k} + e^{iu\lambda^k}}{2} \right)$$

and then we can write

$$\widehat{\nu}_\lambda(u) = \underbrace{\prod k = 0k \neq 2 \pmod{3}\infty \left(\frac{e^{-iu\lambda^k} + e^{iu\lambda^k}}{2} \right)}_{=:\widehat{\nu}'_\lambda} \times \prod k = 0k = 2 \pmod{3}\infty \left(\frac{e^{-iu\lambda^k} + e^{iu\lambda}}{2} \right.$$

Absolute continuity of ν'_λ would imply absolute continuity of ν_λ (since it is a classical fact that convolving an absolutely continuous measure with another measure gives an absolutely continuous measure again). However, modifying the above proof we can replace F_b be $F'_b \subset F_b$ in which the coefficients satisfy $c_{3i+1}c_{3i+2} = 0$ for all $i \geq 0$. For such sequences one can show that the region of transversality can be extended as far as $x_0 = 1/\sqrt{2}$ and so we can deduce that ν_λ is absolutely continuous for a.e. $\frac{1}{2} < \lambda < \frac{1}{\sqrt{2}}$. Finally, since we can write $\widehat{\nu}_\lambda(u) = \widehat{\nu}_{\lambda^2}(u)\widehat{\nu}_{\lambda^2}(\lambda u)$ we can deduce that ν_λ is also absolutely continuous for a.e. $\frac{1}{\sqrt{2}} < \lambda < \frac{1}{2^{1/4}}$. Proceeding inductively completes the proof. $\square$Remark The original proof of Solomyak used another result from Fourier analysis: If $\widehat{\nu}_\lambda \in L^2(\mathbb{R})$ then ν_λ is absolutely continuous and the Radon-Nikodym derivative $\frac{d\nu_\lambda}{dx} \in L^2(\mathbb{R})$. In particular, he showed the stronger result that for a.e. $\frac{1}{2} < \lambda < 1$ one has $\frac{d\nu_\lambda}{dx} \in L^2(\mathbb{R})$.

Remark It is also possible to show that for a.e. λ we have $\frac{d\nu_\lambda}{dx} > 0$ for a.e. $x \in [-\frac{1}{1-\lambda}, \frac{1}{1-\lambda}]$. 7.4 Dimension of the measures ν_λ Unlike the case of the $\{0,1,3\}$-problem, the limit set in the above example is an interval and thus its Hausdorff dimension holds no mystery. However, the dimension of the measure is still of some interest. We shall consider the slightly more general of different Bernoulli measures. Let $\underline{p} = (p_0, p_1)$ be a probability vector (i.e., $0 < p_0, p_1 < 1$ and let $p_0 + p_1 = 1$).

Let $\nu_\lambda = \nu_\lambda^{p_0,p_1}$ now denote the unique probability measure such that

$$\nu_\lambda(J) = p_0 \nu_\lambda(T_0^{-1}(J)) + p_1 \nu_\lambda(T_1^{-1}(J)).$$

for all $J \subset \left[0, \frac{1}{1-\lambda}\right]$.

The main result on these measures is the following.

Theorem 7.4

For almost all $\lambda \in [\frac{1}{2}, y(1) = 0.649\ldots]$,

$$\dim_H \nu_\lambda^{(p_0,p_1)} = \min\left(\frac{p_0 \log p_0 + p_1 \log p_1}{\log \lambda}, 1\right).$$

For almost all $\lambda \in [p_0^{p_0} p_1^{p_1}, y(1) = 0.649]$ we have that ν_λ is absolutely continuous.

Unfortunately, it is not possible to move past the upper bound $y(1)$ on these intervals using properties of the Fourier transform $\widehat{\nu}_\lambda$ (as in the previous section) because this function is not as well behaved in the case of general (p_0, p_1) as it was in the specific case of $(\frac{1}{2}, \frac{1}{2})$ in the Erdös-Solomyak Theorem.

Proof We shall show the lower bound on the dimension of the measure in part (1). The proof of Part (2) is similar to that in the special case $p_0 = p_1 = \frac{1}{2}$.

We let $\mu = \mu_{p_0,p_1} = (p_0, p_1)^{\mathbb{Z}^+}$ denote the usual (p_0, p_1)-Bernoulli measure defined on the sequence space, $\Sigma = \{0,1\}^{\mathbb{Z}^+}$. We again let $\Pi_\lambda : \Sigma \to \mathbb{R}$ be defined by,

$$\Pi_\lambda(\underline{i}) = \sum_{n=0}^{\infty} i_n \lambda^n.$$

As usual, we have that $\nu_\lambda^{(p_0,p_1)} = \mu^{(p_0,p_1)} \circ \Pi_\lambda^{-1}$. We shall use the following lemma.

Claim For any $\alpha \in (0,1]$ we have that for almost all $\lambda \in [0.5, y(1) = 0.649\ldots]$

$$\dim \nu_\lambda^{(p_0,p_1)} \geq \min\left(\frac{\log((p_0^{\alpha+1} + p_1^{\alpha+1})^{\frac{1}{\alpha}})}{\log \lambda}, 1\right).$$

Proof of Claim Fix (p_0, p_1) and let $\epsilon > 0$. For brevity of notation we denote $d(\alpha, \epsilon) = (p_0^{\alpha+1} + p_1^{\alpha+1} + \epsilon)^{\frac{1}{\alpha}}$. Let us write $S_\epsilon(\lambda) = \min\left(\frac{\log(d(\alpha,\epsilon))}{\log \lambda}, 1-\epsilon\right)$. We can first rewrite

$$I = \int_{0.5}^{y(1)} \int \left(\int \frac{\mathrm{d}\nu_\lambda(x)}{|x-y|^{S_\epsilon(\lambda)}} \right)^\alpha \mathrm{d}\nu_\lambda(y) \mathrm{d}\lambda = \int_{0.5}^{y(1)} \int \left(\int \frac{\mathrm{d}\mu(\underline{i})}{|\Pi_\lambda(\underline{i}) - \Pi_\lambda(\underline{j})|^{S_\epsilon(y)}} \right)^\alpha \mathrm{d}\mu(\underline{j}) \mathrm{d}\lambda.$$

To prove the claim it suffices to show that $I < +\infty$. Next we apply Fubini's theorem and Hölder's inequality $\int f^\alpha \leq C(\int f)^\alpha$ for $\alpha \in (0,1]$) to get

$$I \leq C \int \left(\int_{0.5}^{y(1)} \int \frac{\mathrm{d}\mu(\underline{i})\mathrm{d}\lambda}{|\Pi_\lambda(\underline{i}) - \Pi_\lambda(\underline{j})|^{s_\epsilon(\lambda)}} \right)^\alpha \mathrm{d}\mu(\underline{j}) \leq C_1 \int \left(\int_{0.5}^{y(1)} \int \frac{\mathrm{d}\mu(\underline{i})\mathrm{d}\lambda}{\left(\lambda^{|\underline{i}\wedge\underline{j}|} \, |a_0 + \sum_{n=1}^\infty a_n \lambda^n|^{s_\epsilon(\lambda}\right.}\right.$$

for some $C_1 > 0$, where $a_n \in \{-1, 0, 1\}$ for $n \geq 1$ and $a_0 \in \{-1, 1\}$. By transversality,

$$I \leq C_1 \int \left(\int_{0.5}^{y(1)} \int \frac{\mathrm{d}\mu(\underline{i})\mathrm{d}\lambda}{\left(d(\alpha,\epsilon)^{|\underline{i}\wedge\underline{j}|} \, |a_0 + \sum_{n=1}^\infty a_n \lambda^n|\right)^{s_\epsilon(\lambda)}} \right)^\alpha \mathrm{d}\mu(\underline{j}) \leq C_1 \int \left(\int_{0.5}^{y(1)} \frac{\mathrm{d}\lambda}{|a_0 + \sum_{n=1}^\infty a_n}\right.$$

for some $C_2 > 0$. Consider the inequality $(\sum_i b_i)^\alpha \leq \sum_i b_i^\alpha$ for $b_i > 0$ and $\alpha \in (0,1]$, then

$$I \leq C_2 \sum_{k=0}^\infty \sum_{w \in W_k} \frac{\mu(W)^{\alpha+1}}{d(\alpha,\epsilon)^{\alpha k}} \leq C_2 \sum_{k=0}^\infty d(\alpha,\epsilon)^{-\alpha k} (p_0^{\alpha+1} + p_1^{\alpha+1})^k.$$

Thus since $d(\alpha,\epsilon)^\alpha > p_0^{\alpha+1} + p_1^{\alpha+1}$ we have $I < \infty$ and hence, since the integrand must be finite almost everywhere, we deduce that

$$\dim \nu_\lambda \geq \min\left(\frac{d(\alpha,\epsilon)}{\log \lambda}, 1-\epsilon \right)$$

for almost all $\lambda \in [\frac{1}{2}, y(2)]$. To complete the proof of the claim we let $\epsilon = \frac{1}{n}$ for $n \in \mathbb{N}$ and let $n \to \infty$. $\square$

To complete the proof of the Theorem we let $\alpha_n = \frac{1}{n}$ for $n \in$ and observe that,

$$\lim_{n\to\infty} \frac{\log(p_0^{\alpha_n+1} + p_1^{\alpha_n+1})}{\alpha_n \log \lambda} = \frac{p_0 \log p_0 + p_1 \log p_1}{\log \lambda}.$$

7.5 The $\{0,1,3\}$ problem revisited: the measure ν_λ Finally, We can also consider the question of absolute continuity for the $\{0,1,3\}$ problem in the region $\lambda \in [\frac{1}{3}, y(2)]$. Let ν_λ be defined as before. The analogue of the Erdös-Solomyak theorem is the following.

Theorem 7.5 For a.e. $\lambda \in [\frac{1}{3}, y(2)]$ the measure ν_λ is absolutely continuous. In particular, $\Lambda(\lambda)$ has positive Lebesgue measure.

This result was also proved by Solomyak. The method of proof is very similar to that in the case of section 7.3 and we only outline the main steps. Thus to show that ν_λ is absolutely continuous for a.e. $\lambda \in \left(\frac{1}{3}, y(3)\right)$ it is sufficient to show for any $\epsilon > 0$

$$I = \int_{\frac{1}{3}+\epsilon}^{y(2)} \left(\int \liminf_{r \to 0} \frac{\nu_\lambda(B(x,r))}{2r} \mathrm{d}\nu_\lambda(x) \right) \mathrm{d}\lambda < \infty.$$

The first step is to apply Fatou's Lemma (to take the lim inf outside of the integral) and to rewrite this as an integral on Σ. Thus

$$I \leq \liminf_{r \to 0} \frac{1}{2r} \int_{\frac{1}{3}+\epsilon}^{y(2)} \left(\int \nu_\lambda(B(x,r)) \mathrm{d}\nu_\lambda(x) \right) \mathrm{d}\lambda \leq \liminf_{r \to 0} \frac{1}{2r} \int_{\frac{1}{3}+\epsilon} \left(\int_\Sigma^{y(2)} \int_\Sigma {}_{\{\omega,\tau: |\Pi_\lambda(\omega) - \Pi_\lambda(\tau)| \leq r\}} \mathrm{d}\mu(\omega) \mathrm{d}\mu(\tau$$

Applying Fubini's Theorem (to switch the order of the integrals) gives

$$I \leq \liminf_{r \to 0} \frac{1}{2r} \int_\Sigma \int_\Sigma \mathcal{L} \left\{ \lambda \in \left(\frac{1}{3} + \epsilon, y(2) \right) \, : \, |\Pi_\lambda(\omega) - \Pi_\lambda(\tau)| \leq r \right\} \mathrm{d}\mu(\omega) \mathrm{d}\mu(\tau).$$

As usual, one can write

$$|\Pi_\lambda(\omega) - \Pi_\lambda(\tau)| = \lambda^{|\omega \wedge \tau|} g(\lambda)$$

where $g(\lambda) \in F_2$ for all $\omega, \tau \in \Sigma$. Thus transversality gives that

$$\mathcal{L}\rceil\lfloor \left\{ \lambda \in \left(\frac{1}{3} + \epsilon, y(2) \right) : \, |\Pi_\lambda(\omega) - \Pi_\lambda(\tau)| \leq r \right\} \leq C \left(\frac{1}{3} + \epsilon \right)^{|\omega \wedge \tau|} r$$

for some $C > 0$. This gives,

$$I \leq \frac{C}{2} \int_\Sigma \int_\Sigma \left(\frac{1}{3} + \epsilon \right)^{-|\omega \wedge \tau|} \mathrm{d}\mu(\omega) \mathrm{d}(\tau) < +\infty$$

which can easily be seen to be finite, as in the earlier proofs.

Finally, we can consider a general Bernoulli measure $\mu = (p_0, p_1, p_2)^{\mathbb{Z}^+}$ on Σ and associate the probability measure $\nu_\lambda^{p_0,p_1,p_2} = \Pi_\lambda \mu$. In particular, $\nu = \nu_\lambda^{p_0,p_1,p_2}$ will be the self-similar measure such

$$\nu(J) = p_0 \nu(T_0^{-1}(J)) + p_1 \nu(T_1^{-1}(J)) + p_2 \nu(T_2^{-1}(J)),$$

that for all $J \subset \left[0, \frac{1}{1-\lambda}\right]$.

The analogue of Theorem 7.4 is the following:

Theorem 7.5

For almost all $\lambda \in [\frac{1}{3}, y(2) = 0.5]$,

$$\dim_H \nu_\lambda^{(p_0,p_1,p_2)} = \min \left(\frac{p_0 \log p_0 + p_1 \log p_1 + p_2 \log p_2}{\log \lambda}, 1 \right).$$

For almost all $\lambda \in [p_0^{p_0} p_1^{p_1} p_2^{p_2}, y(2) = 0.5]$ we have that ν_λ is absolutely continuous.

This is the standard method for using transversality that for a.e parameter a family of measures are absolutely continuous. This method has been successfully used in many contexts. These include self-affine sets ([**?**]), Bernoulli convolutions ([**?**],[**?**],[**?**]), non linear hyperbolic IFS ([**?**]), Parabolic IFS and random continued fraction expansions ([**?**]).

8. Iterated function schemes with overlaps: Higher dimensions

We now turn to the study of Iterated Function systems in $\mathbb{R}^2$. The starting point is the study of classical Sierpinski carpets. However, we want to modify the construction to allow for overlaps (i.e., where the Open Set Condition fails) by increasing the scaling factor λ. This can be viewed as a multidimensional version of the results from the previous chapter. More precisely, for some range of scaling values we can study the Hausdorff dimension of the limit set for typical values (as in the $\{0,1,3\}$-problem) and for another range of scaling values we can study the Lebesgue measure on the limit set (as in the Erdös problem).

8.1 Fat Sierpinski Gaskets Let $0 < \lambda < 1$ and natural numbers $n > k$. We consider a family of n contractions given by,

$$T_i(x,y) = (\lambda x, \lambda y) + (c_i^{(1)}, c_i^{(2)}),$$

$i = 0, \dots, n-1$ where $(c_i^{(1)}, c_i^{(2)}) \in \{(j,l) \in \mathbb{Z}^2 : 0 \leq j, l \leq k-1\}$ are n distinct points in a $k \times k$ grid. If $\lambda \in (0, \frac{1}{k}]$ then it immediately follows from Moran's Theorem that the attractor $\Lambda(\lambda)$ has dimension $-\frac{\log n}{\log \lambda}$.

Example 1 Our first example is the fat Sierpiński carpet. Here we take $n = 8$ and $k = 3$ and choose $c_0 = (0,0), c_1 = (0,1), c_2 = (0,2), c_3 = (1,0), c_4 = (1,2), c_5 = (2,0), c_6 = (2,1), c_7 = (2,2)$. In Theorem 8.1, we can take $s = \left(\frac{2}{3}\right)^{\frac{2}{3}} 0.338\ldots$. Thus we have that for almost all $\lambda \in [\frac{1}{3}, 0.338\cdots]$ that

$$\dim_H \Lambda(\lambda) = -\frac{\log 8}{\log \lambda}.$$

=2.0in carpet1.eps =2.0in carpet2.eps

The usual Sierpinski carpet (with $\lambda = \frac{1}{3}$) and the Fat Sierpinski carpet (with $\lambda = 0.338$) Example 2 Our next example is the Vicsek set. Here we take $n = 5$ and $k = 3$ and $c_0 = (1,0), c_1 = (0,1), c_2 = (1,1), c_3 = (2,1), c_4 = (1,2)$. We can take $s = \left(\frac{3}{5}\right)^{\frac{3}{5}} \left(\frac{1}{5}\right)^{\frac{2}{5}} = 0.3866\ldots$. Thus we have that for almost all $\lambda \in [\frac{1}{3}, 0.386]$ that

$$\dim_H \Lambda(\lambda) = -\frac{\log 5}{\log \lambda}.$$

=2.0in vicsek1.eps =2.0in vicsek2.eps

The Vicsek cross (with $\lambda = \frac{1}{3}$) and the Fat Vicsek (with $\lambda = 0.386$)

Our main results are rather similar in nature to those in the last chapter. However, our approach requires a detailed study of the measures supported on fat Sierpiński carpets.

Theorem 8.1 There exists $\frac{1}{k} \leq s \leq \frac{1}{\sqrt{n}}$ such that for almost all $\lambda \in (\frac{1}{k}, s)$ we have,

$$\dim_H \Lambda(\lambda) = -\frac{\log n}{\log \lambda}.$$

There are a dense sets of values in $(\frac{1}{k}, \frac{1}{\sqrt{n}}]$ where this inequality is strict.

Of course, for Theorem 8.1 to have any value we need to give an explicit estimate for s which, in most cases, satisfies $s > \frac{1}{k}$. Let denote the number of images in the jth row by

$$n_j = \text{Card}\{1 \le l \le k : c_i^{(2)} = j\},$$

for $1 \le j \le n$. If we assume that each $n_i \ge 1$ then, as we see from the proof, we can take

$$s = \min\left\{\frac{1}{n}\left(\prod_{j=1}^{k} n_j^{n_j}\right), \left(\prod_{j=1}^{k} n_j^{-n_j}\right)^{\frac{1}{n}}\right\}.$$

It should be noted that if all the values of n_j are the same then $s = \frac{1}{k}$ and then Theorem 8.1 yields no new information.

8.2 Measures on Fat Sierpinski Carpets As usual, upper bounds on the Hausdorff Dimension are easier. In particular, it follows immediately from a consideration of covers that $\dim_H \Lambda(\lambda) \le \dim_B \Lambda(\lambda) \le -\frac{\log n}{\log \lambda}$. Moreover, for the sets which we consider an argument analagous to that in the previous chapter that there are a dense sets of values $\lambda \in (\frac{1}{k}, \frac{1}{\sqrt{n}}]$ where this inequality is strict.

To complete the proof Theorem 8.1 by the now tried and tested method of studying measures supported on the fat Sierpiński carpets and using these to get lower bounds on $\dim_H \Lambda(\lambda)$. More precisely, let μ be a shift invariant ergodic measure defined on $\Sigma_n = \{1, \cdots, n\}^{\mathbb{Z}^+}$ and define a map $\Pi_\lambda : \Sigma_n \to \Lambda(\lambda)$ by,

$$\Pi_\lambda(\underline{i}) = \lim_{j\to\infty} T_{i_0} \circ \cdots \circ T_{i_{n-1}}(0,0) = \sum_{j=0}^{n} c_{i_j}\lambda^j.$$

Thus we can define a measure ν_λ supported on $\Lambda(\lambda)$ by $\nu_\lambda = \mu\Pi_\lambda^{-1}$ (i.e., $\nu_\lambda(A) = \mu(\Pi_\lambda^{-1}A)$, for Borel sets $A \subset \mathbb{R}$). We also introduce a map $p : \Sigma_n \to \Sigma_k$ which is given by,

$$p(i_0, i_1, \ldots) = (c_{i_0}^{(2)}, c_{i_1}^{(2)}, \ldots)$$

(i.e., we associated to symbol i the label for the vertical coordinate of $(c_i^{(1)}, c_i^{(2)})$).

We define a shift invariant measure $\overline{\mu}$ on Σ_k by $\overline{\mu} = \mu p^{-1}$ (i.e., $\overline{\mu}(B) = \mu(p^{-1}B)$, for Borel sets $B \subset \Sigma_k$). We have already defined the entropies $h(\mu)$ and $h(\overline{\mu})$ (in a previous chapter) and we can obtain the following technical estimates on the Hausdorff Dimension of the measure of ν_λ.

Proposition 8.2 For almost all $\lambda \in \left[\frac{1}{k}, \frac{1}{\sqrt{n}}\right]$ we have that,

$\dim_H(\nu_\lambda) = -\frac{h(\mu)}{\log\lambda}$ if $\max\left\{-\frac{h(\overline{\mu})}{\log\lambda}, -\frac{h(\mu)-h(\overline{\mu})}{\log\lambda}\right\} \le 1$; $\dim_H(\nu_\lambda) \in \Big[\min\left\{1 - \frac{h(\overline{\mu})}{\log\lambda}, 1 - \frac{h(\mu)-h(\overline{\mu})}{\log\lambda}\right\}$

Example (Bernoulli measure) In fact, for the proof of Theorem 8.1, it suffices to consider only Bernoulli measures. If $\mu = (\frac{1}{n}, \cdots, \frac{1}{n})^{\mathbb{Z}^+}$ then $h(\mu) = \log n$. If there are $n_1, \cdots, n_k$ squares in the k-rows then $\overline{\mu} = (\frac{n_1}{n}, \cdots, \frac{n_k}{n})^{\mathbb{Z}^+}$ and

$$h(\overline{\mu}) = -\sum_i \frac{n_i}{n} \log \frac{n_i}{n} = \log n - \frac{1}{n} \sum_i n_i \log n_i.$$

This is then used to prove the following:

Proposition 8.3 For almost all λ in the set,

$$\left\{ \left[\frac{1}{k}, y(k) \right] \ : \ \min\{h(\mu) - h(\overline{\mu}), h(\overline{\mu})\} \geq -\log\lambda \right\}$$

the measure ν_λ is absolutely continuous.

Examples

Our final example is contrived to have a region of values of λ where the dimension is definitely not equal to $-\frac{\log n}{\log \lambda}$ for almost all λ. We take $k = 3$ and $n = 5$ and choose $c_0 = (0,0), c_1 = (1,0), c_2 = (2,0), c_3 = (0,2), c_4 = (2,2)$. In Theorem we can take $s = 2^{-2/5} * 3^{-3/5} = 0.3920\ldots$. However if we added to the iterated function system the map $T_5(x,y) = \lambda(x,y) + (1,2)$ the attractor would simply be the cartesian product of an interval with the middle $(1-2\lambda)$ cantor set and thus has dimension $1 - \frac{\log 2}{\log \lambda}$. The attractor of our original system must be contained inside this set and so the dimension must be bounded above by $1 - \frac{\log 2}{\log \lambda}$. For $\lambda > 0.4$ we have that $1 - \frac{\log 2}{\log \lambda} < -\frac{\log 5}{\log \lambda}$ and thus $dim\Lambda(\lambda) < -\frac{\log 5}{\log \lambda}$ for all $\lambda > 0.4$. In fact if we take μ to be $\left(\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{4}, \frac{1}{4}\right)$-Bernoulli measure we can use Theorem to show that $\dim \nu_\lambda \geq 1 - \frac{\log 2}{\log \lambda}$ for almost all $\lambda \geq 0.4082$. This gives $\dim \Lambda(\lambda) = 1 - \frac{\log 2}{\log \lambda}$ for almost all $\lambda \geq 0.4082$. It is not clear whether the other examples we have considered also have regions where the dimension drops below $-\frac{\log n}{\log \lambda}$ for a set of λ with positive measure.

The rest of this section is devoted to the proof of this Proposition. In the next section we shall deduce Theorem 8.1. For $\xi \in \Sigma$ we define μ_ξ to be the conditional (probability) measure on $p^{-1}(\xi)$ defined

$$\mu(A) = \int_{\Sigma_k} \mu_\xi(A \cap p^{-1}\xi) \mathrm{d}\overline{\mu}(\xi),$$

for any Borel set $A \subseteq \Sigma_n$. Let $B(\Sigma_n)$ and $B(\Sigma_k)$ denote the Borel sigma algebras for Σ_n and Σ_k, respectively. Let $A = p^{-1}B(\Sigma_k) \subset B(\Sigma_n)$ be the corresponding σ-invariant sub-sigma algebra on Σ_n. In particular, this is a smaller sigma algebra which cannot distinguish between symbols in $\{0, 1, \ldots, n-1\}$ that project under p to the same symbol in Σ_k.

We recall the following result:

Ledrappier-Young Lemma For μ almost every $\underline{x} \in \Sigma_n$

$$\lim_{N \to \infty} -\frac{\log(\mu_\xi([x_0, \ldots, x_{N-1}]))}{N} = h(\mu) - h(\overline{\mu}) := h(\mu|A).$$

Proof We omit the proof in the general case, but observe that for Bernoulli measures it is fairly straight forward to see this. In particular, for a.e. (μ), $x \in \Sigma_n$ the symbols in $p^{-1}(i)$ occur with frequency $\frac{n_i}{n}$ and have associated weight $\frac{n_i}{n}$. Thus the limit is

$$h(\mu|A) = \frac{n_i}{n} \log\left(\frac{n_i}{n}\right),$$

as required. □

Let us define $\overline{\Pi}_\lambda : \Sigma_k \to \mathbb{R}$ by

$$\overline{\Pi}_\lambda(\underline{i}) = \sum_{j=0}^{\infty} c^{(2)}_{i_j} \lambda^j.$$

In particular, $\overline{\Pi}_\lambda$ corresponds to mapping sequences from Σ_k to points on $\mathbb{R}$ by first mapping the sequence $\underline{i}$ to the limit set $\Lambda(\lambda) \subset \mathbb{R}^2$ followed by the horizontal projection of $\Lambda(\lambda)$ to the y-axis. For any sequence $\xi \in \Sigma_k$ it is convenient to write $y_\xi = \overline{\Pi_\lambda}(\xi)$. It is easy to see that $\Pi_\lambda(p^{-1}\xi) \subset \Lambda(\lambda) \subset \mathbb{R}^2$ is actually the part of the limit set $\Lambda(\lambda)$ lying on the horizontal line $L_{y_\xi} := \{(x, y) : y = y_\xi\}$. [2]

We define two new measures. Firstly, $\overline{\nu}_\lambda = \overline{\mu} \circ \overline{\Pi}_\lambda$ on the vertical axis $\mathbb{R}$ and, secondly, on the horizontal axis $\nu_{\lambda,\xi} = \mu_\xi \circ \Pi_\lambda^{-1}$ on $\Lambda(\lambda) \cap L_{y_\xi}$. The following lemma allows us to relate the dimensions of these various measures.

Lemma 8.3 Let $s \geq 0$. If for a.e. $(\overline{\mu})$ $\xi \in \Sigma_k$ we have that $\dim_H \nu_{\lambda,\xi} \geq s$ then

$$\dim_H \nu_\lambda \geq \dim_H \overline{\nu_\lambda} + s.$$

Proof Let $A \subseteq \mathbb{R}^2$ be any Borel set such that $\nu_\lambda(A) = 1$. It follows that $\mu(\Pi_\lambda^{-1}(A)) = 1$ and thus by the decomposition of μ, we have that

$$1 = \mu(\Pi_\lambda^{-1}(A)) = \int \mu_\xi(\Pi_\lambda^{-1} A \cap p^{-1}\xi) d\overline{\mu}(\xi).$$

Thus for a.e. $(\overline{\mu})$ $\xi \in \Sigma_k$ we have $\mu_\xi(\Pi_\lambda^{-1}(A) \cap p^{-1}\xi) = 1$ and, hence, again from the definitions, $\nu_{\lambda,\xi}(A \cap L_{\Pi_\lambda(\xi)}) = 1$. However, $\dim \nu_{\lambda,\xi} \geq s$ for a.e.$(\overline{\mu})$ ξ and thus $\dim_H(A \cap L_{\Pi_\lambda(\xi)}) \geq s$ for a.e.$(\overline{\mu})$ ξ. In particular, $\dim_H(A \cap L_y) \geq s$ for a.e. $(\overline{\nu}_\lambda)$ y. By applying Marstrand's Slicing Theorem to the set $B = \{y : \dim_H(A \cap L_y) \geq s\}$, which is of full $\overline{\nu}_\lambda$ measure, we deduce that $\dim A \geq s + \dim \overline{\nu}_\lambda$. Since this holds for all Borel sets A where $\nu_\lambda(A) = 1$we conclude that $\dim \nu_\lambda(A) \geq s + \dim \overline{\nu}_\lambda$. □

[2] To see this, let $\omega \in \Sigma_n$ satisfy $p\omega = \xi$ then we know that $c^{(2)}_{\omega_i} = \xi_i$. Thus if we consider

$$\Pi_\lambda(\omega) = \sum_{i=0}^{n} (c^{(1)}_{\omega_i}, c^{(2)}_{\omega_i})\lambda^i = \sum_{i=0}^{\infty} (c^{(1)}_{\omega_i}, c^{(2)}_{\xi_i})\lambda^i$$

then the y-co-ordinate of $\Pi_\lambda(\omega)$ is equal to $\overline{\Pi_\lambda}(\xi)$. Thus any point in $(x, y) \in \Pi_\lambda(p^{-1}\xi)$ lies on the line $y = y_\xi = \overline{\Pi_\lambda}(\xi)$ which we denote $L_{\overline{\Pi_\lambda}(\xi)}$.

Since $\overline{\nu}_\lambda$ is a measure on the real line, its properties are better understood. In particular, we have the following result. Lemma 8.4 For almost all $\lambda \in \left[\frac{1}{k}, y(k-1)\right]$ we have that

$$\dim(\overline{\nu}_\lambda) = \min\left(1, -\frac{h(\overline{\mu})}{\log\lambda}\right).$$

Proof The proof makes use of transversality and the Shannon-McMullen-Brieman theorem, and follows the general lines of Theorem 7.3.

Firstly, it is easy to see from the definitions that $\dim_H \Lambda(\lambda) \leq \dim_B \Lambda(\lambda) \leq -\frac{h(\overline{\mu})}{\log\lambda}$. We now consider the opposite inequality. Given $\epsilon > 0$ let $s_\epsilon(\lambda) = -\frac{h(\overline{\mu})}{\log(\lambda+\epsilon)}$. Note that the proof can be completed (as in the proofs in the previous chapters) if it can be shown that,

$$I = \int_{\frac{1}{k}}^{y(k-1)} \left(\int\int \frac{\mathrm{d}\nu_\lambda(x)\mathrm{d}\nu_\lambda(y)}{|x-y|^{s_\epsilon(\lambda)}}\right)\mathrm{d}\lambda < \infty,$$

for all $\epsilon > 0$. In particular, the finiteness of the integrand, for almost all λ, allows us to deduce that for these values $\dim_H \Lambda(\lambda) \geq s_\epsilon(\lambda)$. Since the value of $\epsilon > 0$ is arbitrary, we get the required lower bound $\dim_H \Lambda(\lambda) \geq -\frac{h(\overline{\mu})}{\log\lambda}$.

The inner two integrals can be rewritten in terms of the measure μ on Σ and we can rewrite this as

$$I = \int_{\frac{1}{k}}^{y(k-1)} \left(\int\int \frac{\mathrm{d}\mu(\underline{i})\mathrm{d}\mu(\underline{j})}{|\overline{\Pi}_\lambda(\underline{i}) - \overline{\Pi}_\lambda(\underline{j})|^{s_\epsilon(\lambda)}}\right)\mathrm{d}\lambda.$$

Let $t = \max_{\frac{1}{k}\leq\lambda\leq y(k-1)} s_\epsilon(\lambda)$ and note that $t < 1$. In particular, if $\underline{i} \neq \underline{j}$ then they agree until the $|\underline{i}\wedge\underline{j}|$-th term and we can write

$$|\Pi_\lambda(\underline{i}) - \Pi_\lambda(\underline{j})|^{s_\epsilon(\lambda)} = \lambda^{|\underline{i}\wedge\underline{j}|s_\epsilon(\lambda)}\left(\sum_{k=0}^{\infty} a_k\lambda^k\right)^{s_\epsilon(\lambda)} \geq \left(e^{-h(\overline{\mu})}+\epsilon\right)^{s_\epsilon(\lambda)|\underline{i}\wedge\underline{j}|}\left(\sum_{k=0}^{\infty} a_k\lambda^k\right)^t,$$

where $\{a_k\}_{k\in\mathbb{Z}^+}$ is the sequence $a_k := i_{k+|\underline{i}\wedge\underline{j}|} - j_{k+|\underline{i}\wedge\underline{j}|} \in \{0, \pm1, \ldots, \pm(k-1)\}$ and $a_0 \neq 0$. Substituting this back into the integrand in I and using Fubini's Theorem we get

$$I \leq \int_\Sigma\int_\Sigma \frac{\mathrm{d}\mu(\underline{i})\mathrm{d}\mu(\underline{j})}{\left(e^{-h(\overline{\mu})}+\epsilon\right)^{|\underline{i}\wedge\underline{j}|}}\left(\int_{\frac{1}{k}}^{y(k-1)} \frac{\mathrm{d}\lambda}{\left(\sum_{k=0}^{\infty} a_k\lambda^k\right)^t}\right). \tag{8.1}$$

We can estimate the first integral in (8.1) by

$$\int\int \frac{\mathrm{d}\mu(\underline{i})\mathrm{d}\mu(\underline{j})}{\left(e^{-h(\overline{\mu})}+\epsilon\right)^{|\underline{i}\wedge\underline{j}|}} \leq \sum_{m=0}^{\infty}\sum_{[i_0,i_1,\ldots,i_{k-1}]} \frac{\mu([i_0,i_1,\ldots,i_{m-1}])}{\left(e^{-h(\overline{\mu})}+\epsilon\right)^m} = \sum_{m=0}^{\infty}\frac{e^{-mh(\overline{\mu})}}{\left(e^{-h(\overline{\mu})}+\epsilon\right)^m} < \infty.$$

Thus to show that $I < \infty$ it remains to bound the second integral in (8.1) by
$$\int \frac{\mathrm{d}\lambda}{\left(\sum_{k=0}^{\infty} a_k \lambda^k\right)^t} < \infty$$
for any sequence $\{a_k\}_{k\in\mathbb{Z}^+}$, where $a_k \in \{0, \pm 1, \ldots \pm (k-1)\}$ and $a_0 \neq 0$. Let $f(\lambda) = 1 + \sum_{k=0}^{\infty} \left(\frac{a_k}{a_0}\right) \lambda^k$ then we can apply part (1) of Proposition 7.1 to deduce that the integral is finite. $\square$

The next lemma allows us to associate to the measure $\overline{\nu}_\lambda$ a set $Y \subset \mathbb{R}$.

Lemma 8.5 For almost every $\lambda \in [\frac{1}{k}, y(k-1)]$ there exists a set $Y \subset \mathbb{R}$ with $\dim_H(Y) = \dim_H(\overline{\nu}_\lambda)$ such that for any $\xi \in (\overline{\Pi}_\lambda)^{-1} Y \subset \Sigma_k$ we can bound
$$\dim_H(\nu_{\lambda,\xi}) \geq \min\left\{-\frac{h(\nu|A)}{\log\lambda}, 1\right\}.$$

Proof Given $\delta > 0$, it is enough to show that for almost all $\lambda \in [\frac{1}{k}, y(1)]$ there exists a set $X = X_\delta \subset \Sigma_k$ with $\overline{\mu}(X) \geq 1 - \delta$ and such that for any $\xi \in X$, $\dim_H(\nu_{\xi,\lambda}) \geq \frac{-h(\mu|A)}{\log\lambda}$. In particular, we can take $Y = \cap_{n=1}^{\infty} X_{\frac{1}{n}}$.

Fix $\epsilon, \epsilon' > 0$. By Ergorov's Theorem there exist sets $X_{\epsilon'} \subset \Sigma_k$ and a constant $K > 0$ such that:

$\overline{\mu}(X_{\epsilon'}) > 1 - \epsilon'$; and

for any $\underline{\xi} \in X_{\epsilon'}$ there exists $Y_{\epsilon'}$ such that for any $\underline{x} \in X_{\epsilon'}$ we can bound
$$\mu_\xi[x_0, \ldots, x_N] \leq K \exp\left(-\left(h(\mu|A) - \epsilon\right) N\right), \text{ for } N \geq 1.$$

Let us denote $s = s_\epsilon(\lambda) = -\frac{h(\mu|A)}{\log\lambda} - 2\epsilon$. We want to consider the measure $\overline{\mu}$ restricted to $X_{\epsilon'}$ and the measure $\nu_{\lambda,\xi}$ restricted to $\Pi_\lambda(Y_{\epsilon'}) \cap L_\xi$, where $\xi \in X_{\epsilon'}$. This allows us to use the explicit bound in (2). Consider the multiple integral
$$I = \int_{\frac{1}{k}}^{y(k-1)} \int_{X_{\epsilon'}} \left(\int_{\Pi_\lambda Y_{\epsilon'}} \int_{\Pi_\lambda Y_{\epsilon'}} \frac{d\nu_{\xi,\lambda}(x) d\nu_{\xi,\lambda}(y)}{|x-y|^s} \right) d\overline{\mu}(\xi) d\lambda$$

We want to prove finiteness of this integral by lifting $\nu_{\xi,\lambda}$ to μ_ξ on $p^{-1}\xi$ and then using Fubini's Theorem to rewrite the integral as:
$$I = \int_{X_{\epsilon'}} \int_{Y_{\epsilon'}} \int_{Y_{\epsilon'}} \int_{\frac{1}{k}}^{y(k-1)} \frac{d\lambda}{|\Pi_\lambda(\underline{i}) - \Pi_\lambda(\underline{j})|^s} d\mu_\xi(\underline{i}) d\mu_\xi(\underline{j}) d\overline{\mu}(\xi) = \int_{X_{\epsilon'}} \int_{Y_{\epsilon'}} \int_{Y_{\epsilon'}} \int_{\frac{1}{k}}^{y(k-1)} \frac{d\lambda}{|\sum_{n=1}^{\infty}(i_n -$$

where we have that $a_n \in \{0, \pm 1, \ldots, \pm(k-1)\}$ and $a_0 \neq 0$. Thus we can use transversality to write
$$I \leq C \int_{X_{\epsilon'}} \int_{Y_{\epsilon'}} \int_{Y_{\epsilon'}} e^{-(h(\mu|A)+2\epsilon)\underline{i}\wedge\underline{j}} d\mu_\xi(\underline{i}) d\mu_\xi(\underline{j}) d\overline{\mu}(\xi) \leq C \sum_{m=0}^{\infty} e^{-m(h(\mu|A)+2\epsilon)} (\mu_\xi \times \mu_\xi)\left(\{(\underline{i}, \underline{j}) \in Y$$

In particular, from this we deduce that that for almost every $\lambda \in [\frac{1}{k}, y(k-1)]$, there is a set $Y = Y(\lambda) \subset \Pi_\lambda(X)$ of $\overline{\nu}$ measure $1-\epsilon'$ such that for $y \in Y$ one can choose $\xi \in \overline{\Pi}_\lambda^{-1}(y)$ such that

$$\int_{\Pi_\lambda Y_{\epsilon'}} \int_{\Pi_\lambda Y_{\epsilon'}} \frac{d\nu_{\lambda,\xi}(x) d\nu_{\xi,\lambda}(y)}{|x-y|^s} < +\infty.$$

By results in a previous chapter, this allows us deduce that $\dim_H(\nu_{\lambda,\xi}) \geq s$. Finally, since $\epsilon > 0$ was arbitrary, the result follows. □

Proof of Proposition 8.2 By combining the estimates in Lemma 8.4 and 8.5 and the Marstrand Slicing Lemma we can see that for almost every $\lambda \in [\frac{1}{k}, y(k-1)]$

$$\dim_H \nu_\lambda \geq \min\left\{-\frac{h(\mu|A)}{\log\lambda}, 1\right\} + \min\left(1, -\frac{h(\overline{\mu})}{\log\lambda}\right).$$

Thus if $-\frac{h(\mu|A)}{\log\lambda} < 1$ and $-\frac{h(\overline{\mu})}{\log\lambda} < 1$ we have that

$$\dim \nu_\lambda \geq -\frac{h(\mu|A)}{\log\lambda} - \frac{h(\overline{\mu})}{\log\lambda}$$

for almost every $\lambda \in [\frac{1}{k}, b_{k-1}]$. However, from the definitions:

$$h(\mu) = h(\overline{\mu}) + h(\mu|A)$$

and thus for almost every $\lambda \in [\frac{1}{k}, b_{k-1}]$ we have,

$$\dim \nu_\lambda \geq -\frac{h(\mu)}{\log\lambda}.$$

This completes the proof of Proposition 8.2. □

8.3 Proof of Theorem 8.1 To prove Theorem 8.1 it remains to apply Proposition 8.2 with a suitable choice of μ to get the lower bound.

More precisely, let μ denote the Bernoulli measure $\mu = \left(\frac{1}{n}, \ldots, \frac{1}{n}\right)^{\mathbb{Z}^+}$. Thus $h(\mu) = \log n$. We saw before that transversality gives $b_{k-1} \geq (1+\sqrt{k-1})^{-1}$ and thus since $k < n$ we have that $y(k-1) \geq (1+\sqrt{k-1})^{-1} \geq n^{-\frac{1}{2}}$. We need to find conditions on λ such that $-\log\lambda \geq h(\overline{\mu})$ and $-\log\lambda \geq h(\mu|A)$ and then we can calculate

$$h(\overline{\mu}) = -\sum_{i=0}^{k-1} \frac{n_i}{n} \log\left(\frac{n_i}{n}\right) = -\frac{1}{n}\sum_{i=0}^{k-1}(n_i \log n_i - n_i \log n) = -\frac{1}{n}\sum_{i=0}^{k-1} \log n_i^{n_i} + \log n = -\frac{1}{n}\left(\log \prod_{i=0}^{k-1} n_i^{n_i}\right) + \mathrm{lo}$$

We can write

$$h(\mu|A) = \sum_{i=0}^{k-1} \frac{n_i}{n} \log n_i = \log\left(\prod_{i=0}^{k-1} n_i^{n_i}\right)^{\frac{1}{n}}.$$

Thus, if we choose

$$s = \min \left\{ \frac{1}{n} \left(\prod_{i=0}^{k-1} n_i^{n_i} \right)^{\frac{1}{n}}, \left(\prod_{i=0}^{k-1} n_i^{-n_i} \right)^{\frac{1}{n}} \right\}$$

then for almost every $\lambda \in [\frac{1}{k}, s]$ we have that,

$$\dim_H \nu \geq -\frac{h(\mu)}{\log \lambda} = -\frac{\log n}{\log \lambda}.$$

In particular, for almost every $\lambda \in [\frac{1}{k}, s]$ we have that

$$\dim_H \Lambda(\lambda) \geq -\frac{\log n}{\log \lambda},$$

as required. □

8.4 Fat Sierpinski Carpets As the value of λ increases the limit set $\Lambda(\lambda)$ becomes larger. Eventually, we have a similar type of result where for typical λ the set $\Lambda(\lambda)$ has positive measure.

More precisely, we have the following result we obtain concerning the two dimensional measure of the attractor.

Theorem 8.6 There exists $\frac{1}{\sqrt{n}} \leq t \leq y(k-1)$ such that for almost all $\lambda \in [t, y(k-1)]$ we have that $\text{leb}(\Lambda(\lambda)) > 0$.

Examples For the Sierpinski Carpet, we can take $t = 0.357\ldots$. For the Vicsek cross we can take and $t = 0.4541$. =2.0in carpet3.eps =2.0in vicsek3

A fat Sierpinski carpet (with $\lambda = 0.357$) and a fat Vicsek cross (with $\lambda = 0.455$)

The following simple lemma shows how we can show absolute continuity of ν_λ using absolute continuity of the conditional measures.

Lemma 8.7 If $\overline{\nu}_\lambda$ is absolutely continuous and $\nu_{\lambda,\xi}$ is absolutely continuous for a.e. $(\overline{\mu})$ ξ then ν_λ is absolutely continuous. Proof Let $A \subset \mathbb{R}^2$ be any set such that $\text{Leb}(A) = 0$. We need to show that $\nu_\lambda(A) = 0$. Using the definiton of ν_λ and the decomposition of μ we get that

$$\nu_\lambda(A) = \mu(\Pi_\lambda^{-1} A) = \int_{\Sigma_k} \mu_\xi(\Pi_\lambda^{-1} A \cap p^{-1}\xi) d\overline{\mu}(\xi).$$

From the definition of $\nu_{\xi,\lambda}$ we have that

$$\mu_\xi(\Pi_\lambda^{-1} A \cap p^{-1}\xi) = \nu_{\lambda,\xi}(\Pi_\lambda(\Pi_\lambda^{-1} A \cap p^{-1}\xi)).$$

Since $\text{Leb}(A) = 0$, we know that the set $\{y \in \mathbb{R} : \text{Leb}(L_y \cap A) > 0\}$ has zero Lebesgue measure. Thus from the absolute continuity of ν_λ we have

$$\overline{\mu}\{\xi \in \Sigma_k : \text{Leb}(L_{\Pi_\lambda \xi} \cap A) > 0\} = \overline{\nu}_\lambda\{y \in \mathbb{R} : \text{Leb}(L_y \cap A) > 0\} = 0.$$

Since $\nu_{\lambda,\xi}$ is absolutely continuous for $\overline{\mu}$ almost all ξ we know that $\nu_{\lambda,\xi}(\Pi_\lambda(\Pi_\lambda^{-1}A \cap p^{-1}\xi)) = 0$ for $\overline{\mu}$ almost all ξ. Thus we have that $\nu_\lambda(A) = 0$, as required. □We now need to determine when the measures $\overline{\nu}_\lambda$ and $\nu_{\lambda,\xi}$ are absolutely continuous. The following result concerning $\overline{\nu}_\lambda$ is useful.

Lemma 8.8 For almost all $\lambda \in [e^{-h(\overline{\mu})}, b_{k-1}]$ the measure $\overline{\nu}_\lambda$ is absolutely continuous with respect to one dimensional Lebesgue measure.

Proof We omit the proof since it is similar to the proof of the next lemma. □Of course, it is possible that $e^{-h(\overline{\mu})} > b_{k-1}$. In this case the lemma does not give any new information. We now prove a result about the absolute continuity of measures supported on the fibre.

lemma 8.9 For almost all λ in

$$\left\{\lambda \in \left[\frac{1}{k}, b_{k-1}\right] \,:\, h(\mu|A) > -\log\lambda\right\}$$

there exists a set $X \subseteq \Sigma_k$ such that $\overline{\mu}(X) = 1$ and for any $\xi \in X$ the measure $\nu_{\lambda,\epsilon}$ is absolutely continuous on $L_{\overline{\Pi}_\lambda(\xi)}$. Proof It suffices to show that given $\epsilon' > 0$, there exists a set $X_{\epsilon'} \subseteq \Sigma_k$ such that $\overline{\mu}(X_{\epsilon'}) \geq 1-\epsilon'$ and for any $\xi \in X_{\epsilon'}$ there exists a set $Y_{\epsilon',\xi} \subset L_{\overline{\Pi}_\lambda(\xi)}$ where $\mu_\xi(Y'_\epsilon) \geq 1-\epsilon'$ and $\nu_{\lambda,\epsilon}$ is absolutely continuous on $Y_{\epsilon',\xi}$. We can then take $X = \cap_{N=1}^{\infty} X_{\frac{1}{N}}$.

Let $\epsilon, \epsilon' > 0$. From Ergorov's Theorem we know that there exists $K > 0$ and a set $X_{\epsilon'} \subseteq \Sigma_k$ such that $\overline{\mu}(X_{\epsilon'})$ and for $\xi \in X_{\epsilon'}$ there exists $Y_{\epsilon',\xi} \subseteq p^{-1}\xi$ with $\mu_\xi(Y_{\epsilon',\xi}) > 1-\epsilon'$ and for $\underline{x} \in Y_{\epsilon',\xi}$ we have that

$$\mu_\xi[x_0, \ldots, x_{N-1}] \leq K \exp\left(-\left(h(\mu|A) - \epsilon\right)N\right), \text{ for } N \geq 1.$$

We recall that to show that $\nu_{\xi,\lambda}$ is absolutely continuous it suffices to show that $\underline{D}(\nu_{\xi,\lambda})(x)$ is finite, for a.e.$(\nu_{\xi,\lambda})$ $x \in \Pi_\lambda Y_{\epsilon',\xi}$. In particular, it suffices to show that

$$\int_{\Pi_\lambda Y_{\epsilon',\xi}} \underline{D}(\nu_{\xi,\lambda})(x) d\nu_{\xi,\lambda}(x) < +\infty.$$

Moreover, to show that for almost every λ there exists a set of ξ of $\overline{\mu}$ measure at least $1-\epsilon'$ such that $\nu_{\xi,\lambda}$ is absolutely continuous, it suffices to show that

$$I := \int_t^{b_y(k-1)} \int_{X_{\epsilon'}} \left(\int_{\Pi_\lambda Y_{\epsilon',\xi}} \underline{D}(\nu_{\xi,\lambda})(x) d\nu_{\xi,\lambda}(x)\right) d\overline{\mu}(\xi) d\lambda < +\infty,$$

providing t is sufficiently large. We take $t > e^{h(\mu|A)+2\epsilon}$. For $\omega, \tau \in p^{-1}\xi$ we define

$$\phi_r(\omega,\tau) = \{\lambda : |\Pi_\lambda(\omega) - \Pi_\lambda(\tau)| \leq r\},$$

for $r > 0$. We start by lifting to the shift space, applying Fatou's Lemma and Fubini's Theorem

$$I \leq \liminf_{r\to 0} \frac{1}{2r} \int_t^{b_y(k-1)} \int_{X_{\epsilon'}} \int_{Y_{\epsilon',\xi}} \int_{Y_{\epsilon',\xi}} (\omega,\tau) \mu_\xi(\omega) d\mu_\xi(\tau) d\overline{\mu}(\xi) d\lambda \leq \liminf_{r\to 0} \frac{1}{2r} \int_{X_{\epsilon'}} \int_{Y_{\epsilon',\xi}} \int_{Y_{\epsilon',\xi}} \text{leb}(\phi_r(\omega,\tau))$$

where is the characteristic function for $\{(\omega,\tau) : |\Pi_\lambda(\omega) - \Pi_\lambda(\tau)| \le r\}$. We can deduce

$$I \le C \int_{X_{\epsilon'}} \int_{Y_{\epsilon',\xi}} \int_{Y_{\epsilon',\xi}} t^{-|\omega\wedge\tau|} d\mu_\xi(\omega) d\mu_\xi(\tau) d\overline{\mu}(\xi) \le C \int_{X_{\epsilon'}} \int_{Y_{\epsilon',\xi}} \int_{Y_{\epsilon',\xi}} e^{-|\omega\wedge\tau|(h(\mu|A)+2\epsilon)} d\mu_\xi(\omega) d\mu_\xi($$

where $\Delta_m = \{(\tau,\omega) \in Y_{\epsilon',\xi} \times Y_{\epsilon',\xi} : \omega_1 = \tau_1, \ldots, \omega_m = \tau_m\}$. This completes the proof. □

We can give an explicit value for t by,

$$t = \sup \left\{ \prod_{j=1}^{k} n_j^{-q_j} : \sum_{j=1}^{k} q_j \log\left(\frac{q_j}{n_j}\right) = 0, \sum_{j=1}^{k} q_j = 1 \text{ and } q_j \ge 0 \right\}.$$

Of course is possible that in some examples $t \ge y(k-1)$, in which case Theorem 8.6 tells us nothing new.

Proof of Theorem 8.6 Of course, to prove Theorem 8.6 we want to use Lemma 8.7 once we know that $\overline{\nu}_\lambda$ and $\lambda_{\xi,\lambda}$ are absolutely continuous. It remains to relate the value of t to the entropies in Lemma 8.8 and Lemma 8.9. Let $\underline{q} = (q_0, \ldots, q_{k-1})$ be a probability vector. Let $p_i = \frac{q_{p(i)}}{n_{p(i)}}$ for $i = 1, \ldots, n$ and μ be the $\underline{p}$-Bernoulli measure on Σ_n. If we let $\overline{\mu} = \mu p^{-1}$ then we have that

$$h(\overline{\mu}) = \sum_{i=0}^{k-1} \text{ and } h(\mu|A) = \sum_{i=0}^{k-1} q_i \log n_i.$$

If we let t be defined as above then for $\epsilon > 0$ let $\underline{q}$ satisfy $\sum_{j=0}^{k-1} n_j^{-q_j} \ge t - \epsilon$ then for any $\lambda \ge t - \epsilon$ we have that $-\log\lambda \le h(\overline{\mu}) = h(\mu|A)$. Thus for almost every $\lambda \ge t - \epsilon$ the measure ν_λ is absolutely continuous and hence $\mathrm{Leb}(\Lambda(\lambda)) > 0$. The proof is completed by letting $\epsilon \to 0$. □

Example: Higher dimension The results in this chapter can be generalised without difficulty to higher dimensional setting. We consider two such setting in $\mathbb{R}^3$. Firstly we consider the Sierpiński tetrahedron. This consists of the following four similarities.

$$T_0(x,y,z) = \lambda(x,y,z) + (0,0,0) T_1(x,y,z) = \lambda(x,y,z) + (1,0,0) T_2(x,y,z) = \lambda(x,y,z) + (0,1,0)$$

In the case where $\lambda = \frac{1}{2}$ this iterated function system would satisfy the open set condition and the attractor, $\Lambda(\lambda)$ would have dimension $\frac{\log 4}{\log 2} = 2$. We consider the case when $\lambda > \frac{1}{2}$. Let μ be evenly distributed Bernoulli measure on Σ_4 and ν_λ be the natural projection of μ to $\Lambda(\lambda)$. We can define a map $p : \Sigma_4 \to \Sigma_3$ which maps symbols $0, 1, 2$ to themselves but maps 4 to 0. If we let $\overline{\mu} = \mu p^{-1}$ and project it onto the tetrahedron as $\overline{\nu}_\lambda$ then we can see that it is supported on the perpendicular projection to the (x, y)-plane. This would be a Sierpinski gasket. We can then define a set of conditional measures which would be supported on vertical fibres.For almost all $\lambda \in \left[\frac{1}{2}, 0.569\ldots\right]$ that $\dim \nu_\lambda = -\frac{\log(h(\overline{\mu}))}{\log \lambda}$. We can also show that the conditional measures will for almost every $\lambda \in [0.5, y(1)]$ and for $\overline{\mu}$-almost every $\xi \in \Sigma_3$ have dimension $-\frac{h(\mu)-h(\overline{\mu})}{\log \lambda}$. Thus by using Marstrand's Slicing Theorem we can see that for almost every $\lambda \in [0.5, 0.569\ldots]$ we have $\dim \nu_\lambda \geq -\frac{log4}{\log \lambda}$. This immediately gives A similar argument shows that

$$\dim \Lambda(\lambda) = -\frac{\log 4}{\log \lambda}$$

for almost every $\lambda \in [0.5.0.569\ldots]$.

The menger sponge is another example of a self-similar set in $\mathbb{R}^3$. In the standard case it consists of 20 contractions of ratio $\frac{1}{3}$. The values of c_i consists of all triples of $(x, y, z) \in (0, 1, 2)^3$ where at most one of x, y of z takes the value 1. The perpendicular projection of the attractor to any of the (x, y)-plane, the (x, z)-plane and the (y, z)-plane is the standard Sierpiński carpet. If we consider the case where the contraction ratio (λ) are bigger than $\frac{1}{3}$ we have that $\dim \Lambda(\lambda) = -\frac{\log 20}{\log \lambda}$ for almost all $\lambda \leq 0.348$ and that $\Lambda(\lambda)$ has positive measure for almost every $\lambda \geq 0.393$.

8.5 Limits sets with positive measure and no interior Consider the following problem (posed by Peres and Solomyak): *Can one find examples of self-similar sets with positive Lebesgue measure, but with no interior?*

A variant of the method in the preceding section leads to families of examples of such sets.

The construction Let $\underline{t} = (t_1, t_2)$ with $0 \leq t_1, t_2 \leq 1$. We consider ten similarities (with the same contraction rate $\frac{1}{3}$) given by

$$T_0(x, y) = \left(\frac{1}{3}x, \frac{1}{3}y\right) T_1(x, y) = \left(\frac{1}{3}x, \frac{1}{3}y + t_1\right) T_2(x, y) = \left(\frac{1}{3}x, \frac{1}{3}y + t_2\right) T_3(x, y) = \left(\frac{1}{3}x, \frac{1}{3}y + 1\right) T_4(x, y$$

This construction is similar in spirit to those in the previous section. To see that the associated limit set $\Lambda_{\underline{t}}$ has empty interior, we need only observe that the intersection of $\Lambda_{\underline{t}}$ with each of vertical lines $\{(k + \frac{1}{2})3^{-n}\} \times \mathbb{R}$, with $n \geq 0$ and $0 \leq k \leq 3^n - 1$ has zero measure. It remains to show that typically $\Lambda_{\underline{t}}$ has positive measure.

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A typical limit set $\Lambda_{\underline{t}}$

Let $\Sigma_{10} = \{1, 2, \cdots, 10\}^{\mathbb{Z}^+}$ denote the full shift on 10 symbols and let $\Pi_{\underline{t}} : \Sigma_{10} \to \Lambda_{\underline{t}}$ be the usual projection map. Let

$$\mu = \left(\frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{6}, \frac{1}{6}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}\right)^{\mathbb{Z}^+}$$

be a Bernoulli measure on Σ_{10}. To show that $\Lambda_{\underline{t}}$ has non-zero Lebesgue measure it suffices to show that $\nu := \mu\Pi_{\underline{t}}^{-1}$ is absolutely continuous. By construction, ν projects to Lebesgue measure on the unit interval in the x-axis, thus it suffices to show the conditional measure $\nu_{\underline{t},x}$ on Lebesgue almost every vertical line $\{x\} \times \mathbb{R}$ is absolutely continuous.

Let $\Sigma_3 = \{1, 2, 3\}^{\mathbb{Z}^+}$ be a full shift on 3 symbols corresponding to coding the horizontal coordinate. As before, there is a natural map $p : \Sigma_{10} \to \Sigma_3$ corresponding to the map on symbols given by

$$p(1) = p(2) = p(3) = p(4) = 1 p(5) = p(6) = 2 p(7) = p(8) = p(9) = p(1) = 3.$$

Then $\mu p^{-1} = \overline{\mu} = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)^{\mathbb{Z}^+}$ is the Bernoulli measure on Σ_3. Given $\xi \in \Sigma_3$ let μ_ξ denote the induced measure on $p^{-1}(\xi)$. Clearly, if $\Pi_{\underline{t},\xi} : p^{-1}(\xi) \to \{x\} \times \mathbb{R}$ is the restriction of $\Pi_{\underline{t}}$, then by construction $\mu_\xi \Pi_{\underline{t},\xi}^{-1} = \nu_{\underline{t},x}$. We also let $\pi : \Sigma_3 \to [0, 1]$ be the natural projection from Σ_3 to the x-axis given by

$$\pi(\xi) = \sum_{n=0}^{\infty} \xi_n \left(\frac{1}{3}\right)^{n+1}.$$

The analogue of transversality is the following:

Lemma 8.10 There exists $C > 0$ such that

$$\Delta_\xi(r; \omega, \tau) := \mathrm{Leb}\{\underline{t} \in [0, 1]^2 : |\Pi_{\underline{t},\xi}(\omega) - \Pi_{\underline{t},\xi}(\tau)| \leq r\} \leq C 3^{|\omega \wedge \tau|} r, \text{ for } r > 0.$$

Proof Let $\omega, \tau \in p^{-1}(\xi)$ with $|\omega \wedge \tau| = n$ (i.e., $\tau_i = \omega_i$ for $i < n$ and $\tau_n \neq \omega_n$). Since $\omega, \tau \in p^{-1}(\xi)$ we have $i(\omega_n) = i(\tau_n)$ for all n, and $\Pi_{\underline{t},\xi}(\omega) - \Pi_{\underline{t},\xi}(\tau) = (0, \phi_{\underline{t},\xi}(\omega, \tau))$, where

$$\phi_{\underline{t},\xi}(\omega, \tau) = 3^{-n} \left((t_{j(\omega_n)} - t_{j(\tau_n)}) + \sum_{k=1}^{\infty} 3^{-k} (t_{j(\omega_{k+n})} - t_{j(\tau_{k+n})}) \right)$$

and $j|_{\{0,4,6\}} \equiv 0, j|_{\{1,7\}} \equiv 1, j|_{\{2,8\}} \equiv 2, j|_{\{3,5,9\}} \equiv 3$, and $t_0 = 0, t_3 = 1$ for convenience. If $\{j(\omega_n), j(\tau_n)\} = \{0, 3\}$, then

$$|\phi_{\underline{t},\xi}(\omega, \tau)| \geq 3^{-n} \left(1 - \sum_{k=1}^{\infty} 3^{-k}\right) = 3^{-n}/2,$$

in view of $t_j \in \{0, 1\}$ for all j, and (1) follows. Otherwise, let $j \in \{j(\omega_n), j(\tau_n)\} \cap \{1, 2\}$. Then

$$\left|\frac{\partial \phi_{\underline{t},\xi}(\omega, \tau)}{\partial t_j}\right| \geq 3^{-n} \left(1 - \sum_{k=1}^{\infty} 3^{-k}\right) = 3^{-n}/2,$$

which also implies (1). $\square$

Now we use Lemma 8.11 to prove that $\nu_{\underline{t},x}$ is absolutely continuous for a.e. x. For a sequence $\xi \in \Sigma_3$ we define $n_i(\xi)$ to be the number of i's in the first n terms of ξ. By the Strong Law of Large Numbers, given $\epsilon, \delta > 0$ we can use Egorov's theorem to choose a set $X \subset [0,1]$ of measure $\mathrm{leb}(X) > 1-\epsilon$ (equivalently $\overline{\mu}(\pi^{-1}X) > 1-\epsilon$) such that there exists $N \in \mathbb{N}$ where for $n \geq N$, $n_i(\xi) \geq \left(\frac{1}{3}-\delta\right)^n$, for $i = 0,1,2$. We can bound

$$\int_{[0,1]^2}\int_X \left(\int_{\{x\}\times\mathbb{R}} \underline{D}(\nu_{\underline{t},x})(y)d\nu_{\underline{t},x}(y)\right) d(\mathrm{leb})(x)d\underline{t} \leq \liminf_{r\to 0}\frac{1}{2r}\int_{\pi^{-1}X}\left(\int_{p^{-1}(\xi)}\int_{p^{-1}(\xi)} \Delta_\xi(r;\omega,\tau)d\mu_\xi(\omega)d\mu_\xi$$

for some $C_1 > 0$ bounding the first N terms of the series, and observe that the series is finite for δ sufficiently small. This implies the absolute continuity for a.e. $\underline{t}$.

We have proved the following result.

Theorem 8.12 For almost every $\underline{t} \in [0,1]^2$ the limit set $\Lambda_{\underline{t}}$ has positive Lebesgue measure and empty interior.

We can also construct examples with fewer similarities using different contraction rates. Let $0 < \lambda < \frac{1}{2}$ and $\underline{t} = (t_1,t_2,t_3) \in [0,1]^3$. Consider the six similarities of $\mathbb{R}^2$ defined by

$$T_0(x,y) = (\lambda x, \lambda y)T_1(x,y) = (\lambda x, \lambda y + t_1)T_2(x,y) = (\lambda x, \lambda y + t_2) \qquad T_3(x,y) = (\lambda + \lambda x, \lambda y)T_4(x,y) = ($$

Let $\Lambda_{\underline{t}}$ again denote the self-similar set. Let $\mu = \left(\frac{\lambda}{3},\frac{\lambda}{3},\frac{\lambda}{3},\frac{\lambda}{2},\frac{\lambda}{2},(1-2\lambda)\right)^{\mathbb{Z}^+}$ be the Bernoulli measure on Σ_6. Let $\overline{\mu} = (\lambda,\lambda,(1-2\lambda))^{\mathbb{Z}^+}$ denote the induced measure on Σ_3. The proof of Theorem A can be adapted to this setting provided

$$-(h(\mu) - h(\overline{\mu})) = -\lambda\log 2 - \lambda\log 3 \leq 2\lambda\log\lambda + (1-2\lambda)\log(1-2\lambda),$$

which is true provided λ is sufficiently close to $\frac{1}{2}$. More precisely, we have the following result. Theorem B If $\lambda \in (0.4759, \frac{1}{2})$ then for almost every $\underline{t} \in [0,1]^3$ the limit set $\Lambda_{\underline{t}}$ has positive Lebesgue measure and empty interior.

Remark In General

We can also obtain results about some overlapping self-affine fractals in ^{2}. Let $m > k > 2$ and write $\beta = \frac{\log k}{\log m}$. We consider n affine contractions $\{T_0,\dots,T_{n-1}\}$ $:^2\to^2$ given by,

$$T_i(x,y) = (\lambda^{\frac{1}{\beta}}x, \lambda y) + c_i$$

where $c_i \in \{0,\dots,m-1\}\times\{0,\dots,k-1\}$. In the case where $\lambda = \frac{1}{m}$ these are exactly the self-affine maps considered by Bedford and McMullen. The Hausdorff and Box counting dimensions of the attractor $\Lambda(\lambda)$ are given by Theorem **??**. We wish to calculate the Hausdorff dimension of $\Lambda(\lambda)$ for larger values of λ where the images overlap. For $0 \leq j \leq k-1$ we define

$$t_j = \mathrm{Card}\{c_i : c_i^{(2)} = j\}.$$

We obtain the following result about the Hausdorff dimension of the attractor.

Theorem 34. *There exists s such that for almost every $\lambda \in \left[\frac{1}{m}, s\right]$ we have that*

$$\dim \Lambda(\lambda) = -\frac{log\left(\sum_{i=0}^{k-1} n_i^{\beta}\right)}{\log \lambda}.$$

As was the case with the fat Sierpiński carpets there will be cases where $s = \frac{1}{m}$ and hence Theorem 34 gives no new information. However in most cases this will not be the case. An explicit values for s will be given in the proof.

Proof of Theorem 34

We start by showing that

$$\dim \Lambda(\lambda) \geq -\frac{log\left(\sum_{i=0}^{k-1} n_i^{\beta}\right)}{\log \lambda}$$

holds for almost every $\lambda \in \left[\frac{1}{m}, s\right]$. Let $\Pi_\lambda : \Sigma_n \to \Lambda(\lambda)$ be given by,

$$\Pi_\lambda(\underline{i}) = \sum_{j=0}^{\infty} c_{i_j} \lambda^j.$$

We then let μ be a shift invariant Ergodic measure on Σ_n. As before we define $\nu_\lambda = \mu \circ \Pi_\lambda^{-1}$. We define $p : \Sigma_n \to \Sigma_m$ by

$$p(j_0, j_1, \ldots) = (c_{j_0}^{(2)}, c_{j_0}^{(3)}).$$

Now let $\overline{\mu} = \mu \circ p^{-1}$. Once again we can use the Rohlin decompsition of measures. We define a family of measures μ_ξ where $\xi \in \Sigma_m$ and for all Borel subsets $A \subset \Sigma_n$

$$\mu(A) = \int_{\Sigma_m} \mu_\xi(A \cap p^{-1}\xi) \mathrm{d}\overline{\mu}(\xi).$$

Projections $\overline{\Pi_\lambda} : \Sigma_m \to$ and $\Pi_{\lambda,\xi} : p^{-1}\xi \to r$ are given by

$$\overline{\Pi_\lambda}(\xi) = \sum_{j=0}^{\infty} \xi_j \lambda^j \text{ and } \Pi_{\lambda,\xi}(\omega) = \sum_{j=0}^{\infty} \omega_j \lambda^{\beta j}.$$

By the definition of p we have that if $\omega \in p^{-1}\xi$ then for all $j \in_0$ there exists c_i such that $(\omega_j, p_j) = \omega_j$. Thus

$$(\Pi_{\lambda,\xi}(\omega), \overline{\pi}_\lambda(\xi)) \in \Lambda(\lambda).$$

Let

$$\overline{\nu_\lambda} = \overline{\mu} \circ \overline{\Pi}_\lambda^{-1} \text{ and } \nu_{\lambda,\xi} = \mu_\xi \circ \Pi_{\lambda,\xi}$$

be measures defined on . using exactly the same methods as for the Sierpiński carpets we can obtian the following Lemma.

Lemma 50.

1. *For almost all* $\lambda \in \left[\frac{1}{k}, b(m-1)^{\frac{1}{\beta}}\right]$ *we have that,*
$$\dim \overline{\nu}_\lambda \geq \min\left\{1, -\frac{h(\overline{\mu})}{\log\lambda}\right\}.$$
2. *For almost all* $\lambda \in \left[\frac{1}{k}, b(k-1)\right]$ *we have that for* $\overline{\mu}$ *almost all* ξ*,*
$$\dim \nu_{\lambda,\xi} \geq -\frac{h(\mu)-h(\overline{\mu})}{\frac{1}{\beta}\log\lambda}.$$
3. *For almost all* $\lambda \in \left[\frac{1}{k}, \min\left\{b(k-1), e^{-h(\overline{\mu})}, e^{h(\overline{\mu})-h(\mu)}\right\}\right]$ *we have*
$$\dim \nu_\lambda \geq -\frac{e^{h(\overline{\mu})}}{\log\lambda} - \frac{e^{h(\mu)-h(\overline{\mu})}}{\frac{1}{\beta}\log\lambda}$$

To complete the proof of the almost sure lower bound we let $p_i = \frac{n^\beta_{c_i^{(2)}}}{n_{c_i^{(2)}}\sum_{j=0}^{m-1} n_j^\beta}$ for $i = 0,\dots,n_i$. We then let μ be $(p_0,\dots,p_{m-1})$-Bernoulli measure on Σ_m. If we let
$$s = \min\{b(m-1), e^{-h(\overline{\mu})}, e^{h(\overline{mu})-h(\mu)}\}$$
then for almost every $\lambda \in [\frac{1}{m}, s]$ we have that
$$\dim \nu_\lambda \geq -\frac{h(\overline{\mu})}{\log\lambda} + \frac{h(\overline{\mu})-h(\mu)}{\frac{1}{\beta}\log\lambda}.$$
However if we let $q_i = \frac{n_i^\beta}{\sum_{j=0}^m -1 n_j^\beta}$ for $i = 0,\dots,m-1$ we can calculate,
$$h(\overline{\mu}) = \log\prod_{j=0}^{m-1} q_j^{q_j} \text{ and } h(\mu)-h(\overline{\mu}) = \log\prod_{j=0}^{m-1} n_j^{q_j}.$$
Thus for almost every $\lambda \in [\frac{1}{k}, s]$
$$\begin{aligned}
\dim\Lambda(\lambda) &\geq -\frac{h(\overline{\mu})}{\log\lambda} + \frac{h(\overline{\mu})-h(\mu)}{\frac{1}{\beta}\log\lambda} \\
&= -\frac{\log\prod_{j=0}^{m-1} q_j^{q_j} - \beta\log\prod_{j=0}^{m-1} n_j^{q_j}}{\log\lambda} \\
&= \frac{\log\prod_{j=0}^{m-1} \frac{n_j^\beta}{\sum_{i=0}^{m-1} n_j^\beta}^{q_j} - \beta\log\prod_{j=0}^{m-1} n_j^{q_j}}{\log\lambda} \\
&= \frac{\log\left(\frac{1}{\sum_{j=0}^{m-1} n_j^\beta}\right)}{\log\lambda} + \frac{\log\prod_{j=0}^{m-1} n_j^{\beta q_j} - \log\prod_{j=0}^{m-1} n_j^{\beta q_j}}{\log\lambda} \\
&= -\frac{\log\left(\sum_{j=0}^{m-1} n_j^\beta\right)}{\log\lambda}
\end{aligned}$$

We complete the proof by showing that McMullen's arguments from [?] can be adjusted to give us a uniform upper bound in the overlapping case.

12.2 Non-linear Contractions

In this section we show how the results can be generalised to certain specific families of non-linear contractions. Let $T_{ij}^{(\lambda,\alpha)} :^2\to$, where $0 \leq i \leq k-1$, be defined by

$$T_{ij} : (x,y) = (f_i^{(\lambda)}, g_{ij}^{(\chi)}(y))$$

where $f_i^{(\lambda)} : [0,1] \to [0,1]$ and $g_{ij}^{(\chi)} : [0,1] \to [0,1]$ are $C^[1+\alpha]$ parameterized by some λ and ϕ respectively. λ and ϕ will be from ^{m} for some m; when Lebesgue measure is referred to it will be m dimensional. Suppose that there are n different maps and let Σ_n be the space of sequences with symbols (i,j) and $\sigma : \Sigma_n \to \Sigma_n$ the usual shift map. Let $\Pi_{\lambda,\phi} : \Sigma_n \to^2$ be defined by,

$$\Pi_{\lambda,\phi}(\underline{x}) = \lim_{m\to\infty} T_{x_0}\cdots T_{x_{m-1}}(0,0).$$

This is the natural projection from the shift space to the attractor of the iterated function system, $\Lambda(\lambda,\phi)$. Let $\overline{\Pi}_\lambda : \Sigma_k \to \Sigma_k$ be defined by,

$$\overline{\Pi}_\lambda(\underline{x}) = \lim_{m\to\infty} f_{x_0}\circ\cdots\circ f_{x_{m-1}}(0).$$

We deifne $p : \Sigma_n \to \Sigma_k$ by mapping each element (i,j) of a sequence in Σ_n to j. Let μ be an Ergodic shift invariant map measure on Σ_n and let $\overline{\mu} = p\mu$. We then decompose μ so that for any Borel set $A \subset \Sigma_n$,

$$\mu(A) = \int_{\Sigma_k} \mu_\xi(A)\mathrm{d}\overline{\mu}(\xi).$$

Let $\Pi_{\phi,\xi}$ be the restriction of $\Pi_{\lambda,\phi}$ to $p^{-1}\xi$. We define measures by

$$\nu_{\lambda,\phi} = \mu\circ\Pi_\lambda, \overline{\nu}_\lambda = \overline{\mu}\circ\overline{\Pi}_\lambda^{-1}, \nu_{\chi,\xi} = \mu_\xi\circ\Pi_{\phi,\xi}.$$

Note that $\nu_{\phi,\xi}$ is entirely supported on a vertical line with x-coordinate $\overline{\Pi}_\lambda(\xi)$. In this setting we need to define Lyapunov exponents both on the projection and along the fibres. These are the analogues of the contraction rates in the linear cases. Let

$$\chi_1 = \int_{\Sigma_k} \log|f'_{x_0}|\circ\overline{\Pi}_\lambda\mathrm{d}\overline{\mu}(x) \text{ and } \chi_2 = \int_{\Sigma_k}\int_{p^{-1}\xi} \log|g'_{\xi_0 x_0}|\circ\Pi_{\lambda,\xi}\mathrm{d}\mu_\xi(\underline{x})\mathrm{d}\overline{\mu}(\xi).$$

To use these Lyapunov exponents we need the following two Lemmas.

Lemma 51.

1. *For $\overline{\mu}$ almost all $\underline{x}$ we have that*

$$\lim_{m\to\infty}\frac{1}{m}\log(\overline{\Pi}_\lambda([x_0,\ldots,x_{m-1}]))=\chi_1$$

2. *For $\overline{\mu}$ almost all ξ for μ_ξ almost all x*

$$\lim_{m\to\infty}\frac{1}{m}\log(\Pi_\lambda([x_0,\ldots,x_{m-1}]))=\chi_2.$$

Proof.

1. By the Mean Value Theorem for all $\underline{x}\in\Sigma_k$ there exists x such that,

$$(\overline{\Pi}_\lambda([x_0,\ldots,x_{n-1}]=f_{x_0}\circ\cdots\circ)f_{x_{m-1}})'(x).$$

By the Hölder continuity of each f_i' and the chain rule there exists a constant $c>0$ such that,

$$\frac{1}{m-1}\log|f_{x_0}\circ\cdots\circ f'_{x_{m-1}}(x)|=\frac{1}{m-1}\left|\log C+\sum_{l=0}^{m-1}\log f'_{x_l}(f_{x_{l+1}}\circ\cdots\circ f_{i_k}(0))\right|. \tag{12.1}$$

If we let $h:\Sigma_k\to$ be defined by $h(\underline{i})=\log f'_{i=0}\circ\overline{\Pi}_\lambda(\sigma(\underline{i}))$ then we can rewrite the right hand side of 12.1 as

$$\frac{1}{m-1}\left|\log C+\sum_{l=0}^{m-1}h(\sigma^l(\underline{x}))\right|.$$

The proof can then be completed by the Birkhoff Ergodic Theorem.

2. A similar method can be used to prove this part.

□

We need the following two transversality conditions to be satisfied. There exists a constant $C_1>0$ such that for $\omega,\tau\in\Sigma_k$ with $\omega_0\neq\tau_0$:

$$L\{\lambda:|\overline{\Pi}_\lambda(\omega)-\overline{\Pi}_\lambda(\tau)|\leq r\}\leq C_1r$$

There exists a constant C_2 so that for $\omega,\tau\in p^{-1}\xi$ where $\omega_0\neq\tau_0$ we have

$$L\{\chi:|\Pi_{\lambda,\phi,\xi}(\omega)-\Pi_{\lambda,\phi,\xi}(\tau)|\leq r\}\leq C_2r.$$

Let $A_{\lambda,\phi}$ be the set where both transversality conditions are satisfied. The following results hold when $A_{\lambda,\phi}$ has positive measure. A is the Borel sigma algebra defined by $p^{-1}B(\Sigma_n)$.

Proposition 19. *For almost all $(\lambda,\phi)\in A_\lambda$ we have*

$$\dim(\nu_{\lambda,\phi}) \;\geq\; -\left(\frac{h(\overline{\mu})}{\chi_1}+\frac{h(\mu|A)}{\chi_2}\right)\ \textit{if}\ \max\left\{-\frac{h(\overline{\mu})}{\chi_1},-\frac{h(\mu|A)}{\chi_2}\right\}$$

$$\dim(\nu_\lambda)\geq 1+\min\left\{-\frac{h(\overline{\mu})}{\chi_1},-\frac{h(\mu|A)}{\chi_2}\right\}\ \textit{otherwise.}$$

Proposition 20. *For almost all* (λ, ϕ) *in the set*

$$\left\{(\lambda, \phi) \in A_{\lambda,\phi} : \min\left\{-\frac{h(\mu)}{\chi_1}, -\frac{h(\mu|A)}{\chi_2}\right\} \geq 1\right\}$$

$\nu_{\lambda,\chi}$ *is absolutely continuous.*

Both of these Theorem can be proved using the same transversality techniques as used earlier in this chapter. Details of the slight changes needed to deal with the non-linear case can be found in [**?**] and [**?**].

Families where Proposition 19 can be applyed

We now find a family of iterated function systems on ^{2} where Proposition 19 can be applied. Let $f, g : [0,1] \to [0,1]$ be $C^{1+\alpha}$ contractions where $||F||_\infty, ||g||_\infty < \frac{1}{2}$. Let $(t_0, \ldots, t_{k-1}) \in^k$. For $0 \leq i \leq k-1$

Chapter 13

Games and Dimension

Chapter 14

Estimating dimensions

We now come to one of the main themes we want to discuss: *How can one compute the Hausdorff Dimension of a set*?

14.1 A basic approach

In the case of linear contractions there is a very effective way to estimate the dimension using the Moran formula.

More generally, assume that $T_1, \cdots, T_k : I \to I$ are a family of (non-linear) C^2 contractions of an interval I. We can associate to each $1 \leq i \leq k$ the lower and upper bounds on the derivative of the form:

$$0 < \alpha_i = \inf_{x \in I} |T_i'(x)| \leq \beta_i = \sup_{x \in I} |T_i'(x)|.$$

Let $\Lambda = \Lambda(T_1, \cdots, T_k)$ be the associated limit set then we have the following elementary bounds.

Proposition 21. *Let $0 < d_- \leq d_+ \leq 1$ be the solutions to:*

$$\sum_{i=1}^{k} \alpha_i^{d_-} = 1 \text{ and } \sum_{i=1}^{k} \beta_i^{d_+} = 1$$

then

$$d_- \leq \dim_H(\Lambda) \leq d_+$$

To proceed we need to prove basic distortion bounds.

Lemma 52 (Distortion bounds)**.** *There exists a constant $A > 0$ such that for any $i_1, \cdots, i_n \in \{1, \cdots, k\}$ and all $x, y \in I$:*

$$\frac{1}{A} \leq \frac{|(T_{i_1} \circ \cdots \circ T_{i_n})'(x)|}{|(T_{i_1} \circ \cdots \circ T_{i_n})'(y)|} \leq A$$

Proof. Since the maps T_i are C^2 we have that $\log |T_i'|$ is C^1 and this is Lipschitz, i.e.,

$$\big|\log |T_i'(x)| - \log |T_i'(x)|\big| \leq C|x-y| \text{ where } C = \sum_{x \in I} \frac{|f_i''(x)|}{|f_i'(x)|} < +\infty.$$

where we use the Mean Value Theorem. By the chain rule we can write

$$(T_{i_1} \circ \cdots \circ T_{i_n})'(x) = \prod_{j=1}^{n} T_{i_j}(T_{i_{j+1}} \circ \cdots \circ T_{i_n})(x))$$

and then for $x, y \in I$ we have

$$\begin{aligned}
&\big|\log |(T_{i_1} \circ \cdots \circ T_{i_n})'(x)| - \log |(T_{i_1} \circ \cdots \circ T_{i_n})'(y)|\big| \\
&\leq \sum_{j=1}^{n} \big|\log |T_{i_j}(T_{i_{j+1}} \circ \cdots \circ T_{i_n})(x))| - \log |T_{i_j}(T_{i_{j+1}} \circ \cdots \circ T_{i_n})(y))|\big| \\
&\leq C \sum_{j=1}^{n} \big|T_{i_{j+1}} \circ \cdots \circ T_{i_n}(x) - T_{i_{j+1}} \circ \cdots \circ T_{i_n}(y)\big| \\
&\leq C \sum_{j=1}^{n} (\beta_i)^{n-j-1} \leq \frac{C}{1-\beta}.
\end{aligned} \tag{1}$$

where we have used the Mean Value Theorem to bound

$$\begin{aligned}
\big|T_{i_{j+1}} \circ \cdots \circ T_{i_n}(x) - T_{i_{j+1}} \circ \cdots \circ T_{i_n}(y)\big| &\leq \int_x^y |(T_{i_{j+1}} \circ \cdots \circ T_{i_n})'(t)| dt \\
&\leq \beta^{n-j-1}
\end{aligned}$$

Exponentiating both sides of (1) and writing $C = \exp\left(\frac{C}{1-\beta}\right)$.

Given $\underline{i} = (i_1, \cdots, i_n) \in \{1, \cdots, k\}^n$ we denote $I_{\underline{i}} = T_{i_1} \circ \cdots \circ T_{i_n}(I)$ denote the images of I. We have the following corollary.

Corollary 11. *The length $|I_{\underline{i}}|$ of the interval $|I_{\underline{i}}|$ satsifies*

$$\frac{1}{A} \leq \frac{|I_{\underline{i}}|}{|(T_{i_1} \circ \cdots \circ T_{i_n})'(x_0)|} \leq A$$

for any $x_0 \in I$.

Proof. We can use the change of variables formula to write

$$\begin{aligned}
\left(\inf_{x \in I} |(T_{i_1} \circ \cdots \circ T_{i_n})'(x)|\right) |I| \leq |I_{\underline{i}}| &= \int_I |(T_{i_1} \circ \cdots \circ T_{i_n})'(t)| dt \\
&\leq \left(\sup_{x \in I} |(T_{i_1} \circ \cdots \circ T_{i_n})'(x)|\right) |I|
\end{aligned}$$

then the result follows by the lemma. □

Recall that the limit set can be written as

$$X = \cap_{n=1}^{\infty} \cup_{|\underline{i}|=n} I_{\underline{i}}.$$

Let $\delta = \inf_{i \neq j} \inf_{x \in I_i, y \in I_j} |x-y| > 0$ be the smallest gap between different images $T_i(I)$ and $T_j(I)$.

Lemma 53. *For $x, y \in I$ with $x \neq y$ we can choose $n \geq 1$ and $\underline{i} = (i_1, \cdots, i_n) \in \{1, \cdots, k\}^n$ with*

1. *$x, y \in I$, and*
2. *there exists $i \neq j$ such that $x \in I_{\underline{i}i}$ and $y \in I_{\underline{i}j}$*

where $\underline{i}i = (i_1, \cdots, i_n, i) \in \{1, \cdots, k\}^{n+1}$. Moreover, for any $x_0 \in I$ we can bound

$$\frac{\delta}{A} \leq \frac{|x-y|}{|(T_{i_1} \circ \cdots \circ T_{i_n})'(x_0)|} \leq A.|I|$$

Proof. We can choose an interval J from $I - \cup_{i=1}^{k} I_i$ which lies between I_i and I_j. We denote $J_{\underline{i}} = T_{i_1} \circ \cdots \circ T_{i_n}(J)$. In particular, $|x-y| \geq |J_{\underline{i}}|$ and by the change of variable formula and the corollary

$$|J_{\underline{i}}| = \int_J |(T_{i_1} \circ \cdots \circ T_{i_n})'(t)| dt \geq \frac{|J|}{|A|} |(T_{i_1} \circ \cdots \circ T_{i_n})'(x_0)|$$

and by definition $|J| \geq \delta > 0$. On the other hand

$$|x-y| \leq |I_{\underline{i}}| \leq A |(T_{i_1} \circ \cdots \circ T_{i_n})'(x_0)|.|I|.$$

This complete the proof. □

Let us assume that $\beta_1 + \cdots + \beta_k < 1 - \delta < 1$. We can then associate an affine linear function scheme $S_i : [0,1] \to [0,1]$ $(i = 1, \cdots, k)$ by

$$S_i(x) = \beta_i x + \gamma_i$$

where we choose the $\gamma_i = \beta_1 + \cdots \beta_{i-1} + i\delta/k$ $(i = 1, \cdots, k)$. Let $\overline{X}$ be the limit set associated to $\{S_i\}_{i=1}^{k}$, i.e., $\overline{X}$ is the smallest closed non-empty set such that $\overline{X} = \cup_{i=1}^{k} T_i \overline{X}$.

This leads to the following.

Lemma 54. *The natural map $\overline{\pi} : \overline{X} \to X$ given by*

$$\overline{\pi}\left(\lim_{n \to +\infty} S_{i_1} S_{i_2} \circ \cdots \circ S_{i_n}(x_0)\right) = \lim_{n \to +\infty} T_{i_1} T_{i_2} \circ \cdots \circ T_{i_n}(x_0)$$

is Lipschitz.

In particular, this implies that $\dim_H(X) \leq \dim_H(\overline{X})$. But by Moran's theorem we have $\dim_H(\overline{X}) = \overline{d}$.

Similarly, we can then associate an affine linear function scheme $R_i : [0,1] \to [0,1]$ $(i = 1, \cdots, k)$ by

$$R_i(x) = \alpha_i x + \gamma_i$$

where we choose the $\gamma_i = \alpha_1 + \cdots \alpha_{i-1} + i\delta/k$ $(i = 1, \cdots, k)$. Let $\underline{X}$ be the limit set associated to $\{R_i\}_{i=1}^k$, i.e., $\underline{X}$ is the smallest closed non-empty set such that $\underline{X} = \cup_{i=1}^k T_i \underline{X}$.

This leads to the following.

Lemma 55. *The natural map $\overline{\pi} : X \to \overline{X}$ given by*

$$\underline{\pi}\left(\lim_{n\to+\infty} T_{i_1} T_{i_2} \circ \cdots \circ T_{i_n}(x_0)\right) = \lim_{n\to+\infty} R_{i_1} R_{i_2} \circ \cdots \circ R_{i_n}(x_0)$$

is Lipschitz.

In particular, this implies that $\dim_H(X) \leq \dim_H(\underline{X})$. But by Moran's theorem we have $\dim_H(\underline{X}) = \underline{d}$.

This completes the proof of the proposition.

Example 59. *Let $2 \leq a < b$ be integers and let $T_1, T_2 : [0,1] \to [0,1]$ be defined by*

$$T_1(x) = \frac{1}{a+x} \text{ and } T_1(x) = \frac{1}{b+x}$$

In particular, we see that

$$\frac{1}{(a+1)^2} \leq |T_1'(x)| = \frac{1}{(a+x)^2} \leq \frac{1}{a^2} \text{ and } \frac{1}{(b+1)^2} \leq |T_2'(x)| = \frac{1}{(b+x)^2} \leq \frac{1}{b^2}.$$

(a) For example, when $a = 2$ and $b = 3$ we have that

$$\frac{1}{16} \leq |T_1'(x)| \leq \frac{1}{9} \text{ and } \frac{1}{9} \leq |T_2'(x)| \leq \frac{1}{4}$$

and we can solve for $0 < \underline{d} < \overline{d} < 1$ with

$$\left(\frac{1}{9}\right)^{\underline{d}} + \left(\frac{1}{16}\right)^{\underline{d}} = 1 \text{ and } \left(\frac{1}{4}\right)^{\overline{d}} + \left(\frac{1}{9}\right)^{\overline{d}} = 1$$

and get

$$\underline{d} = 0.2802\ldots \text{ and } \underline{d} = 0.3939\cdots.$$

(b) For example, when $a = 12$ and $b = 13$ we have that

$$\frac{1}{169} \leq |T_1'(x)| \leq \frac{1}{196} \text{ and } \frac{1}{169} \leq |T_2'(x)| \leq \frac{1}{144}$$

and we can solve for $0 < \underline{d} < \overline{d} < 1$ *with*

$$\left(\frac{1}{169}\right)^{\underline{d}} + \left(\frac{1}{196}\right)^{\underline{d}} = 1 \text{ and } \left(\frac{1}{144}\right)^{\overline{d}} + \left(\frac{1}{169}\right)^{\overline{d}} = 1$$

and get

$$\underline{d} = 0.1332\ldots \text{ and } \underline{d} = 0.1372\cdots.$$

In fact, we don't need to bound the derivatives of $|T_1'(x)|$ *and* $|T_2'(x)|$ *for all* $x \in I$*, but only on sub-intervals*

$$I_1 = \big[[\overline{ba}], [\overline{b}]\big] \text{ and } I_2 = \big[[\overline{a}], [\overline{ab}]\big]$$

where we have periodic continued fraction expansions

$$\begin{aligned} \overline{ba} &= [bababa\cdots] \\ \overline{b} &= [bbbbbb\cdots] \\ \overline{a} &= [aaaaaa\cdots] \\ \overline{ab} &=]ababab\cdots] \end{aligned}$$

Since $[\overline{a}] = \frac{1}{a+[\overline{a}]}$ *and* $[\overline{b}] = \frac{1}{b+[\overline{b}]}$ *we can solve to get*

$$[\overline{a}] = \frac{1}{2}\left(-a + \sqrt{a^2 - 4(a-1)}\right) \text{ and } [\overline{b}] = \frac{1}{2}\left(-b + \sqrt{b^2 - 4(b-1)}\right)$$

Since $[\overline{ab}] = \frac{1}{a+\frac{1}{b+[\overline{ab}]}}$ *and* $\overline{b} = \frac{1}{b+\frac{1}{a+[\overline{ba}]}}$ *we can solve to get*

$$[\overline{ab}] = \frac{1}{2}\left(-ab + \sqrt{(ab)^2 - 4ab}\right) \text{ and } [\overline{ba}] = \frac{1}{2}\left(-ba + \sqrt{(ab)^2 - 4ab}\right)$$

14.2 Algorithms

In some of the simpler examples, particularly those constructed by affine maps, it was possible to give explicit formulae for the Hausdorff dimension. In this chapter we shall consider more general cases. Typically, it is not possible to give a simple closed form for the dimension and it is necessary to resort to algorithms to compute the dimension as efficiently as we can. The original definition of Haudorff Dimension isn't particularly convenient for computation in the type of examples we have been discussing. However, the use of pressure for interated function schemes provides a much more promising approach.

We shall describe a couple of different variations on this idea. The main hypotheses on the compact X is that there exists a transformation $T : X \to X$ such that:

1. *Markov dynamics:* There is a Markov partition (to help describe the local inverses as an interted function scheme);
2. *Hyperbolicity:* There exists some $\lambda > 1$ such that $|T'(x)| \geq \lambda$ for all $x \in X$;
3. *Conformality:* T is a conformal map;
4. *Local maximality:* For any sufficiently small open neighbourhood U of the invariant set X we have $X = \cap_{n=0}^{\infty} T^{-n}U$ (such an X is sometimes called a *repeller*).

Our two main examples are the following:

Example 3.1.1 Consider a hyperbolic rational map $T : \widehat{C} \to \widehat{C}$ of degree $d \geq 2$ and let J be the Julia set. This satisfies the hypotheses (1)-(4). We let U be a sufficiently small neighbourhood of J.

Using the Markov partitions can write $J = \cup_{i=1}^{k} J_i$ and inverse branches $T_i : J \to J_i$ such that $T \circ T_i(z)$ and $i = 1, \ldots, k$ for all $z \in J_i$. J is the limit set for this iterated function schemes. □

Example 3.1.2 Consider a Schottky group $\Gamma = \langle g_1, \cdots, g_n, g_{n+1} = g_1^{-1}, \cdots, g_{2n} = g_n^{-1} \rangle$ and let Λ be the limit set. We let $U = \cup_{i=1}^{2n} U_i$ be the union of the disjoint open sets $U_i = \{z \in \ : |g_i'(z)| > 1\}$ of isometric circles. We define $T : \Lambda \to \Lambda$ by $T(z) = g_i(z)$, for $z \in U_i \cap \Lambda$ and $i = 1, \ldots, 2n$. This satisfies the hypotheses (1)-(4).

We can define inverse branches $T_i : g_i(U_i \cap \Lambda) \to U_i \cap \Lambda$ such that $T \circ T_i(z)$ and $i = 1, \ldots, n$ for all $z \in U_i \cap \Lambda$. The limit set Λ is the same as that given by the iterated function scheme.

We now describe three different approaches to estimating Hausdorff dimension.

A first approach: Using the definition of pressure. The most direct approach is to try to estimate the pressure directly from its definition, and thus the dimension from the last chapter.

Lemma 3.1 For each $n \geq 1$ we can choose s_n to be the unique solution to

$$\frac{1}{n} \log \left(\sum_{T^n x = x} |(T^n)'(x)|^{-s_n} \right) = 1.$$

Then $s_n = \dim_H(X) + O\left(\frac{1}{n}\right)$.

Proof. Fix a point x_0. There exists $C > 0$, we can associate to each preimage $y \in T^{-n}x_0$ a periodic point $T^n x = x$ with $|(T^n)'(y)|/|(T^n)'(x)| \leq C$ (in the last chapter). We can estimate

$$e^{-Cs} \sum_{T^n y = x_0} |(T^n)'(y)|^{-s} \leq \sum_{T^n x = x} |(T^n)'(x)|^{-s} \leq e^{Cs} \sum_{T^n y = x_0} |(T^n)'(y)|^{-s}$$

We can identify

$$L_s 1^n(x) = \sum_{T^n y = x} |(T^n)'(y)|^{-s}. \tag{3.1}$$

Recall that the Ruelle operator theorem allows us to write that $L_s^n 1(x) = \lambda_s^n(1 + o(1))$, where $s > 0$, and thus

$$\log \lambda_s = \frac{1}{n} \log \left(\sum_{T^n x = x} |(T^n)'(x)|^{-s} \right) + O\left(\frac{1}{n}\right).$$

We can deduce the result from the the Bowen-Ruelle Theorem (since the derivative of $\log \lambda_s$ is non-zero). □ □

In particular, in order to get an estimate with error of size $\epsilon > 0$, say, one expects to need the information on periodic points of period approximately $1/\epsilon$. This does not suggest itself as a very promising approach for very accurate approximations, since the number of periodic points we need to consider grows exponentially quickly with $n \asymp \frac{1}{\epsilon}$.

A second approach: Using the transfer operator. McMullen observed that working with the transfer operator one can quite effectively compute the pressure and the dimension. In practise, the numerical competition uses the approximation of the operator by matrices. Some of the flavour is given by the following statement.

Proposition 3.2 Given $x \in X$, and then for each $n \geq 1$ we can choose s_n to be the unique solution to $\sum_{T^n y = x} |(T^n)'(y)|^{-s_n} = 1$. Then $s_n = \dim_H(X) + O(\theta^n)$, for some $0 < \theta < 1$.

Proof. We begin from the identity (3.1). The stronger form of the Ruelle operator theorem means we can write that $L_s^n 1(x) = \lambda_s^n(1 + O(\alpha^n))$ where $0 < \alpha < 1$. The derivative $\frac{1}{\lambda_s} \frac{\partial \lambda_s}{\partial s}$ of $\log \lambda_s$ can be seen to be non-zero, and so we can deduce the result from the Bowen-Ruelle Theorem. □ □

For many practical purposes, this gives a pretty accurate approximation to the Hausdorff dimension of X. However, we now turn to the main method we want to discuss.

A third approach: Using determinants. Finally, we want to consider an approach based on determinants of transfer operators. The advantage of this approach is that it gives very fast, super-exponential, convergence to the Hausdorff dimension of the compact set X. This is based on the map $T : X \to X$ satisfying the additional assumption:

1. "(5)" *Analyticity:* T is real-analytic.

We need to introduce some notation.

Definition Let us define a sequence of real numbers

$$a_n = \frac{1}{n}\sum_{|\underline{i}|=n} \frac{|T_{\underline{i}}(z_{\underline{i}})|^{-s}}{\det\left(I - \left[T_{\underline{i}}(z_{\underline{i}})\right]^{-1}\right)}, \text{ for } n \geq 1,$$

where the summation is over all n-strings of contractions, $T'_{\underline{i}}(z_{\underline{i}})$ denotes the derivative of $T_{\underline{i}}$ at the fixed point $z_{\underline{i}} = T_{\underline{i}}(z_{\underline{i}})$, and $|T'_{\underline{i}}(z_{\underline{i}})|$ denotes the modulus of the derivative. Next we define a sequence of functions by

$$\Delta_N(s) = 1 + \sum_{n=1}^{N} \sum_{\substack{(n_1,\ldots,n_m)\\ n_1+\ldots+n_m=n}} \frac{(-1)^m}{m!} a_{n_1}\ldots a_{n_m},$$

where the second summation is over all ordered m-tuples of positive integers whose sum is n.

The main result relating these functions to the Hausdorff dimension of X is the following.

Theorem 3.3 Let $X \subset \mathbb{R}^d$ and assume that $T : X \to X$ satisfies conditions (1)-(5). We can find $C > 0$ and $0 < \theta < 1$ such that if s_N is the largest real zero of Δ_N then

$$|\dim(X) - s_N| \leq C\theta^{N^{\left(1+\frac{1}{d}\right)}} \text{ for each } N \geq 1.$$

In the case of Cantor sets in an interval then we would take $d = 1$. In the case of Julia sets and Kleinian group limit sets we would take $d = 2$.

Practical points

1. In practise, we can get estimates for $C > 0$ and $0 < \theta < 1$ in terms of T. For example, θ is typically smaller for systems which are more hyperbolic.

2. To implement this on a desktop computer, the main issue is amount memory required. In most examples it is difficult to get N larger than 18, say.

3.2 Examples

Example 1: E_2 We can consider the non-linear Cantor set

$$E_2 = \left\{ \frac{1}{i_1 + \frac{1}{i_2 + \frac{1}{i_3+\ldots}}} : i_n \in \{1,2\} \right\}.$$

For $X = E_2$, we can define $Tx = \frac{1}{x} \pmod 1$. This forms a Cantor set in the line, contained in the interval $[\frac{1}{2}(\sqrt{3}-1), \sqrt{3}-1]$, of zero Lebesgue measure. [1]

[1] It represents sets of numbers with certain diophantine approximatibility conditions and its Hausdorff dimension has other number theoretic significance in terms of the Markloff spectrum in diophantine approximation, as we shall see in the next chapter.

A number of authors have considered the problem of estimating the Hausdorff dimension $\dim_H(E_2)$ of the set E_2. In 1941, Good showed that $0.5194 \leq \dim_H(E_2) \leq 0.5433$. In 1982, Bumby improved these bounds to $0.5312 \leq \dim_H(E_2) \leq 0.5314$. In 1989 Hensley showed that $0.53128049 \leq \dim_H(E_2) \leq 0.53128051$. In 1996, he improved this estimate to 0.5312805062772051416.

We can apply Theorem 3.3 to estimating $\dim_H(E_2)$. In practice we can choose $N = 16$, say, and if we solve for $\Delta_{16}(s_{16}) = 0$ then we derive the approximation

$$\dim_H(E_2) = 0.5312805062772051416244686\ldots$$

which is correct to the 25 decimal places given.

Example 2: Julia sets We can consider Julia sets for quadratic polynomials $f_c(z) = z^2 + c$ with different values of c.

Example 2(a). Inside the main cardioid of the Mandelbrot set Let $c = -0.06$, which is in the main cardioid of the Mandelbrot set. Thus the quadratic map T_c is hyperbolic and its Julia set is a quasi-circle (which looks quite "close" to a circle).

=2.5in frog6.eps

The Julia set for $z^2 - 0.06$ is the boundary between the white and black regions. (The white points are those which do not escape to infinity)

Bodart & Zinsmeister estimated the Hausdorff dimension of the Julia set to be $dim_H(J_c) = 1.001141$, whereas McMullen gave an estimate of $dim_H(J_c) = 1.0012$. Using Theorem 3.3 we can recover and improve on these estimates. Working with $N = 8$ we obtain the approximation

$$dim_H(J_c) = 1.00121366248017464642\ldots$$

Example 2(b). Outside the Mandelbrot set Let $c = -20$, which is outside the Mandelbrot set. Thus the quadratic map T_c is hyperbolic and its Julia set is a Cantor set. With $N = 12$ this gives the approximation

$$dim_H(J_c) = 0.3185080957\ldots$$

which is correct to ten decimal places. This improves on an earlier estimate of Bodart & Zinsmeister.

=2.5in trig.eps

Figure 9 Contraction (a) in the r-plane; and (b) in the θ-plane

3.3 Proof of Theorem 3.3 (outline) The proof of this Theorem is based on the study of the transfer operator on Hilbert spaces of real analytic functions. To explain the ideas, we shall first outline the main steps in the general case (without proofs) and then restrict to a special case (where more proofs will be provided). The difficulties in extending from the particular case to the general case are more notational than technical.

(i) Real Analytic Functions We have a natural identification

$$\mathbb{R}^d = \mathbb{R}^d \times \{0\} \subset \mathbb{R}^d \times i\mathbb{R}^d = \mathbb{C}^d.$$

A function $f : U \to \mathbb{R}^k$ on an neighbourhood $U \subset \mathbb{R}^d$ is real analytic if about every point $x \in U$ there is a convergent power series expansion. Equivalently, it has a complex analytic extension to a function $f : D \to \mathbb{C}^k$, where $U \subset D \subset \mathbb{C}^d$ is an open set in $\mathbb{C}^d$.

(ii) Expanding maps and Markov Partitions We start from an expanding map $T : X \to X$ with a Markov Partition $P = \{X_j\}$, say. For each $1 \leq j \leq k$, let us assume that U_j is an open neighbourhood of a element X_j of the Markov Partition. We may assume that for each (i, j), the local inverse $T_{ji} : X_j \to X_i$ for $T : X_i \cap T^{-1}X_j \to X_j$ are contracting maps in an interated function scheme. Using analyticity (and choosing a smaller Markov partition P, if necessary) we can assume that $U_j \times \{0\} \subset D_j$ where $\mathrm{D}_j = D_j^{(1)} \times \ldots \times D_j^{(d)} \subset \mathbb{C}^d$ is chosen is an open polydisc, i.e., a product of open discs $D_j^{(l)}$ in $\mathbb{C}$. Thus, we can assume that these extend holomorphically to maps $T_{ji} : D_i \to D_j$, and $|DT_{ji}(\cdot)| : D_i \to \mathbb{C}$ too, such that both

$$\overline{T_{ji}(D_i)} \subset D_j \qquad \text{and} \qquad \sup_{z \in D_i} |DT_{ji}(z)| < 1, \tag{3.1}$$

i.e., the discs are mapped are mapped so that their closures are contained inside the interior of the range disk, and the derivative is smaller than 1.

(iii) A Hilbert space and a linear operator For any open set $U \subset \mathbb{C}^d$, let $A_2(U)$ denote the Hilbert space of square integrable holomorphic functions on U equipped with the norm

$$||f||_{A_2(U)} = \sqrt{\int_U |f|^2 d(\mathrm{vol})}.$$

For any $s \in \mathbb{R}$, and any admissible pair (i, j), define the analytic *weight function* $w_{s,(j,i)} \in H(D_i)$ by $w_{s,(j,i)}(z) = |DT_{ji}(z)|^s$. [2] We then define the bounded linear operator $L_{s,(j,i)} : H(D_j) \to H(D_i)$ by

$$L_{s,(j,i)}g(z) = g(T_{ji}z)w_{s,(j,i)}(z).$$

For a fixed i we sum over all (admissible) composition-type operators $L_{s,(j,i)}$ to form the *transfer operator* $L_{s,i}$, i.e.,

$$L_{s,i}h(z) = \sum_{j:A(i,j)=1} h(T_{ji}z)w_{s,(j,i)}(z). \tag{3.2}$$

[2] It is here that we need to consider real analyticity, because of the need for the modulus $|\cdot|$.

Finally, let $D = \coprod_i D_i$ be the disjoint union of the disks, then we define the *transfer operator* $L_s : A_2(D) \to A_2(D)$ by setting

$$L_s h|_{D_i} = L_{s,i} h$$

for each $h \in A_2(D)$ and each $i \in \{1, \ldots, k\}$.

The strategy we shall follow is the following. The operators L_s are defined on analytic functions on the disjoint union of the disks D_i. This in turn allows us to define their Fredholm determinants det$(I - zL_s)$. These are entire function of z which, in particular, have as a zero the value $z = 1/\lambda_s$. In this context we can get very good approximations to det$(I - zL_s)$ using polynomials whose coefficients involve the traces tr(L_s^n). Finally, these expressions can be evaluated in terms of fixed points of the iterated function scheme, leading to the functions $\Delta_N(s)$ introduced above.

(iv) Nuclear operators and approximation numbers Given a bounded linear operator $L : H \to H$ on a Hilbert space H, its i^{th} *approximation number* $s_i(L)$ is defined as

$$s_i(L) = \inf\{||L - K|| : \mathrm{rank}(K) \leq i - 1\},$$

where K is a bounded linear operator on H.

Definition A linear operator $L : H \to H$ on a Hilbert space H is called *nuclear* if there exist $u_n \in H$, $l_n \in H^*$ (with $||u_n|| = 1$ and $||l_n|| = 1$) and $\sum_{n=0}^{\infty} |\rho_n| < +\infty$ such that

$$L(v) = \sum_{n=0}^{\infty} \rho_n l_n(v) u_n, \qquad \text{for all } v \in H. \tag{3.4}$$

The following theorem is due to Ruelle.

Proposition 3.4 The transfer operator $L : A_2(D) \to A_2(D)$ is nuclear.

(iv) Determinants We now associate to the transfer operators a function of a two complex variables.

Definition For $s \in \mathbb{C}$ and $z \in \mathbb{C}$ we define the *Fredholm determinant* $\det(I - zL_s)$ of the transfer operator L_s by

$$\det(I - zL_s) = \exp\left(-\sum_{n=1}^{\infty} \frac{z^n}{n} \mathrm{tr}(L_s^n)\right) \tag{3.5}$$

This is similar to the way in which one associates to a matrix the determinant.

We can compute the traces explicitly.

The key to our method is the following explicit formula for the traces of the powers L_s^n in terms of the fixed points of our iterated function scheme.

Proposition 3.5 If $L_s : A_\infty(D) \to A_\infty(D)$ is the transfer operator associated to a conformal iterated function scheme then

$$\operatorname{tr}(L_s^n) = \sum_{|\underline{i}|=n} \frac{|T_{\underline{i}}'(z_{\underline{i}})|^s}{\det(I - T_{\underline{i}}'(z_{\underline{i}}))},$$

where $T_{\underline{i}}'(\cdot)$ is the (conformal) derivative of the map $T_{\underline{i}}$. This allows us to compute the determinant:

$$\det(I - zL_s) = \exp\left(-\sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{\underline{i} \in \operatorname{Fix}(n)} \frac{|DT_{\underline{i}}(z_{\underline{i}})|^s}{\det(I - DT_{\underline{i}}(z_{\underline{i}}))} \right).$$

(iv) Pressure, Hausdorff Dimension and Determinants We can now make the final connection with the Hausdorff dimension.

Proposition 3.6 For any $s \in \mathbb{C}$, let $\lambda_r(s)$, $r = 1, 2, \ldots$ be an enumeration of the non-zero eigenvalues of L_s, counted with algebraic multiplicities. Then

$$\det(I - zL_s) = \prod_{r=1}^{\infty} (1 - z\lambda_r(s)).$$

In particular, the set of zeros z of the Fredholm determinant $\det(I - zL_s)$, counted with algebraic multiplicities, is equal to the set of reciprocals of non-zero eigenvalues of L_s, counted with algebraic multiplicities.

This brings us to the connection we want.

Proposition 3.7 Given an iterated function scheme, the Hausdorff dimension $\dim(\Lambda)$ of its limit set Λ is the largest real zero of the function $s \mapsto \det(I - L_s)$.

Proof. If s is real then by the previous section the operator L_s has simple maximal eigenvalue λ_s, which equals 1 if and only if $s = \dim(\Lambda)$. But Proposition 3.7 tells us that 1 is an eigenvalue of L_s if and only if s is a zero of $\det(I - L_s)$.

To see that $\dim_H(\Lambda)$ is actually the *largest* real zero of $\det(I - L_s)$, observe that if $s > \dim(\Lambda)$ then the spectral radius of L_s is less than 1, so that 1 cannot be an eigenvalue of L_s, and hence cannot be a zero of $\det(I - L_s)$. □

The reason that $\det(I - zL_s)$ is particularly useful for estimating λ_s is because of the following result.

Proposition 3.8 The function $\det(I - zL_s)$ is entire as a function of $z \in \mathbb{C}$ (i.e., it has an analytic extension to the entire complex plane). In particular, we can expand

$$\det(I - zL_s) = 1 + \sum_{n=1}^{\infty} b_n(s) z^n$$

where $|b_n(s)| \leq C\theta^{n^{1+1/d}}$, for some $C > 0$ and $0 < \theta < 1$.

We can rewrite $\det(I - L_s)$ by applying the series expansion for $e^{-x} = 1 + \sum_{m=1}^{\infty}(-1)^m \frac{x^m}{m!}$ to the trace formula representation of $\det(I - zL_s)$, and then regrouping powers of z. More precisely, we can expand the presentation

$$\det(I - zL_s) = \exp\left(-\sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{|\underline{i}|=n} \frac{|T_{\underline{i}}(\underline{z}_{\underline{i}}^*)|^{-s}}{\det(I - T_{\underline{i}}(\underline{z}_{\underline{i}}^*))}\right) = 1 + \sum_{n=1}^{\infty} b_n(s) z^n \tag{3.6}$$

using the Taylor series $e^{-x} = 1 + \sum_{m=1}^{\infty}(-1)^m \frac{x^m}{m!}$. Collecting together the coefficients of z^N we have the following:

Proposition 3.9 Let $det(I - zL_s) = 1 + \sum_{N=1}^{\infty} d_N(s) z^N$ be the power series expansion of the Fredholm determinant of the transfer operator L_s. Then

$$b_N(s) = \sum_{\substack{(n_1,\ldots,n_m) \\ n_1+\ldots+n_m=N}} \frac{(-1)^m}{m!} \prod_{l=1}^{m} \frac{1}{n_l} \sum_{|\underline{i}|=n_l} \frac{|DT_{\underline{i}}(z_{\underline{i}})|^s}{\det(I - DT_{\underline{i}}(z_{\underline{i}}))}, \tag{3.7}$$

where the summation is over all ordered m-tuples of positive integers whose sum is N.

In conclusion, (3.7) allows an *explicit* calculation of any coefficient $d_N(s)$, in terms of fixed points of compositions of at most N contractions.

3.4 Proof of Theorem 3.3 (special case) We shall try to illustrate the basic ideas of the proof, by proving these results with in the simplest setting: $d = 1$. Let $\Delta_r = \{z \in \mathbb{C} : |z| < r\}$ denote the open disk of radius r centered at the origin in the complex plane. Assume that X is contained in the unit disk Δ_1 and that $T : X \to X$ has two inverse branches T_1, T_2 which have analytic extensions $T_1 : \Delta_1 \to \Delta_1$ and $T_2 : \Delta_1 \to \Delta_1$ which have analytic extensions to $\Delta_{1+\epsilon}$ satisfying $T_1(\Delta_{1+\epsilon}) \cup T_2(\Delta_{1+\epsilon}) \subset \Delta_1$. Thus T_1 and T_2 are strict contractions of $\Delta_{1+\epsilon}$ into Δ_1 with the radii being reduced by a factor of $\theta = 1/(1+\epsilon) < 1$.

Let $A_2(\Delta_r)$ denote the Hilbert space of analytic functions on Δ_r with inner product $\langle f, g \rangle := \int_{\Delta_r} f(z)\overline{g(z)}\, dx\, dy$.

Let us assume that $|T_1'(z)|$ and $|T_2'(z)|$ have analytic extensions from X to $\Delta_{1+\epsilon}$. We define the transfer operator $L_s : A_2(\Delta_1) \to A_2(\Delta_1)$ by

$$L_s h(z) = |T_1'(z)|^s h(T_1 z) + |T_2'(z)|^s h(T_2 z), \text{ for } z \in \Delta_{1+\epsilon}.$$

Observe that $L_s(A_2(\Delta_1)) \subset A_2(\Delta_{1+\epsilon})$ and then

$$L_s h(z) = \int_{|\xi=1+\epsilon|} \frac{L_s h(\xi)}{z - \xi} d\xi = \frac{1}{2\pi i} \int_{|\xi|=1+\epsilon} L_s h(\xi) \left(\frac{1}{\xi} \sum_{n=0}^{\infty} \left(\frac{z}{\xi}\right)^n\right) d\xi = \sum_{n=0}^{\infty} z^n \frac{1}{2\pi i} \int_{|\xi|=1+\epsilon} \frac{L_s h(\xi)}{\xi^{n+1}} d\xi,$$

where $u_n(z) = z^n \in A_2(\Delta_{1+\epsilon})$ and $l_n(h) = \frac{1}{2\pi i} \int_{|\xi|=1+2\epsilon} \frac{L_s h(\xi)}{\xi^{n+1}} \in A_2(\Delta_{1+\epsilon})^*$ is a linear functional. We can deduce that L_s is a nuclear operator, the uniform convergence of the series coming from $|z/\xi| = \theta < 1$. *Aside on Operator Theory.* A bounded linear operator $T : H \to H$ on a Hilbert space H is called *compact* if the image $T(B) \subset H$ of the unit ball $\{x \in H : ||x|| \leq 1\}$ has a compact closure. In particular, a nuclear operator is automatically compact.

We denote the norm of the operator by $||T||_H = \sup_{||f||=1} ||T(f)||$.

We recall the following classical result.

Weyl's Lemma Let $A : H \to H$ be a compact operator with eigenvalues $(\lambda_n)_{n=1}^{\infty}$. We can bound $|\lambda_1 \lambda_2 \cdots \lambda_n| \leq s_1 s_2 \cdots s_n$

Proof. Given a bounded linear operator $A : H \to H$ on a Hilbert space H we can associate a bounded self-adjoint linear operator $B : H \to H$ by $B = A^* A$. Since B is non-negative (i.e, $\langle Bf, f \rangle = ||Af||^2 \geq 0$ for all $f \in H$) the eigenvalues $\mu_1 \geq \mu_2 \geq \cdots$ for B are described by the minimax identity:

$$\mu_1 = \max_{f \neq 0} \frac{\langle Bf, f \rangle}{||f||^2} \text{ and} \mu_{n+1} = \max_{\dim L = n} \max_{f \in L^\perp} \frac{\langle Bf, f \rangle}{||f||^2} \text{ for } n \geq 1,$$

where L denotes an n-dimensional subspace.

Claim 1 $\mu_n \leq s_n(A)$ Proof of Claim 1 For any linear operator $K : H \to H$ with n-dimensional image $K(H) \subset H$ we can use the minimax identity to write

$$\mu_n \leq \max_{f \in \ker(K)} \frac{\langle Bf, f \rangle}{||f||^2} = \max_{f \in \ker(K)} \frac{\langle (B-K)f, f \rangle}{||f||^2} \leq ||B - K||$$

Taking the infimum over all such K proves the claim. □ □

Claim 2 Given an orthonormal set $\{\phi_i\}_{i=1}^n \subset H$ we can write

$$\det(\langle A\phi_i, A\phi_j \rangle)_{i,j=1}^n \leq s_1^2 s_2^2 \cdots s_n^2 \det(\langle \phi_i, \phi_j \rangle)_{i,j=1}^n$$

Proof of Claim 2 Let $\{e_n\}_{m=0}^{\infty}$ be a complete orthonormal basis of eigenvectors for B. We can write $\langle A\phi_i, A\phi_j \rangle = \langle B\phi_i, \phi_j \rangle = \sum_{m=0}^{\infty} \mu_m \langle \phi_j, e_m \rangle \langle e_m, \phi_k \rangle$. In particular, we can write the original matrix as a product of two infinite matrices.

$$(\langle A\phi_i, A\phi_j \rangle)_{i,j=1}^n = (\sqrt{\mu_m} \langle \phi_j, e_m \rangle)_{m=1}^{\infty}{}_{j=1}^{n} \times (\sqrt{\mu_m} \overline{\langle e_m, \phi_k \rangle})_{k=1}^{n}{}_{m=1}^{\infty}. \quad (3.7)$$

Considering determinants gives:

$$\det(\langle A\phi_i, A\phi_j \rangle)_{i,j=1}^n = \sum_{C,C'} \det(C) \det(C'),$$

where the sum is over all possible $n \times n$ submatrices C and C' of the two matrices on the rights hand side of (*3.7), respectively. In this latter expression, we can take out a factor of $\sqrt{\mu_1 \mu_2 \cdots \mu_n}$ from each matrix to leave $\det(\langle \phi_i, \phi_j \rangle)_{i,j=1}^n$. Since, by Claim 1, $\mu_1 \mu_2 \cdots \mu_n \leq s_1 s_2 \cdots s_n$ this gives the desired result. □

It remains to complete the proof of Weyl's Lemma. Since A is a compact operator we can choose an orthonormal basis $(e)_{n=0}^{\infty}$ for H such that $Ae_n = a_{n1}e_1 + a_{n2}e_2 + \cdots + a_{nn}e_n$, (i.e., the matrix (a_{nm}) is triangular) and $a_{nn} = \lambda_n$ is an eigenvalue. In particular, if $i < j$ than

$$\langle Ae_i, Ae_j\rangle = \sum_{k=1}^{i} \langle A\phi_i, \phi_k\rangle \overline{\langle A\phi_k, A\phi_j\rangle}$$

and thus

$$\det\left(\langle Ae_i, Ae_j\rangle\right)_{i,j=1}^{n} = \det\left(\langle Ae_i, e_j\rangle\right)_{i,j=1}^{n} \det\left(\overline{\langle Ae_i, e_j\rangle}\right)_{i,j=1}^{n} = \left|\det\left(\langle Ae_i, e_j\rangle\right)_{i,j=1}^{n}\right|^2 = |\lambda_1 \cdots \lambda_n|^2.$$

This completes the proof $\square$

We now return to the explicit case of analytic functions.

Lemma 3.10 The singular values of the transfer operator $L_s : A_2(\Delta_1) \to A_2(\Delta_1)$ satisfy

$$s_j(L_s) \leq \frac{||L_s||_{A_2(\Delta_{1+\epsilon})}}{1-\theta}\theta^j, \text{ for all } j \geq 1.$$

Proof. Let $g \in A_2(\Delta_1)$ and write $L_s g = \sum_{k=0}^{\infty} l_k(g)p_k$, where $p_k(z) = z^k$. We can easily check $||p_k||_{A_2(\Delta_1)} = \sqrt{\frac{\pi}{k+1}}$ and $||p_k||_{A_2(\Delta_{1+\epsilon})} = \sqrt{\frac{\pi}{k+1}}(1+\epsilon)^{k+1}$. The functions $\{p_k\}_{k=0}^{\infty}$ form a complete orthogonal family for $A_2(\Delta_{1+\epsilon})$, and so $\langle L_s g, p_k\rangle_{A_2(\Delta_{1+\epsilon})} = l_k(g)||p_k||^2_{A_2(\Delta_{1+\epsilon})}$. The Cauchy-Schwarz inequality implies that

$$|l_k(g)| \leq ||L_s g||_{A_2(\Delta_{1+\epsilon})} \, ||p_k||^{-1}_{A_2(\Delta_{1+\epsilon})}.$$

We denote the rank-j projection operator $L_s^{(j)}$ by $L_s^{(j)}(g) = \sum_{k=0}^{j-1} l_k(g)p_k$. For any $g \in A_2(\Delta_1)$ we can estimate

$$|| \left(L_s - L_s^{(j)}\right)(g)||_{A_2(\Delta_1)} \leq ||L_s g||_{A_2(\Delta_1)} \sum_{k=j}^{\infty} \theta^{k+1}.$$

It follows that

$$||L_s - L_s^{(j)}||_{A_2(\Delta_1)} \leq \frac{||L_s||_{A_2(\Delta_1)}}{1-\theta}\theta^{j+1} \text{ and so } s_j(L_s) \leq \frac{||L_s||_{A_2(\Delta_1)}}{1-\theta}\theta^{j+1},$$

and the result follows. $\square$ $\square$

We now show that the coefficients of the power series of the determinant decay to zero with super-exponential speed.

Lemma 3.11 If we write $\prod_{j=1}^{\infty}(1 + zs_j) = 1 + \sum_{m=1}^{\infty} c_m z^m$, then

$$|c_m| \leq B\left(||L_s||_{A_2(\Delta_1)}\right)^m \theta^{\frac{m(m+1)}{2}},$$

where $B = \prod_{m=1}^{\infty}(1-\theta^m)^{-1} < \infty$.

Proof. The coefficients c_n in the power series expansion of the determinant have the form $c_m = \sum_{i_1<\ldots<i_m} s_{i_1}\cdots s_{i_m}$, the summation is over all m-tuples $(i_1,\ldots,i_m)$ of positive integers satisfying $i_1<\ldots<i_m$. Thus by Lemma 3.10 we can bound

$$|c_m| \le \left(\frac{||L_s||_{A_2(\Delta_1)}}{1-\theta}\right)^m \frac{\theta^{m(m+1)/2}}{(1-\theta)(1-\theta^2)\cdots(1-\theta^m)}. \le B\left(\frac{||L_s||_{A_2(\Delta_1)}}{1-\theta}\right)^m \theta^{m(m+1)/2}.$$

For some $B>0$. □

□

The coefficients of $\det(I-zL_s) = 1+\sum_{n=1}^{\infty} b_n z^n$ are given by Cauchy's Theorem:

$$|b_n| \le \frac{1}{r^n}\sup_{|z|=r}|\det(I-zL_s)|, \text{ for any } r>0. \tag{3.8}$$

We recall the following standard bound of Hardy, Littlewood and Polya: Let $\{a_n\}$, $\{b_n\}$ be not increasing sequences of real numbers such that $\sum_{j=1}^n a_j \le \sum_{j=1}^n b_j$ and let $\Phi:\mathbb{R}\to\mathbb{R}$ be a convex function then $\sum_{j=1}^n \Phi(a_j) \le \sum_{j=1}^n \Phi(b_j)$. Letting $a_j = \log|\lambda_j|$, $b_j = \log s_j$ and $\Phi(x) = \log(1+tx)$ (and letting $n\to+\infty$) we deduce that if $|z|=r$ then

$$|\det(I-zL_s)| \le \prod_{j=1}^{\infty}(1+|z|\lambda_j) \le \prod_{j=1}^{\infty}(1+|z|s_j) \le \left(1+B\sum_{m=1}^{\infty}(r\alpha)^m\theta^{\frac{m(m+1)}{2}}\right) \tag{3.9}$$

where $\alpha = ||L_s||_{A_2(\Delta_1)}$. If we choose $r = r(n) := \frac{\theta^{-n/2}}{\alpha}$ then we can bound

$$(r\alpha)^m\theta^{m^2/2} \le \theta^{n^2/2} \text{ for } m=1,\ldots,\left[\frac{n}{2}\right] \theta^{((m-n)^2+nm)/2} \le (\theta^{n/2})^m \text{ for } m>\left[\frac{n}{2}\right] \tag{3.10}$$

Comparing (3.8), (3.9) and (3.10) we can bound

$$|b_n| \le \left[\frac{n}{2}\right]\theta^{n^2/2} + \frac{(\theta^{n/2})^{n/2}}{1-\theta^{n/2}}$$

This proves the super-exponential decay of the coefficients provided we replace θ by a value larger than $\theta^{1/4}$.

Lemma 3.12 We can compute the traces:

$$\mathrm{tr}\,(L_s^n) = \sum_{|\underline{i}|=n}\frac{|T_{\underline{i}}'(x)|^s}{1-|T_{\underline{i}}'(x)|^{-1}}$$

Proof. For each string $\underline{i} = (i_1,\ldots,i_n) \in \prod_{j=1}^n\{0,1\}$ let us first define operators $L_{s,\underline{i}} : A_2(\Delta_1)\to A_2(\Delta_1)$ by $L_{s,\underline{i}}g(z) = g(T_{\underline{i}}z)w_{s,\underline{i}}(z)$, where the analytic weight functions $w_{s,\underline{i}}$ are given by $w_{s,\underline{i}}(z) = |DT_{\underline{i}}(z)|^s$. The n^{th} iterate of the transfer operator L_s is given by

$$L_s^n = \sum_{|\underline{i}|=n} L_{s,\underline{i}}.$$

The additivity of the trace means we can write

$$\mathrm{tr}(L_s^n) = \sum_{|\underline{i}|=n} \mathrm{tr}(L_{s,\underline{i}}). \tag{3.11}$$

For each $\underline{i}$ there is a unique fixed point $z_{\underline{i}}$ of the contraction $T_{\underline{i}} : \Delta_1 \to \Delta_1$. We can compute the trace of $L_{s,\underline{i}}$ by evaluating the eigenvalues of this operator and summing. In particular, consider the eigenfunction equation $L_{s,\underline{i}}h(z) = \lambda h(z)$. We can evaluate this at $z = z_{\underline{i}}$ to deduce that $w_{s,\underline{i}}(z_{\underline{i}})h(z_{\underline{i}}) = \lambda h(z_{\underline{i}})$. If $h(z_{\underline{i}}) \neq 0$ then we see that the only solution corresponds to $\lambda = 1$. If $h(z_{\underline{i}}) = 0$, then we can differentiate the eigenvalue equation to get that

$$w'_{s,\underline{i}}(z)h(z) + w_{s,\underline{i}}(z)h'(z) = \lambda h'(z)$$

Evaluating this at $z = z_{\underline{i}}$ (and recalling that $h(z_{\underline{i}}) = 0$) we get that

$$w_{s,\underline{i}}(z_{\underline{i}})h'(z_{\underline{i}}) = \lambda h'(z_{\underline{i}})$$

If $h'(z_{\underline{i}}) \neq 0$ then we see that the only solution corresponds to $\lambda = w_{s,\underline{i}}(z_{\underline{i}})$. Proceeding inductively, we can see that the only eigenvalues are $\{\lambda_n\}_{n=1}^{\infty} = \{w_{s,\underline{i}}(z_{\underline{i}})^k : k \geq 0\}$. (Moreover, one can see that these eigenvalues are realized). Summing over these eigenvalues gives:

$$\mathrm{tr}(L_{s,\underline{i}}) = \sum_{n=1}^{\infty} \lambda_n = \frac{w_{s,\underline{i}}(z_{\underline{i}})}{(1 - T'_{\underline{i}}(z_{\underline{i}}))} = \frac{|T'_{\underline{i}}(z_{\underline{i}})|^s}{(1 - T'_{\underline{i}}(z_{\underline{i}}))}. \tag{3.12}$$

Finally, comparing (3.11) and (3.12) completes the proof. □

We will consider the Banach space

$$A := \left\{ h \in C(\coprod_{i=1}^{k} \overline{V}_i, \mathbb{C}) : h \text{ is holomorphic on } \coprod_{i=1}^{k} V_i \right\},$$

equipped with the norm $||h|| = \sup\{|h(z)| : z \in \coprod_{i=1}^{k} \overline{V}_i\} < \infty$.

Proposition 2 The operator $L_s : A \to A$ is nuclear.

Proof. Because property (iii) holds, this follows from Lemma 3.3 in [**?**]. □

In order to study periodic points it is useful to consider a more general setting. Let $V \subset \mathbb{C}^d$ be an open set and consider an analytic contraction

$T : V \to V$ such that $\overline{T(V)} \subset V$. The contraction $T : U \to U$ has a unique fixed point $\underline{z}^* \in U$.

For future reference, we write the components $T = (T_1, \ldots, T_d)$.

The operator $L_T : A \to A$ defined by $L_{\psi,T}h(\underline{z}) = \psi(T\underline{z})h(T\underline{z})$ is nuclear (by the same proof as the above proposition). If $L_{\psi,T}$ has eigenvalues λ_n, $n \geq 0$, then the trace $\mathrm{tr}(L_{\psi,T}) := \sum_{n=0}^{\infty} \lambda_n$ is well defined.

Proof. For each admissible string $\underline{i} = (i_1, \ldots, i_{n+1})$ let us first define composition-like operators $L_{s,\underline{i}} : A_\infty(D_{i_{n+1}}) \to A_\infty(D_{i_1})$ by

$$L_{s,\underline{i}}g(z) = g(T_{\underline{i}}z)w_{s,\underline{i}}(z), \tag{3.22}$$

where the weight functions $w_{s,\underline{i}} \in A_\infty(D_{i_1})$ are given by

$$w_{s,\underline{i}}(z) = |DT_{\underline{i}}(z)|^s.$$

For a fixed $i_1 = i$, the n^{th} iterate of the component transfer operator $L_{s,i}$ (see (3.7)) is given by

$$L^n_{s,i} = \sum_{\substack{|\underline{i}|=n+1 \\ i_1=i}} L_{s,\underline{i}},$$

where the summation is over those length-$(n+1)$ admissible strings $\underline{i} = (i_1, \ldots, i_{n+1})$ with $i_1 = i$.

Then note that the n^{th} iterates of the operators $M_{s,i} : A_\infty(D) \to A_\infty(D)$ (defined by (3.10)) satisfy

$$M^n_{s,i}u|_{D_i} = L^n_{s,i}u, \quad M^n_{s,i}u|_{D_j} = 0 \text{ if } j \in \{1, \ldots, k\} \setminus \{i\},$$

so we can express

$$L^n_s = \sum_{i=1}^{k} M^n_{s,i}. \tag{3.23}$$

The additivity of the trace means we then have

$$\mathrm{tr}(L^n_s) = \sum_{i=1}^{k} \mathrm{tr}(M^n_{s,i}) = \sum_{i=1}^{k} \mathrm{tr}(L^n_{s,i}) = \sum_{i=1}^{k} \sum_{\substack{|\underline{i}|=n+1 \\ i_1=i}} \mathrm{tr}(L_{s,\underline{i}}) = \sum_{|\underline{i}|=n+1} \mathrm{tr}(L_{s,\underline{i}}) = \sum_{\underline{i} \in \mathrm{Fix}(n)} \mathrm{tr}(L_{s,\underline{i}}). \tag{3.24}$$

The last equality in the above follows because if $i_1 \neq i_{n+1}$ then the domain and target spaces of the operator $L_{s,\underline{i}} : A_\infty(D_{i_{n+1}}) \to A_\infty(D_{i_1})$ are not the same, so it has no eigenvalues.

If $\underline{i} \in \mathrm{Fix}(n)$, however, we have the following trace formula for the operators $L_{s,\underline{i}}$ in terms of the fixed point $z_{\underline{i}}$ of the composition $T_{\underline{i}}$,

$$\mathrm{tr}(L_{s,\underline{i}}) = \frac{w_{s,\underline{i}}(z_{\underline{i}})}{\det(I - DT_{\underline{i}}(z_{\underline{i}}))} = \frac{|DT_{\underline{i}}(z_{\underline{i}})|^s}{\det(I - DT_{\underline{i}}(z_{\underline{i}}))}. \tag{3.25}$$

The above formula (3.25) has its origins in the work of Atiyah & Bott [?] on the Lefschetz fixed point theorem, and in our context is proved in [?] (see also [?]). Note that since $T_{\underline{i}} : U_{i_1} \to U_{i_n}$ is a contraction, then the determinant $\det(I - DT_{\underline{i}}(\underline{z}_{\underline{i}})) > 0$.

Combining (3.24) and (3.25) completes the proof. □

Remark Fried actually corrected a minor error in Grothendieck's original paper which was reproduced in Ruelle's paper. □

Combining the above gives us the following bound of Fried.

Lemma 2

$$|b_n(s)| \leq C^n n^{n/2} \exp\left(cn - bn^{1+1/d}\right), \text{ for } n \geq 0 \tag{3.3}$$

for some $C = C(s) > 1$, and $c, b > 0$.

3.5 Julia sets For practical purposes, our algorithm is effective in computing the dimension $\dim_H(J_c)$ of the Julia set J_c if we choose c either in the main cardioid of the Mandelbrot set M, or c outside of M, say. In the latter case all periodic points are repelling, while in the former case all periodic points are repelling except for a single attractive fixed point. We can give explicitly estimate $\gamma = \gamma_c$ for c close to 0.

For quadratic maps we know that $T'(z) = 2z$ and if $T^n(z) = z$ then by the chain rule

$$(T^n)'(z) = T'(T^{n-1}z) \cdots T'(Tz).T'(z) = 2^n (T^{n-1}z) \cdots (Tz).z$$

and so the coefficients in the expansions take a simpler form.

Example 3.5.1 ($c = i/4$) First we consider the purely imaginary value $c = i/4$, which lies in the main cardioid of the Mandelbrot set. Table 1 illustrates the successive approximations s_N to $\dim_H(J_{i/4})$ arising from our algorithm.
=2.5in frog2.eps

The Julia set for $z^2 + i/4$ is the boundary between the white and black regions. (The white points are those which do not escape to infinity)

N	N^{th} approximation to $\dim(J_{i/4})$
3	1.1677078534172827136
4	0.9974580934808979848
5	1.0169164188641603339
6	1.0218764720532313644
7	1.0230776911089017648
8	1.0232246810534996595
9	1.0232072525392922127
10	1.0231992637099065199
11	1.0231993120941968028
12	1.0231992857944621198
13	1.0231992888227184780
14	1.0231992890455073830
15	1.0231992890300189633
16	1.0231992890307255210
17	1.0231992890309781268
18	1.0231992890309686742
19	1.0231992890309691466
20	1.0231992890309691251

Table 1 Successive approximations to $\dim(J_{i/4})$

Example 3.5.2 ($c = -\frac{3}{2} + \frac{2}{3}i$) If we take the parameter value $c = -\frac{3}{2} + \frac{2}{3}i$, which lies outside the Mandelbrot set, then the sequence of approximations to the dimension of J_c are given in Table 2. =2.5in frog4.eps

The Julia set for $z^2 - \frac{3}{2} + \frac{2}{3}i$ is a zero measure Cantor set - so invisible to the computer. The lighter regions are points "nearer" the Julia set which take longer to escape.

N	N^{th} approximation to $\dim(J_{-3/2+2i/3})$
1	0.7149355610391974853
2	0.9991996994914223217
3	0.8948837401931045135
4	0.8990693400138277172
5	0.9048525377869365908
6	0.9040847144651654898
7	0.9038472818583009063
8	0.9038738383368002502
9	0.9038748469934538668
10	0.9038745896021979531
11	0.9038745956441220338
12	0.9038745968650866636
13	0.9038745968171929578
14	0.9038745968108846487
15	0.9038745968111623979
16	0.9038745968111848616

Table 2 Successive approximations to $\dim_H(J_{-3/2+2i/3})$

Example 3.5.3 $c = -5$ For real values of c which are strictly less than -2, the Julia set J_c is a Cantor set completely contained in the real line. For such cases we have, by Corollary 3.1, the faster $O(\delta^{N^2})$ convergence rate to $\dim(J_c)$, as illustrated in Table 3 for the case $c = -5$.

N	N^{th} approximation to $\dim(J_{-5})$
1	0.4513993584764174609675959101241383349
2	0.4841518684194122992464635900326070715
3	0.4847979587486975778612282908975662571
4	0.4847982943561895699730717563576367090
5	0.4847982944381635057518511943420942957
6	0.4847982944381604305347487891271825909
7	0.4847982944381604305383984765793729512
8	0.4847982944381604305383984781726830747

Table 3 Successive approximations to $\dim_H(J_{-5})$

Example 3.5.4 ($c = -20$) For larger negative real values of c, the hyperbolicity of $f_c : J_c \to J_c$ is more pronounced, so that the constant $0 < \delta < 1$ in the $O(\delta^{N^2})$ estimate is closer to zero, and the convergence to $\dim_H(J_c)$ consequently faster. Table 4 illustrates this for $c = -20$.

N	N^{th} approximation to $\dim_H(J_{-20})$
1	0.31485651652009699091265279629753355933688857812644665851918
2	0.31850483144363986562810164826944017431378984622904321285835
3	0.31850809576591085725942984004207253452015913804880055477625
4	0.31850809575800523882867786043747732330759968092023152922729
5	0.31850809575800524988789850335472906645586111530021825766595
6	0.31850809575800524988789848098884346788677292871828344714065
7	0.31850809575800524988789848098884348414792438297975066097358
8	0.31850809575800524988789848098884348414792438305840652044425

Table 4 Successive approximations to $\dim_H(J_{-20})$

Remark Of particular interest are those c in the intersection $M \cap \mathbb{R} = [-2, \frac{1}{4}]$, i.e., the where the real axis intersects the Mandelbrot set. For values $-3/4 < c < 1/4$ (in the main Cartoid) the map T_c is expanding and the dimension $c \mapsto \dim(J_c)$ changes analytically. Indeed, about $c = 0$ we have the asymptotic expansion of Ruelle, mentioned before. However, at $c = 0$ the map $T_{c=\frac{1}{4}}$ is *not expanding* (since $T_{c=\frac{1}{4}}$ has a parabolic fixed point of derivative 1 at the point $z = \frac{1}{2}$). Moreover, $c \mapsto \dim(J_c)$ is actually discontinuous at $c = 1/4$. This phenomenon was studied by Douady, Sentenac & Zinsmeister. Havard & Zinsmeister proved that when restricted to the real line, the left derivative of the map $c \mapsto \dim(J_c)$ at the point $c = 1/4$ is infinite.

One advantage of this method is that it leads to effective estimates on the rate of convergence of the algorithm. This is illustrated by the following result.

Proposition 3.13 For any $\eta > 1/2$ there exists $\epsilon > 0$ such that if $|c| < \epsilon$ then the expansion coefficient for T_c is less than η.

The proof is very easy.

Proof. First consider the (unperturbed) map $T(z) = z^2$, whose Julia set is the unit circle S^1. We have a natural Markov partition consisting of the upper and lower semi-circles, and corresponding local inverse branches $T_0(z) = z^{1/2}$ and $T_1(z) = -z^{1/2}$. Let us think of T, T_0, T_1 as maps defined on subsets of $\mathbb{R}^2$ (and by abuse of notation we will continue to denote f, T_0, T_1). Taking polar coordinates (r, θ), define the rectangular regions

$$U_0 = [1 - \varrho, 1 + \varrho] \times [0, 2\pi] \subset \mathbb{R}^2 \text{ and } U_1 = [1 - \varrho, 1 + \varrho] \times [-2\pi, 0] \subset \mathbb{R}^2,$$

for some as yet undefined $0 < \varrho < 1$. We then have formulae $T_0(r, \theta) = (r^{1/2}, \theta/2)$ and $T_1(r, \theta) = (r^{1/2}, -\theta/2)$. Thus

$$T_0 U_0 = [(1-\varrho)^{1/2}, (1+\varrho)^{1/2}] \times [0, \pi] \text{ and } T_1 U_1 = [(1-\varrho)^{1/2}, (1+\varrho)^{1/2}] \times [-\pi, 0].$$

Both maps $T_i : U_i \to \mathbb{R}^2$ are real-analytic, so we may consider their holomorphic extensions to suitable subsets of $\mathbb{C}^2$. Define the poly-disc

$$V = D_\varrho(1) \times D_{2\pi}(0) \subset \mathbb{C}^2$$

(i.e. the product of the radius-ϱ disc around 1 in the complex r-plane with the radius-2π disc around 0 in the complex θ-plane). Both T_0 and T_1 extend holomorphically to V, and as usual we continue to denote these extensions T_0, T_1. Let us concentrate on the map T_0, the other map being similar.

We see that the image T_0V is contained in the poly-disc $D_{1-(1-\varrho)^{1/2}}(1) \times D_\pi(0)$.

In the θ-plane this gives a contraction ratio of 1/2. In the r-plane the contraction ratio is

$$\frac{1-(1-\varrho)^{1/2}}{\varrho} = 1/2 - \varrho/8 + \ldots,$$

which can be made arbitrarily close to 1/2 by choosing ϱ small.

Therefore the overall contraction ratio is also 1/2, as expected. □ □

3.6 Schottky groups Limit sets

Example 3.6.1 Fix $2p$ disjoint closed discs $D_1, \ldots, D_{2p}$ in the plane, and Möbius maps $g_1, \ldots, g_p$ such that each g_i maps the interior of D_i to the exterior of D_{p+i}. The corresponding Schottky group is defined as the group generated by $g_1, \ldots, g_p$. The associated limit set Λ is a Cantor subset of the union of the interiors of the discs $D_1, \ldots, D_{2p}$. We define a map T on this union by $T|_{int(D_i)} = g_i$ and $T|_{int(D_{p+i})} = g_i^{-1}$. A *reflection group* is a Schottky group with $D_i = D_{p+i}$ for all $i = 1, \ldots, p$.

Example 3.6.2. Quasifuchsian groups Such groups are isomorphic to the fundamental group of a compact Riemann surface, and are obtained by a quasiconformal deformation of a Fuchsian group (a Kleinian group whose limit set is contained in some circle). The limit set Λ of a quasifuchsian group is a simple closed curve. We can associate an expanding map T with the limit set of any Fuchsian group, and the quasiconformal deformation induces an expanding map on Λ.

We show that the Hausdorff dimension of the limit sets Λ of both Schottky and quasifuchsian groups can be efficiently calculated via a knowledge of the derivatives $(T^n)'(z)$, evaluated at periodic points $T^n z = z$.

Theorem 3.14 (Kleinian groups) Let Γ be a finitely generated non-elementary convex cocompact Schottky or quasifuchsian group, with associated limit set Λ. The algorithm applies.

First suppose Γ is a Schottky group. We define a map T on the union $\cup_{j=1}^{2p} D_j$ by $T|_{int(D_j)} = g_j$ and $T|_{int(D_{p+j})} = g_j^{-1}$, for $j = 1, \ldots, p$, A Markov partition for this map just consists of the collection of interiors $\{int(D_i)\}_{i=1}^{2p}$.

The corresponding $2p \times 2p$ transition matrix A has entries $A(i, p+i) = 0 = A(p+i, i)$ for each $i = 1, \ldots, p$, and all other entries are 1 (in the reflection group case, the transition matrix has zeros along the leading diagonal, and 1's elsewhere).

Now T is not quite an expanding map, since the conformal derivative $|Dg_j(z)| = 1$ on the boundary of D_j. However, the second iterate of T *is* expanding. Conformality and real-analyticity are clearly satisfied, so by Theorem 3 we deduce the result for Schottky groups.

Suppose Γ is quasifuchsian, with limit set Λ. Now Γ is quasi-conformally conjugate to some Fuchsian group Γ'. Bowen & Series proved there exists an expanding Markov map $S : S^1 \to S^1$ which faithfully models the action of Γ', and the quasiconformal deformation conjugates this to an expanding Markov map $T : \Lambda \to \Lambda$. Conformality and real-analyticity are clearly satisfied.

Example 3.6.3 The following family of reflection groups was considered by McMullen. Consider three circles $C_0, C_1, C_2 \subset \mathbb{C}$ of equal radius, arranged symmetrically around S^1, each intersecting the unit circle S^1 orthogonally, and meeting S^1 in an arc of length θ. We do not want the C_i to intersect each other, so we ask that $0 < \theta < 2\pi/3$. For definiteness let us suppose each C_i has radius $r = r_\theta = \tan\frac{\theta}{2}$, and that the circle centres are at the points $z_0 = a$, $z_1 = ae^{2\pi i/3}$ and $z_2 = ae^{-2\pi i/3}$ (where $a = a_\theta = \sqrt{1+r^2} = \sec\frac{\theta}{2}$). =2.5in limitplus.eps

Figure 5 Reflection in three circles The reflection ρ_i :ˆ

C →ˆ

C *inthecircle*C$_i$ takes the explicit form

$$\rho_i(z) = \frac{r^2}{|z - z_i|^2}(z - z_i) + z_i.$$

Let $\Lambda_\theta \subset \mathbb{S}^1$ denote the limit set associated to the group Γ_θ of transformations given by reflection in these circles. For example, with the value $\theta = \pi/6$ we show that the dimension of the limit set $\Lambda_{\pi/6}$ is

$$\dim(\Lambda_{\pi/6}) = 0.18398306124833918694118127344474173288\ldots$$

which is empirically accurate to the 38 decimal places given. The approximations are shown in Table 5.

N	Largest zero of Δ_N
2	0.14633481296007741055454748401454596
3	0.18423440272351767688822531747382350
4	0.18399977929621235204864644797773486
5	0.18398305039516509087579859265399133
6	0.18398305988417009403195596234810316
7	0.18398306122261622100816402885866734
8	0.18398306124841998285455137338908131
9	0.18398306124833255797187772764544302
10	0.18398306124833929946685349025674957
11	0.18398306124833918404985469216386875
12	0.18398306124833918700689278881066430
13	0.18398306124833918693967757277042711
14	0.18398306124833918694121655021916395
15	0.18398306124833918694118046846226018
16	0.18398306124833918694118129222351397
17	0.18398306124833918694118127301338345
18	0.18398306124833918694118127345475071
19	0.18398306124833918694118127344451095
20	0.18398306124833918694118127344474707

Table 5 Successive approximations to $\dim(\Lambda_{\pi/6})$

Chapter 15

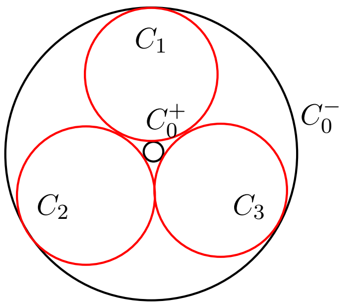
Applications

15.1 Circle packings

We begin with a brief history. Apollonius (c. 240 - 190 BC) who was known as the "The Great Geometer"and was a greek geometer born in Perga (now in Turkey). He proved the following basic theorem.

Theorem 35 (Apollonius)**.** *Given three mutually tangent circles C_1, C_2, C_3 with disjoint interiors there are precisely two new circles C_0^-, C_0^+ which are tangent to each of the original three.*

Proof. One can choose a Möbius map $g : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ which takes the tangency point of two of the circle ($C_1 \cap C_2$, for example) to ∞. The circles C_1 and C_2 are mapped to parallel lines with the image $g(C_3)$ being a circle sitting between then and tangent to both. But we can then translate $g(C_3)$ (twice) to two images tangent to both $g(C_3)$ and the parallel lines. Mapping this configuration back under g^{-1} gives the required result. $\square$

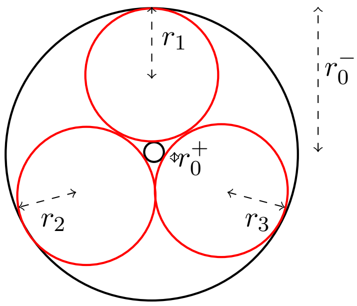


But one can ask: *How are the radii of these circles related?* This problem was studied by royalty. Princess Elizabeth of Bohemia (1618-1680) was the daughter of King Frederick V of Bohemia (whose brief reign lasted 1 year and 4 days). Her education included correspondence with Rene Descartes (1596-1650), the french mathematician and philosopher on many topics, including Apollonian circles. When she wrote to Queen Christina of Sweden asking help regaining her Father's lost lands, the Queen instead invited Descartes to Stockholm, which proved unfortunate for him since he died of pneumonia caught during his 5am audiences in a draughty palace.

In 1643, Descartes set Elizabeth the following problem: *Assume that the radii of the original* 3 *circles are* $r_1, r_2, r_3 > 0$ *determine the radius* r_0 *of a fourth mutually tangent circle.* Her solution was the following.

Theorem 36 (Descartes - Princess Elizabeth)**.** *We can write*

$$2\left(\frac{1}{a_0^2}+\frac{1}{a_1^2}+\frac{1}{a_2^2}+\frac{1}{a_3^2}\right)=\left(-\frac{1}{a_0}+\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}\right)^2 \tag{1}$$



Proof. Given circles

$$C_i = \{(x_1^{(i)}, x_2^{(i)}) : (c_1^{(i)} - x_1^{(i)})^2 + (c_2^{(i)} - x_2^{(i)}) = r_i^2\} \text{ for } i = 1, \cdots, 4,$$

with centres $c_i = (c_1^{(i)}, c_2^{(i)}) \in \mathbb{R}^2$ and radii $r_i > 0$ we associate

$$\langle C_i, C_j \rangle = \frac{d^2 - r_i^2 - r_j^2}{2 r_i r_j}$$

where $d = \|c_i - c_j\|$ is the distance apart of the centres. In particular, if we are assuming C_i and C_j are tangent if $i \neq j$ then we easily see that

$$\langle C_i, C_j \rangle = \begin{cases} -1 & \text{if } i = j \\ 1 & \text{if } i \neq j. \end{cases} \tag{1}$$

We can associate to the circle C_i the vector

$$v_i = \begin{pmatrix} x_1^{(i)}/r_i \\ x_2^{(i)}/r_i \\ 1/r_i \\ ((x_1^{(i)})^2 + (x_2^{(i)})^2)/r_i \end{pmatrix} \text{ and write } g = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 0 \end{pmatrix}$$

and then we can rewrite $\langle C_i, C_j\rangle = v_i^T g v_i$. We can then combine the four column vectors to get a 4×4 matrix $\mathbf{C} = (v_1, v_2, v_3, v_4)$. By (1) we can write

$$\mathbf{C}^T g\mathbf{C} = \begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{pmatrix}.$$

A simple observation is that the square of this matrix is $(\mathbf{C}^T g\mathbf{C})^2 = 4\mathbf{I}$, where $\mathbf{I}$ is the identity matrix, and thus $(\mathbf{C}^T g\mathbf{C})^{-1} = \frac{1}{4}(\mathbf{C}^T g\mathbf{C})$. Taking inverses and rearranging gives $g^{-1} = \frac{1}{4}\mathbf{C}(\mathbf{C}^T g\mathbf{C})\mathbf{C}^T$, i.e.,

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 1/r_1 & 1/r_2 & 1/r_3 & 1/r_4 \\ * & * & * & * \end{pmatrix}\begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{pmatrix}\begin{pmatrix} * & * & 1/r_1 & * \\ * & * & 1/r_2 & * \\ * & * & 1/r_3 & * \\ * & * & 1/r_4 & * \end{pmatrix}.$$

But comparing the entry in row 3 and column 3 gives the result. □

Since this is a quadratic polynomial in c_0, given $r_1, r_2, r_3 > 0$ (and thus c_1, c_2, c_3) we actually have two possible solutions

$$c_0^{\pm} = c_1 + c_2 + c_3 \pm 2\sqrt{c_1c_2 + c_2c_3 + c_3c_1} \tag{2}$$

i.e., the curvatures of the two circles of Apollonius. The solution c_0^+ will be positive, but the solution c_0^- will be negative. We interpret the latter as corresponding to a circle of radius $r_0 = 1/|c_0^-| > 0$.

The formula of Descartes and Elizabeth was later rediscovered by Frederick Soddy (1877-1956) the winner of the Nobel prize for chemistry in 1921. He chose to publish it as a poem in the journal *Nature*.

Adding the two solutions

$$\begin{aligned} c_0^+ &= c_1 + c_2 + c_3 + 2\sqrt{c_1c_2 + c_2c_3 + c_3c_1} \text{ and} \\ c_0^- &= c_1 + c_2 + c_3 - 2\sqrt{c_1c_2 + c_2c_3 + c_3c_1} \end{aligned}$$

from (2) gives

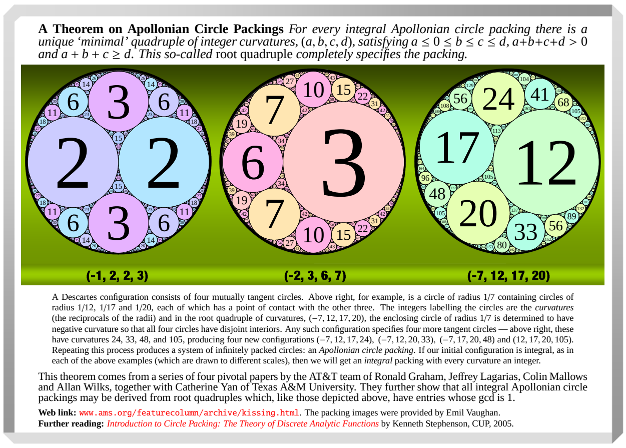
$$c_0^+ + c_0^- = 2\,(c_1 + c_2 + c_3)\,.$$

Therefore, we easily deduce that:

Lemma 56. *If $c_0^-, c_1, c_2, c_3 \in \mathbb{Z}$ then $c_0^+ \in \mathbb{Z}$.*

Proceeding inductively proves the following.

Corollary 12. *If the four initial Apollonian circles have curvatures that are integers then so do all of the others.*



Clearly these curvatures tend to infinity (i.e., the sequence of radii (r_n) tends to zero) since the total area enclosed by the circles is

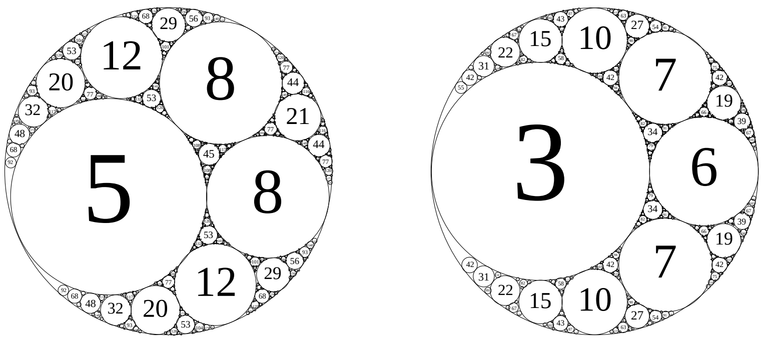
$$\sum_{n=1}^{\infty} \pi r_n^2 = \sum_{n=1}^{\infty} \pi c_n^{-2} < +\infty. \tag{3}$$

Example 60. *Let us start with an example with radii*

$$r_1 = \frac{1}{5}, \quad r_2 = \frac{1}{8}, \quad r_3 = \frac{1}{8} \text{ and } r_0^- = -\frac{1}{3},$$

i.e., curvatures $c_1 = 5$*,* $c_2 = 8$*,* $c_3 = 8$ *and* $c_0^- = -3$*. We can consider the values of the curvatures* c_n $(n \geq 1)$ *we get for all of the circles*

$$(c_n)_{n=1}^{\infty} = 5, 8, 8, 12, 12, 20, 20, 21, 29, 29, 32, 32, \cdots$$



Example 61. *Let us next consider the example with*

$$r_1 = \frac{1}{3}, \quad r_2 = \frac{1}{6}, \quad r_3 = \frac{1}{7}, \quad \text{and } r_0^- = -\frac{1}{2},$$

i.e., curvatures $c_1 = 3$*,* $c_2 = 6$*,* $c_3 = 7$ *and* $c_0^- = -2$*. We can consider the values of the curvatures* c_n $(n \geq 1)$ *we get for all of the circles*

$$(c_n)_{n=1}^{\infty} = 3, 6, 7, 7, 10, 10, 15, 15, 19, 19, 22, 22, \cdots$$

Definition 26. *Let $N(T)$ be the number of circles with curvature at most $T > 0$.*

An improvement on the basic result $N(T) \to +\infty$ as $T \to 0$, which follows from (3). is the following.

Theorem 37 (Kontorovich-Oh, 2009)**.** *There exist $K, \delta > 0$ such that $N(T) \sim KT^{\delta}$ as $T \to +\infty$, i.e.,*

$$\lim_{T \to +\infty} \frac{N(T)}{KT^{\delta}} = 1.$$

This doesn't require integral curvatures. The original proof used spectral theory of the Laplacian and hyperbolic geometry.

Lemma 57. *We denote by $\mathcal{A}$ the closure of the union of all the circles.*

1. *The exponent δ is equal to the Hausdorff dimension of $\mathcal{A}$.*
2. *All of these Apollonian circle packings have the same dimension δ.*

Proof. The second part comes from the fact that any two such circle packings are related by a Möbius maps. This is because Möbius maps take circles to circles and once the initial circles are aligned the remaining circles match up because of this property. Then, in particular, they have the same dimension. □

Curt McMullen calculated $\delta = 1.30568\ldots$.

One might compare this with a similar looking problem. Let

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, \ldots$$

be the *prime numbers*. Let $\pi(T)$ denote the number of prime numbers less than $T > 0$. Since there are infinitely many primes, we see that $\pi(T) \to \infty$ as T tends to infinity.

Theorem 38 (Prime Number Theorem: Hadamard (1896))**.**

$$\pi(T) \sim \frac{T}{\log T} \qquad \left(i.e., \lim_{T \to +\infty} \frac{\pi(T)}{\frac{T}{\log T}} = 1 \right)$$

Hadamard

The Prime Number Theorem was proved by Jacques Hadamard.

For primes: use Riemann ζ-function $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$.

- $\zeta(s)$ converges to a non-zero analytic function on $Re(s) > 1$.
- $\zeta(s)$ has a simple pole at $s = 1$.
- $\zeta(s)$ has no zeros on $Re(s) = 1$.

For circles use complex function $\eta(s) = \sum_{n=1}^{\infty} c_n^{-s}$.

- $\eta(s)$ converges to a non-zero analytic function on $Re(s) > \delta$.
- $\eta(s)$ has a simple pole at $s = \delta$.
- $\eta(s)$ has no poles on $Re(s) = \delta$.

One can apply classical tauberian theorems to get the asymptotic formula (i.e., a theorem which converts properties of series into counting results).

1. using the above strategy from Number Theory, with circle radii replacing prime numbers,
2. using transformations of $\mathcal{A}$ whose images systematically generate circles (originally observed by mancunian Philip Beecroft in 1842, in the wonderfully named journal *Lady's and Gentleman's diary*) and
3. some ideas from dynamical systems to prove the necessary results on $\eta(s)$.

This method is fairly flexible and applies to quite different problems and can be used to prove other types of related results.

15.2 The Zaramba conjecture

The Zaremba conjecture [**?**] was formulated in 1972, motivated by problems in numerical analysis. It deals with the denominators that can occur in finite continued fraction expansions using a uniform bound on the digits. A nice account appears in the very informative survey of Kontorovich [**?**].

Zaremba conjecture. For any natural number $q \in \mathbb{N}$ there exists p (coprime to q) and $a_1, \cdots, a_n \in \{1, 2, 3, 4, 5\}$ such that

$$\frac{p}{q} = [0; a_1, \cdots, a_n] \colon = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\cdots + \cfrac{1}{a_n}}}}.$$

Let us denote for each for $N \geq 1$ and $m \geq 2$,

$D_m(N) :=$

$$\mathrm{Card}\left\{1 \leq q \leq N \mid \exists p \in \mathbb{N}, (p,q)=1,\ a_1, \cdots, a_n \in \{1,2,\cdots,m\} \text{ with } \frac{p}{q} = [0; a_1, \cdots, a_n]\right\},$$

i.e., the number of $1 \leq q \leq N$ which occur as denominators of finite continued fractions using digits $|a_i| \leq m$. The Zaremba conjecture would correspond to $D_5(N) = N$ for all $N \in \mathbb{N}$. The conjecture remains open, but Huang [**?**], building on work of Bourgain and Kontorovich [**?**], proved the following version of Zaremba conjecture.

[Bourgain—Kontorovich, Huang] There is a density one version of the Zaremba conjecture, i.e.,

$$\lim_{N \to +\infty} \frac{D_5(N)}{N} = 1.$$

There have been other important refinements on this result by Frolenkov–Kan [**?**], [**?**], Kan [**?**], [**?**], Huang [**?**] and Magee–Oh–Winter [**?**].

Let us introduce for each $m \geq 2$,

$$E_m : = \{[0; a_1, a_2, \cdots] \mid a_n \in \{1, 2, \cdots, m\} \text{ for all } n \in \mathbb{N}\}$$

which is a Cantor set in the unit interval. Originally, Bourgain—Kontorovich [**?**] proved an analogue to Theorem 15.2 for $D_{50}(N)$. Amongst other things, their argument, related to the circle method, used the fact that the Hausdorff dimension $\dim_H(E_{50})$ is sufficiently close to 1 (more precisely, $\dim_H(E_{50}) > \frac{307}{312}$). In Huang's refinement of their approach, he reduced m to 5, i.e. replaced the alphabet $\{1, 2, \cdots, 50\}$ with $\{1, 2, 3, 4, 5\}$, as in the statement of Theorem 15.2. In Huang's approach, it was sufficient to show that $\dim_H(E_5) > \frac{5}{6}$. In [**?**] there is an explicit rigorous bound on the Hausdorff dimension of this set which confirms this inequality. The approach used there is the periodic point method, whereas in this article we use a different method to confirm and improve on these bounds.

As another example, we recall the following result for $m = 4$ and the smaller alphabet $\{1, 2, 3, 4\}$.

[Kan [**?**]] For the alphabet $\{1, 2, 3, 4\}$ there is a positive density version of the Zaremba conjecture, i.e.,

$$\liminf_{N \to +\infty} \frac{D_4(N)}{N} > 0.$$

The proof of the result is conditional on the lower bound $\dim_H(E_4) > \frac{\sqrt{19}-2}{3}$. In [**?**] this inequality is attributed to Jenkinson [**?**], where this value was, in fact, only heuristically estimated. In [**?**] there is an explicit rigorous bound on the Hausdorff dimension of this set which confirms this inequality. The approach used there is the periodic point method, whereas in this article we give a different method to confirm and improve on these bounds, as well as give new examples. These results are presented in §**??**.

15.3 Diophantine Approximations

Given any irrational number $\alpha \in \mathbb{R}$, we can approximate it arbitrarily closely by rational numbers, since they are dense in the real numbers. The following is a very classical result. [1]

Theorem 39 (Dirichlet, 1840). *Let α be an irrational number. We can find infinitely many distinct $p, q \in \mathbb{Z}$ $(q \neq 0)$ such that*

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2} \tag{4.1}$$

Proof. The proof just uses the "pigeon-hole principle". Let $N \geq 1$. Consider the $N+1$ fractional parts $\{\alpha\}, \{2\alpha\}, \{3\alpha\}, \cdots, \{(N+1)\alpha\} \in [0,1]$ (where $0 \leq \{j\alpha\} < 1$ is the fractional part of $j\alpha$, i.e., $j\alpha = \{j\alpha\} + [j\alpha]$ with $[j\alpha] \in \mathbb{N}$). If we divide up the unit interval into N-intervals $[0, \frac{1}{N}]$, $[\frac{1}{N}, \frac{2}{N}]$, ..., $[\frac{N-1}{N}, 1]$, each of length $\frac{1}{N}$, then one of the intervals must contain at least two terms $\{i\alpha\}, \{j\alpha\}$, say, for some $1 \leq i < j \leq N+1$. In particular, $0 \leq \{i\alpha\} - \{j\alpha\} \leq \frac{1}{N}$ from which we see that

$$0 \leq \alpha \underbrace{(i-j)}_{=:q} - \underbrace{([\alpha i] - [\alpha j])}_{=:p} = \{i\alpha\} - \{j\alpha\} \leq \frac{1}{N}$$

where $0 \leq q \leq N$. In particular, writing $p = [\alpha i] - [\alpha j]$ and $q = i - j$ we have that $|\alpha - \frac{p}{q}| \leq \frac{1}{q^2}$. Moreover, by successively choosing N sufficiently large we can exclude previous choices of $\frac{p}{q}$ and thus generate an infinite sequence of approximations (4.1) $\square$ $\square$

In particular, since almost every number is irrational, almost every $0 < \alpha < 1$ satisfies (4.1). We want to consider what happens if we try still stronger approximations.

First version: Replace exponent in the denominator by a larger value: Considers instead the inequality (4.1) with the Right Hand Side decreased from $\frac{1}{q^2}$ to $\frac{1}{q^{2+\eta}}$, say, for some $\eta > 0$. In this case, the set Λ_η of $0 < \alpha < 1$ for which the stronger inequality

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^{2+\eta}} \tag{4.2}$$

has infinitely many solutions is smaller. In fact, the set has Hausdorff Dimension strictly smaller than 1 and so, in particular, has zero measure. This follows from the following classical result.

[1] Dirichlet was a distinguished mathematician, and was married to the sister of the composer Mendelhson

Theorem 40 (Janik-Besicovitch Theorem)**.** *For $\eta > 0$, the set of α with infinitely many solutions to (4.2) has zero measure. Moreover this set has Hausdorff dimension, i.e.,*

$$dim_H \underbrace{\left\{\alpha : \left|\alpha - \frac{p}{q}\right| < \frac{1}{q^{2+\eta}} \text{ for infinitely } p \in \mathbb{Z}, q \in \mathbb{Z} - \{0\}\right\}}_{=:\Lambda_\eta} = \frac{2}{2+\eta} < 1.$$

Proof. The upper bound on the dimension is easy to prove. Given $\epsilon > 0$, we can choose $q \geq 2$ such that $\frac{1}{q^{2+\eta}} < \delta \leq \frac{1}{(q-1)^{2+\eta}}$. For each $q \geq 1$, we can choose a cover for this set by the $q(q+1)/2$-intervals

$$\left(\frac{p}{q} - \frac{1}{q^{2+\eta}}, \frac{p}{q} + \frac{1}{q^{2+\eta}}\right), \text{ for } 0 \leq p \leq q.$$

Since these each have diameter $q^{-(2+\eta)} < \epsilon$ we deduce that $H_\epsilon^d \leq q^{2-d(2+\eta)}$. In particular, if $d > \frac{2}{2+\eta}$ then we see that $\lim_{\epsilon \to 0} H_\epsilon^d = 0$. We thus deduce that the Hausdorff dimension is at most $\frac{2}{2+\eta}$. We omit the other inequality, referring to the book of Falconer for the details. □ □

Second version: replace 1 *in numerator by a different value* C: A natural question to ask is how big a value of $C = C(\alpha) \geq 1$ we can choose such that we can still find infinitely many distinct $p, q \in \mathbb{Z}$ $(q \neq 0)$ such that

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{Cq^2}. \tag{4.3}$$

To begin with, we recall that there is a slightly stronger version of Dirichlet's theorem due to Hurewicz.

Theorem 41 (Hurewicz's Theorem)**.** *Let α be an irrational number. We can find infinitely many distinct $p, q \in \mathbb{Z}$ $(q \neq 0)$ such that*

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{\sqrt{5}q^2}$$

In particular, we can always choose $C \geq \sqrt{5} = 2.23607\ldots$. (The proof, which is not difficult, uses Continued Fractions and can be found in the book of Hardy and Wright).

Notation For a given irrational number $0 < \alpha < 1$ we define $C(\alpha) \geq \sqrt{5}$ to be the largest C such that $|\alpha - p/q| < 1/(Cq^2)$, for infinitely many p, q, i.e.,

$$C(\alpha) = \liminf_{q \to \infty} \left[\max_{p \in \mathbb{N}} |q^2\alpha - pq|^{-1}\right].$$

We next want to consider the set of all possible values $C(\alpha)$, where α ranges over all irrational numbers between 0 and 1, say. We define the *Lagrange spectrum* to be the set $\mathbb{L} = \{C(\alpha) : \alpha \in (0,1) - \mathbb{Q}\}$.

=2.25in markov.eps

The Lagrange spectrum

By Hurewitz's theorem we know that $\mathbb{L} \subset [\sqrt{5}, +\infty)$. Moreover, it is also known that for $\alpha = 1/\sqrt{2}$, say, we have $C(1/\sqrt{2}) = \sqrt{5} \in \mathbb{L}$. In particular, we see that $\sqrt{5}$ is the smallest point in $\mathbb{L}$. In fact, the portion of the spectra below the value 3 is a countable set which is known exactly. For completeness, we quote the following result without proof.

Proposition 22. *We can identify*

$$\mathbb{L}\cap[0,3] = \left\{ \frac{1}{z}\sqrt{9z^2-4} : x^2+y^2+z^2 = 3xyz, \text{ where } x,y,z \in \mathbb{N} \text{ and } x,y \le z \right\}$$

In particular, the smallest value in the spectrum is $\sqrt{5}$ and the next smallest values (in ascending order) are: $\sqrt{8} = 2.82843\ldots$, $\sqrt{221}/5 = 2.97321\ldots$, $\sqrt{1517}/13 = 2.99605\ldots$, $\sqrt{7565}/29 = 2.99921\ldots$.

Since this portion $\mathbb{L} \cap [0,3]$ is countable, we have the following corollary. Corollary $\dim_H(\mathbb{L} \cap [\sqrt{5}, 3]) = 0$. At the other extreme, the spectrum is known to contain the whole interval $[\mu, +\infty)$, where $\mu \approx 4.527829566$.

It is an interesting question to ask how large an interval $[\sqrt{5}, t]$ $(t > 3)$ we can choose such that we still have $\dim_H(\mathbb{L} \cap [\sqrt{5}, t]) < 1$ or $\mathbb{L} \cap [\sqrt{5}, t]$ has zero Lebesgue measure. We shall return to this in a moment.

There is an alternative definition of $\mathbb{L}$ which is particularly useful in studying the region $\mathbb{L} \cap [\sqrt{5}, 4.527\ldots]$.

Proposition 4.2 The set $\mathbb{L}$ can also be defined in terms of doubly infinite sequences of positive integers. Given $a = (a_n)_{n\in\mathbb{Z}}$ we define

$$\lambda_i(a) = a_i + [a_{i+1}, a_{i+2}\ldots] + [a_{i-1}, a_{i-2}, \ldots], \quad i \in \mathbb{Z}$$

where, as usual, $[c_0, c_1, \ldots] = 1/(c_0 + (1/c_1 + \ldots))$ denotes the continued fraction with $c_0, c_1, \ldots \in \mathbb{N}$. We then have

$$\mathbb{L} = \left\{ L(a) = \limsup_{|i|\to\infty} \lambda_i(a) : a \in \mathbb{N}^{\mathbb{Z}} \right\}.$$

The proof is outside the scope of these notes, and is so omitted.

A little calculation shows:

1. If $a = (a_n)_{n\in\mathbb{Z}}$ has at least one entry greater than 2 then $L(a) \ge \sqrt{13}$. and indeed $L(a) = \sqrt{13}$ if and only if $a = (\ldots, 3, 3, 3, \ldots)$. However,
2. if a has entries only 1's and 2's then $L(a) \le \sqrt{12}$

In particular, we can deduce the following result.

Corollary There are *gaps* in the spectrum (i.e., intervals which don't intersect $\mathbb{L}$

Proof. This is apparent, since $(\sqrt{12}, \sqrt{13}) \cap \mathbb{L} \neq \emptyset$, as we saw above. □ □

We can now consider the problem of finding the Lebesgue measure and Hausdorff dimension of various portions of the spectrum. Let us define $\mathbb{L}_t = \mathbb{L} \cap [0, t]$. We have the following result.

Theorem 4.3 We can estimate

$$\dim_H(\mathbb{L}_{\sqrt{10}}) \approx 0.8121505756228 \text{ and } \dim_H(\mathbb{L}_{\sqrt{689}/8}) \approx 0.9716519526$$

(where $sqrt10 \approx 3.1622\ldots$ and $\sqrt{689}/8 \approx 3.2811\ldots$).

Sketch Proof If we consider $\Lambda_1 \subset E_2$ to be those numbers whose continued fraction expansions do *not* have consecutive triples $(i_k i_{k+1} i_{k+2}) = (121)$ then $\mathbb{L}_{\sqrt{10}} = \mathbb{L} \cap [0, \sqrt{10}] \subset \Lambda_1 + \Lambda_1$ In particular, $\dim_H(\mathbb{L}_{\sqrt{10}}) \leq 2\dim_H(\Lambda_1)$, and we can estimate the numerical value of $\dim_H(\Lambda_1)$ by the method in Chapter 3. Similarly, if we consider $\Lambda_2 \subset E_2$ to be those numbers whose continued fraction expansions do *not* have consecutive quadruples $(i_k i_{k+1} i_{k+2} i_{k+3}) = (1212)$ then $\mathbb{L}_{\sqrt{689}/8} = \mathbb{L} \cap [0, \sqrt{689}/8] \subset \Lambda_2 + \Lambda_2$ and $\dim_H(\mathbb{L}_{\sqrt{10}}) \leq 2\dim_H(\Lambda_2)$. Using degree-16 truncated equations we can estimate $\dim_H(\Lambda_1) \approx 0.4060752878114$ and $\dim_H(\Lambda_2) \approx 0.4858259763$, giving the upper bounds on the dimension in the theorem. On the other hand, a result of Moreira-Yoccoz implies equality. □

In particular the above result implies that:

Corollary $\mathbb{L}_{\sqrt{689}/8}$ has zero Lebesgue measure. Observe that $\sqrt{689}/8 \approx$ 3.2811... The strongest result in this direction is due to Bumby, who showed that $\mathbb{L}_{3.33437}$ has zero Lebesgue measure.

Remark The triples (x, y, z) are known as *Markoff triples.* A closely related notion is that of the Markoff spectrum. $\mathbb{M}$. Consider quadratic forms $f(x, y) = ax^2 + bxy + cy^2$ (with $a, b, c \in \mathbb{Z}$) for which $d(f) := b^2 - 4ac > 0$. If we denote $m(f) = \inf|f(x, y)|$, then *Markoff spectrum* $\mathbb{M}$ is defined to be the set of all possible values of $\sqrt{d(f)}/m(f)$. which can be defined in terms of minima of certain indefinite quadratic forms. The Lagrange spectrum $\mathbb{L}$ is a closed subset of $\mathbb{R}$. It is clear from this definition that the Lagrange spectrum is a subset of the Markoff spectrum. It is in the interval $(3, \mu)$ where the Markof and Lagrange spectra differ. The largest known number in $\mathbb{M}$ but not in $\mathbb{L}$ is $\beta \approx 3.293$ (the number is known exactly).

15.4 Fuchsian groups

15.5 Kleinian groups

The Limit sets of Kleinian groups often have similar features to those of Julia sets. Indeed, in the 1970's Sullivan devised a "dictionary" describing many of the corresponding properties.

Let $\mathbb{H}^3 = \{z + jt \in \mathbb{C} \oplus \mathbb{R} : t > 0\}$ be the three dimensional upper half space. We can equip this space with the Poincare metric

$$ds^2 = \frac{dx^2 + dy^2 + dt^2}{t^2}.$$

With this metric the space has curvature $\kappa = -1$. For a detailed description of the space and its geodesics we refer the reader to Bearden's book on *Discrete groups*.

We can identify the isometries for $\mathbb{H}^3$ and this metric with the (orientation preserving) transformations

$$(z, t) \mapsto \left(\frac{az + b}{cz + d}, t + 2\log|cz + d|\right),$$

where $a, b, c, d \in \mathbb{C}$ with $ad - bc = 1$. In particular, the first component is a linear fractional transformation and we can identify the space of isometries with the matrices $G = SL(2, \mathbb{C})$.

Defintion A Kleinian group $\Gamma < G$ is a finitely generated discrete group of isometries. Let Γ_0 be the generators of Γ.

Although the action of $g \in G$ is an isometry on $\mathbb{H}^3$, the action on the boundary is typically not an isometry. In particular, we can associate to each $g \in \Gamma$ its *isometric circle* $C(g) := \{z \in \mathbb{C} : |g'(z)| = 1\}$. This is a Euclidean circle in the complex plane $\mathbb{C}$.

Defintion We define the limit set $\Lambda = \Lambda_\Gamma \subset \mathbb{C} \cup \{\infty\}$ for Γ to be the set of all limit points (in the Euclidean metric) of the set of points $\{g(j) : g \in \Gamma\}$.

By way of clarification, we should explain that since Γ is a discrete group these limit points must necessarily be in the Euclidean boundary. Moreover, we should really take the limit points using the one point compactification of $\mathbb{C}$ (where the the compactification point is denoted by ∞. Depending on the choice of Γ, the limit set Λ_Γ may have different properties.

These include the possibilities that Λ_Γ is a Cantor set, or all of $\mathbb{C} \cup \{\infty\}$. We begin by considering one of the most famous examples of a Limit set for a Kleinian group - which happens to be neither of these cases.

Example 1.4.1 Apollonian circle packing. Consider three circles C_1, C_2, C_3 in the euclidean plane that are pairwise tangent. Inscribe a fourth circle C_4 which is tangent to all three circles. Within the three triangular region whose sides consist of the new circle and pairs of the other circles inscribe three new circles. Proceed inductively. The limit set is call an Apollonian circle packing. =4.50in apollonian.ps

The Apollonian circle packing

We can associate to each circle $C_i = \{z : |z - z_i| = r_i\}$ (with $z_i \in \mathbb{C}$ and $r_i > 0$) an element $g_i \in G$ associated to the linear fractional transformation

$$g_i : z \mapsto \frac{1}{r_i^2(z - z_i)}.$$

These correspond to generators for a Kleinian group $\Gamma < G$. The limit set is estimated to have dimension $1.305686729\ldots$.

Let us consider some special cases:

Example 1.4.2. Fuchsian Groups: Let $K = \{z \in \mathbb{C} : |z| = 1\}$ be the unit circle in the complex plane $\mathbb{C}$. If each element g preserves K then Γ is a *Fuchsian group.* In this case the isometric circles for each element $g \in \Gamma$ meet K orthogonally.

The standard presentation for a (cocompact) Fuchsian group is of the form

$$\Gamma = \langle g_1, \ldots, g_{2d} \in G : \prod_{i=1}^{d} [g_{2i-1}, g_{2i}] = 1 \rangle.$$

where $[g_{2i-1}, g_{2i}] = g_{2i-1} g_{2i} g_{2i-1}^{-1} g_{2i}^{-1}$. We can also consider the limit sets of such groups.

Theorem 1.4.1 The Limit set of a non-cocompact convex cocompact Fuchsian group is either:

1. a Cantor set lying in the unit circle; or

2. the entire circle.

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For Fuchsian groups (a subclass of Kleianin groups) the limit set could be the entire circle or a Cantor set.

Example 1.4.3. Quasi- Fuchsian Groups: We can next consider a Kleinian group whose generators (and associated isometric circles) are close to that of a Fuchsian group. Such groups are called *quasi-Fuchsian.* In this case the limit set is still homeomorphic to a closed circle. This is called a *quasi-circle.*

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Perturbing the generators of a Fuchsian group changes the limit circle to a quasi-circle. (The dotted circles represent the generators for the Fuchsian group (left) and quasi-Fuchsian group (right).)

However, although the quasi-circle is topologically a circle it can be quite different in terms of geometry.

Theorem 1.4.2 The Hausdorff dimension of a quasi-circles is greater than or equal to 1, with equality only when it is actually a circle. This result was originally proved by Bowen, in one of two posthumous papers published after his death in 1978. Quasi-circles whose Hausdorff dimension is strictly bigger than 1 are necessarily non-rectifiable, i.e., they have infinite length.

15.6 Horseshoes

Example Consider the example of a linear horseshoe. Taking the horizonal and vertical projections we have Cantor sets in the line with smaller Hausdorff dimensions $-\log 2/\log\alpha$ and $-\log 2/\log\beta$.

The next result says that Hausdorff dimension behaves in the way we might have guessed under addition of sets.

Proposition 1.6.3 Let $\Lambda_1, \Lambda_2 \subset \mathbb{R}$ and let

$$\Lambda_1 + \Lambda_2 = \{\lambda_1 + \lambda_2 : \lambda_1 \in \Lambda_1, \lambda_2 \in \Lambda_2\}$$

then $\dim_H(\Lambda_1 + \Lambda_2) \le \dim_H(\Lambda_1) + \dim_H(\Lambda_2)$.

Proof. It is easy to see from the definitions that $\dim_H(\Lambda_1 \times \Lambda_2) = \dim_H(\Lambda_1) + \dim_H(\Lambda_2)$. Since the map $L : \mathbb{R}^2 \to \mathbb{R}$ given by $L(x, y) = x + y$ is Lipshitz, the result follows. □ □

15.7 Kleinian groups

Given any Kleinian group Γ of isometries of n-dimensional hyperbolic space $\mathbb{H}^n$ we can associate the quotient manifold $M = \mathbb{H}^n/\Gamma$. The Laplacian $\Delta_M : C^\infty(M) \to C^\infty(M)$ is a self-adjoint second order linear differential operator. This extends to a self-adjoint linear operator Δ_M on the Hilbert space $L^2(M)$. In particular, the spectrum of $-\Delta_M$ is contained in the interval $[\lambda_0, +\infty)$, where λ_0 is the smallest eigenvalue. If M is compact then the constant functions are an eigenfunction and so $\lambda_0 = 0$. More generally, we can have $\lambda_0 > 0$.

Perhaps surprisingly, λ_0 is related to the Hausdorff dimension $\dim_H(\Lambda)$ of the Limit set by the following result.

Sullivan's Theorem $\lambda_0 = \min\{d(1-d), 1/4\}$

McMullen's Example This problem is very closely related to the geometry of an associated surface of constant curvature $\kappa = -1$. Consider the unit disk

$$\mathbb{D}^2 = \{x + iy \in \mathbb{C} : x^2 + y^2 < 1\}$$

with the Poincaré metric

$$ds^2 = 4\frac{dx^2 + dy^2}{(1 - x^2 - y^2)^2}$$

then $(\mathbb{D}^2, ds^2)$ has constant curvature $\kappa = -1$. Let $C_1, C_2, C_3 \subset \mathbb{C}$ be the three similar circles in the complex plane which meet the unit circle orthogonally and enclose an arc of length θ.

$$\text{radii } r = \sqrt{2} \text{ centres } c_1 = \sqrt{3}, c_2 = \sqrt{3}e^{2\pi i/3}, c_3 = \sqrt{3}e^{4\pi i/3}$$

We can identify the reflections in these circles with isometries $R_1, R_2, R_3 \subset \mathrm{Isom}(\mathbb{D}^2)$ and then consider the Kleinian group Γ_θ they generate. where $R_i : z \to \frac{3(z-c_i)}{|z-c_i|^2} + c_i$ $(i = 1, 2, 3)$. Let $\Gamma = \langle R_1, R_2, R_3 : R_1^2 = R_2^2 = R_3^2 = 1\rangle \subset \mathrm{Isom}(\mathbb{D}^2)$. We can then let $M = \mathbb{D}^2/\Gamma$ be the quotient manifold.

The Laplacian $\Delta_M : C^\infty(M) \to C^\infty(M)$ is given by

$$\Delta_M = (1 - x^2 - y^2)^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right).$$

The smallest eigengvalue of $-\Delta_M$ is related to the dimension d of the boundary by Sullivan's Theorem. In particular, we have the following corollary.

Proposition 4.4 When $\theta = \pi/6$ then we can estimate $\lambda_0 = 0.24922656...$

Proof. In Chapter 3 we estimated that $\dim_H(\Lambda) = 0.4721891278821...$ [2]. By applying Sullivan's Theorem, the result follows. □ □

On can also study the asymptotic behavior of $\dim_H(\Lambda_\theta)$. McMullen showed the following:

Propositon 4.5 The asymptotic behaviour of $\dim_H(\Lambda_\theta)$ is described by the following result:

1. $$\dim_H(\Lambda_\theta) \sim \frac{1}{|\log\theta|} \text{ as } \theta \to 0;$$

2. $$\dim_H(\Lambda_\theta) \sim 1 - \frac{1}{2}\left(\frac{2\pi}{3} - \theta\right) \text{ as } \theta \to \frac{2\pi}{3}.$$

(Equivalently, the associated smallest eigenvalue $\lambda_0(\theta)$ satisfies $\lambda_0(\theta) \sim \frac{1}{|\log\theta|}$ as $\theta \to 0$ and $\lambda_0(\theta) \sim \frac{1}{2}\left(\frac{2\pi}{3} - \theta\right)$ as $\theta \to \frac{2\pi}{3}$.)

Proof. For small θ, the radii of the circles C_i is well approximated by $\theta/2$. The derivative on C_j $(i \neq j)$ of the hyperbolic reflection in C_i is approximately $(\theta/2)^2/|C_i - C_j| \sim \theta^2/12$. Every periodic orbit $T^n x = x$ satisfies a uniform estimate $|(T^n)'(x)|^{1/n} \sim \theta^2/12$ from which we deduce that $P(-t\log|T'|)\ 2 - t(\theta^2/12)$, since there are 32^{n-1} periodic orbits of period n, for $n \geq 2$. Thus, solving for $2 - t\log(\theta^2/12) = 0$ gives that $t \sim \frac{1}{|\log\theta|}$.

[2]McMullen previously estimated $d = \dim_H(X) = 0.47218913...$

The proof for $\theta \sim \frac{2\pi}{3}$ relies of Sullivan's theorem and asymptotic behaviour of the eigenvalues, as controlled by a minimax principle. In particular, $\dim_H(\Lambda_\theta) \sim 1 - \lambda_0(\theta) \to 1$. However, one can write $\lambda_0(\theta) = \inf_f \int |\nabla f|^2 d\mathrm{vol} / \int |f|^2 d\mathrm{vol} \sim l_\theta$, where l_θ is the length of the boundary curves on the quotient surface. For θ close to $2\pi/3$ on can estimate $l_\theta \sim \sqrt{2\pi/3 - \theta}$. □

□

1.5 Horseshoes We now recall a famous Cantor set in Dynamical Systems. The "Horseshoe" was introduced by Smale as an example of invariant set for a (hyperbolic) diffeomorphism $f : S^2 \to S^2$ on the two sphere S^2.

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f bends the rectangle into a horseshoe. The Cantor set Λ is the set of points that never escape from the rectangle.

In the original construction, f is chosen to expand a given rectangle R (sitting on S^2) vertically; contract it horizontally; and bends it over to a horseshoe shape. The points that remain in the rectangle under all iterates of f (and f^{-1}) are an f-invariant Cantor set, which we shall denote by Λ. The rest of the points on S^2 are arranged to disappear to a fixed point.

In an more general construction, let M be a compact manifold and let $f : M \to M$ be a diffeomorphism. A compact set $\Lambda = \Lambda(f) \subset M$ is called invariant if $f(\Lambda) = \Lambda$. We say that $f : \Lambda \to \Lambda$ is *hyperbolic* if there is a continuous splitting $T_\Lambda M = E^s \oplus E^u$ of the tangent space into Df-invariant bundles and there exists $C > 0$ and $0 < \lambda < 1$ such that

$$||D_x f^n(v)|| \leq C\lambda^n ||v|| \text{ and } v \in E^s$$

$$||D_x f^{-n}(v)|| \leq C\lambda^n ||v|| \text{ and } v \in E^u.$$

We say that Λ is locally maximal if we can choose an open set $U \supset \Lambda$ such that $\Lambda = \cap_{n=-\infty}^{\infty} f^n U$. In general, we can take a horseshoe Λ to be an locally maximal f-invariant hyperbolic Cantor sets a diffeomorphism f on M.

Theorem 1.5.1 (Manning-McClusky) For Horseshoes $\Lambda(f)$ on surfaces we have that $\dim_H(\Lambda(f)) = \dim_B(\Lambda(f))$.

Moreover, Manning and McClusky gave an implicit formula for the Hausdorff dimension, which we shall return to in a later chapter.

Example Consider the case of the original Smale horseshoe such that $f : R \cap f^{-1}R \to R$ is a linear map which contracts (in the horizontal direction) at a rate α and expands (in the vertical direction) at a rate $1/\beta$. For a linear horseshoe Λ the work of Manning-McClusky gives that:

$$\dim_H(\Lambda) = \dim_B(\Lambda(f)) = \log 2 \left(\frac{1}{\alpha} + \frac{1}{\beta} \right).$$

Let us now consider the dependence of the dimension X on the diffeomeorphism f. Let $D \subset C^2(M, M)$ be the space of C^2 diffeomorphisms from M to itself. This comes equipped with a standard topology. We can consider a parmeterised family of diffeomorpisms $(-\epsilon, \epsilon) \ni \lambda \mapsto f_\lambda$. The first part of the next result shows smooth dependence of the Hausdorff dimension of horseshoes on surfaces. However, the second part shows this fails dramatically in higher dimensions.

Theorem 1.5.3

1. On surfaces the Hausdorff dimension $\dim_H(\Lambda(f_\lambda))$ of the horseshoe varies continuously (even differentiably).

2. There exist examples of horseshoes on three dimensional manifolds for which the Hausdorff dimension does not change continuously.

Palis and Viana originally showed continuity of the Hausdorff dimension in the case or surfaces, and Mane subsequently showed smoothness. Both results used a study of the "structural stability conjugacy map". Pollicott and Weiss showed the failure in higher dimensions by exploiting number theoretic results of two dimensional expanding maps.

Example Consider an extension of the original construction of Smale where the rectangle is now replaced by a cube C (sitting on the sphere S^3). We can arrange that f expands the cube in one direction; contracts it in the remaining two directions; and maps it back across C is in the Smale construction. In this case, the dimension depends on the alignment of the intersection of $f(C)$ and C in the two dimensional contracting direction.

15.8 Differences of Cantor sets

15.9 Microsets

15.10 Fourier dimension

Appendix: A little belated history

Felix Hausdorff

Felix Hausdorff was born on 8th November 1868 in Breslau, Germany (which is now Wroclaw, Poland) into a wealthy family. His Father was a textile merchant. In fact, Felix grew up in Leipzig after his parents moved there when he was a child. He studied Mathematics at Leipzig University, completing his PhD there in 1891.

He was subsequently a Privatdozent, and then an Extraordinary Professor in Leipzig. However, Hausdorff really wanted to be a writer and actually published books on philosophy and poetry under a pseudonym. In 1904 he even published a farce which, when eventually produced, turned out to be very successful. Following this literary phase, he concentrated again on mathematics, and during the next dozen years he made major contributions to both topology and set theory. In 1910 he moved to Bonn, and then in 1913 he moved again to take up an ordinary professorship in Greifswalf before finally, in 1921, he returned again to Bonn. In 1919 he introduced the notion of Hausdorff dimension in a seminal paper on analysis. This was essentially a generalisation of an idea introduced earlier by Carathéodory, but Hausdorff realised that the construction actually allows a definition of "fractional dimensions". In particular, Hausdorff's paper includes a proof of the famous result that the dimension of the middle-third Cantor set is $\log 2/\log 3$. Unfortunately, the final years of Hausdorff's life were tragic. He had come from a Jewish family, and in 1935 he was forced to retire by the Nazi regime in power in Germany. In 1941 he was scheduled to be sent to an internment camp, but managed to avoid being sent through the intervention of the University. However, this was merely a postponement, and on 26th Januray 1942 Hausdorff, his wife and sister-in-law committed suicide when internment seemed inevitable.

Figure 15.1: Felix Hausdorff (1868-1942)

Constantin Carathéodory

Constantin Carathéodory was born on 13th September 1873, in Berlin. He was of Greek extraction, being the son of a secretary in the Greek embassy in Berlin. As a stundent, he studied as a military engineer at the École Militaire de Belgique. Subsequently, he joined the

Figure 15.2: Constantin Carathéodory (1873-1950)

British colonial service and worked on the construction of the Assiut dam in Egypt in 1900. He then went on to study for his PhD in Berlin, and then Gottingen, before becoming a Provatdozent in Bonn in 1908. The following year he married - his own aunt! In the following years Carathéodory went on to hold chairs at Universities in Hanover, Breslau, Gottingen and Bonn. However, in 1919 the Greek Government asked him to help establish a new university in Smyrna. However, this was not a happy experience since the project was thwarted by a turkish attack. Eventually, following this interlude he was appointed to a chair in Munich, which he held until his retirement in 1938. He died there on 2nd February 1950.

Anton Julia

Anton Julia was born on 3rd February 1893 in Sidi Bel Abbés, in Algeria. As a soldier in the First World War, he was severely wounded during an attack on the western front. This resulted in a disfiguring injury and he had to wear a leather strap across his face for the rest of his life. In 1918 Julia published "Mémoire sur l'itération des fonctions rationnelles" on the iteration of a rational function f, much of the work done while he was in hospital. In this, Julia gave a precise description of the set of those points whose orbits under the iterates of the map stayed bounded. This received the Grand Prix de l'Académie des Sciences. Julia became a distinguished professor at the École Polytechnique in Paris. He died on 19 March 1978 in Paris. His work was essentially forgotten until B Mandelbrot brought it back to prominence in the 1970s through computer experiments.

Figure 15.3: Anton Julia(1893-1978)

Benoit Mandelbrot

Benoit Mandelbrot was born on 20th November 1924, in Warsaw. When his family emigrated to France in 1936 his uncle Szolem Mandelbrojt, who was Professor of Mathematics at the Collége de France, took responsibility for his

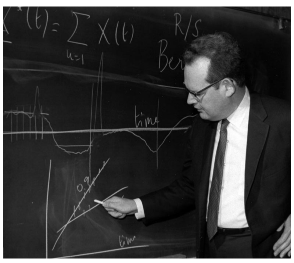

Figure 15.4: Benoit Mandelbrot (1924-20??)

early education. After studying at Lyon, he studied for his PhD at the École Polytechnique and after a brief spell in the CNRS, accepted an appointment with IBM. In 1945 Mandelbrot's uncle had recommended Julia's 1918 paper. However, is wasn't until the 1970s that he had returned to this problem. By this time rudimentary computer graphics allowed a study of the complicated fractal structure of Julia sets and Mandelbrot sets. This, and subsequent work, has provided and immense impetus to the study of Hausdorff Dimension.

Abram Besicovitch

Abram Besicovitch was born on 24th January 1891 in Berdyansk, Russia. His Father used to own a jeweller's shop. He studied mathematics at the University of St Petersburg, taking a chair there in 1991, during the Russian Civil War. Following positions in Copenhagen and Liverpool he moved to Cambridge in 1927, where he worked until his retirement in 1958. His work on sets of non-integer dimension was an early contribution to fractal geometry. Besicovitch extended Hausdorff's work to density properties of sets of finite Hausdorff measure. He died in Cambridge on 2nd November 1970.

Figure 15.5: Abram Besicovitch (1891-1970)