# Appolonius Circle Counting

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Circle packings in the plane date back over 2000 years and have attracted the attention of princesses and Nobel prize winners alike. The most famous are the Appolonian circle packings, consisting of a family of infinitely many circles with disjoint interiors, constructed iteratively by starting from a finite family of mutually tangent circles and successively inscribing circles into the spaces between them. We give a brief history of them and describe some recent developments.

Appolonius and his circles

The famous geometer Apollonius of Perga (circa 240–190 BC) made many important contributions to mathematics, perhaps the best known nowadays being the coining of the modern terms *ellipse*, *hyperbola* and *parabola*. Apollonius also established the following basic result.

Theorem 1 (Apollonius). *Given three tangent circles in the plane there are precisely two more circles each of which mutually tangent to the original three.*
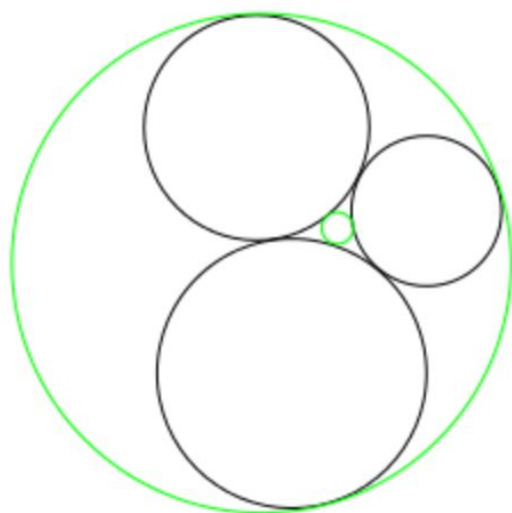
The three initial circles (or radii r_1, r_2, r_3, say) and the new Apollonian circles (of radii r_0, r_4, say)

There is a simple modern proof of this theorem using Möbius maps [7]. By transforming to infinity one of the points where two of the original circles touch the image circles are straight lines. The new circles are then simply translates of the third circle.

Descartes and Princess Elizabeth of Bohemia

Princess Elizabeth (1618–1680) was the daughter of King Frederick of Bohemia, whose ill-fated reign lasted a mere 1 year and 4 days. Her superior education included instruction from the great french mathematician and philosopher Rene Descartes (1596–1650). Included in their correspondence was a formula relating the radii r_0, r_4 of the two additional circles given by Apollonius' theorem to the radii r_1, r_2 and r_3 of the original three circles.

Theorem 2 (Descartes–Princess Elizabeth). *The two possible solutions ξ to the quadratic equation*

$$\frac{1}{\xi^2} + \frac{1}{r_1^2} + \frac{1}{r_2^2} + \frac{1}{r_3^2} = \left(\frac{1}{\xi} + \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} \right)^2$$

correspond to r_0 and $-r_4$.

Sadly, when the princess petitioned Queen Christine of Sweden for support in recovering her lands the only consequence was that Descartes went to Stockholm, dying there of pneumonia due to having 5:00am audiences in a drafty palace. Thwarted in her attempts to recover her family's lands in Bohemia, the princess retreated to a nunnery.

Frederick Soddy

The formula was rediscovered two centuries later by Frederick Soddy (1877–1956), and for this reason the circles are sometimes called *Soddy circles*. Soddy is probably more famous for having won the Nobel Prize for Chemistry in 1921, and having introduced the terms *isotopes* and *chain reaction*.

In 1936 he published the same theorem in the form of a poem in the journal *Nature* entitled *The Kiss Precise* [8]. (See "Part of *The kiss precise*".)

Part of *The kiss precise*

Four circles to the kissing come.
The smaller are the benter.
The bend is just the inverse of
The distance from the center.
Though their intrigue left Euclid dumb
There's now no need for rule of thumb.
Since zero bend's a dead straight line
And concave bends have minus sign,
The sum of the squares of all four bends
Is half the square of their sum.

Counting infinitely many circles

We can continue to add circles ad infinitum by itera-
tively applying Theorem 1 to every triples of mutually
tangent circles we see. Let $(r_n)_{n=0}^{\infty}$ enumerate their
radii.
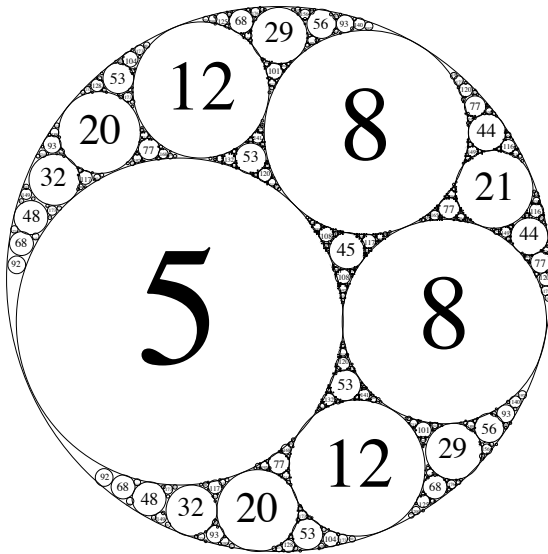
A fractal set $\mathscr{A}$ in the plane

Example. If we start with $\frac{1}{r_0} = 3$, $\frac{1}{r_1} = 5$, $\frac{1}{r_2} = \frac{1}{r_3} = 8$
then we can order the reciprocals of all the new radii.
In the example above this becomes

$$(1/r_n) = 5, 8, 8, 12, 12, 20, 20, 21, 29, 29, 32, 32, \cdots$$

The sequence of radii tends to infinity (or equiva-
lently the sequence of radii (r_n) tends to zero) since
the total area of the disjoint disks enclosed by the

circles $\pi \sum r_n^2 < +\infty$ which is bounded by the area
inside the outer circle.

Why do we get natural numbers?

It is an interesting property of the identity that if
r_0, r_1, r_2, r_3 are reciprocals of natural numbers then
so is r_4 since

$$\frac{1}{r_0} + \frac{1}{r_4} = 2\left(\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_0} + \frac{1}{r_3}\right)$$

by the formula of Descartes–Princess Elizabeth. Pro-
ceeding iteratively, we see that r_n is the reciprocal
of a natural numbers for all $n \geq 0$.

It is a natural question to ask how quickly such the
sequence $(1/r_n)$ grows (see "The sequence looks a
bit like. . .").

The sequence looks a bit like...

One might compare this with a similar looking
result for prime numbers

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, \ldots$$

For any $x > 0$ let $\pi(x)$ denote the finite num-
ber of primes numbers less than x. Since
there are infinitely many primes we see that
$\pi(x)$ tends to infinity as x tends to infinity. A
more refined estimate comes from the follow-
ing famous theorem proved in 1896.

Theorem 3 (Prime Number Theorem). *The
function $\pi(x)$ is asymptotic to $\frac{x}{\log x}$ as $x \to +\infty$,
i.e., $\lim_{x\to+\infty} \frac{\pi(x)}{x/\log x} = 1$.*

One might (rightly) imagine that these two
results might be proved in a similar way.

Kontorovich-Oh

We denote by $N(\epsilon) = \#\{n : r_n > \epsilon\}$ the finite num-
ber of circles with radii greater than $\epsilon > 0$. We have
already observed that $N(\epsilon) \to +\infty$ as $\epsilon \to 0$. This
was considerably strengthened by A. Kontorovich
and H. Oh in the following theorem.

Theorem 4 (Kontorovich–Oh). *There exist $C > 0$ and $\delta > 0$ such that the number $N(\epsilon) \sim \epsilon^{-\delta}$ as $\epsilon \to 0$, i.e.,*

$$\lim_{\epsilon \to 0} \frac{N(\epsilon)}{\epsilon^{-\delta}} = C.$$

The value δ

We can consider the compact set $\overline{\mathscr{A}}$ in the plane given by the closure of the countable union of circles $\mathscr{A}$. The value of δ is equal to the Hausdorff Dimension of this set $\mathscr{A}$ (i.e., the natural notion of "size" for fractal sets). Although no explicit expression is known for δ it has been numerically estimated to be $\delta = 1 \cdot 30568 \ldots$ [4].
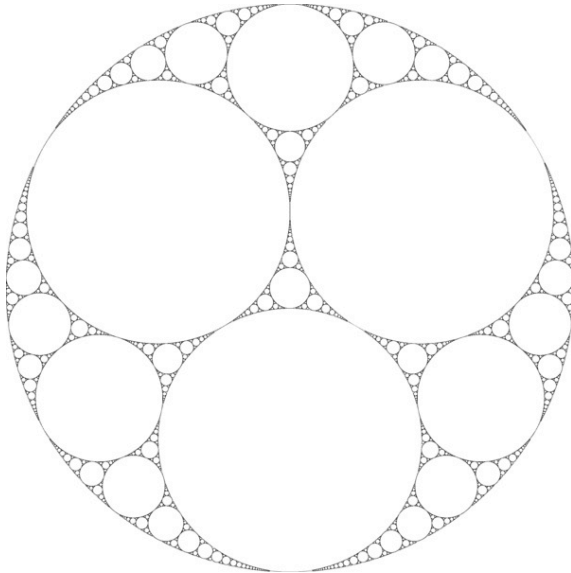
A fractal set $\mathscr{A}$ in the plane

Any initial configuration of circles in the plane can be mapped onto any others by a Möbius map, which will also map the corresponding circles in each packing. Since the Hausdorff Dimension is preserved by any Lipschitz map we deduce that the same value of δ occurs for any initial choice of r_1, r_2 and r_3.

A pinch of complex analysis

Rather than describe the original proof in [5] it is possible to give a proof of the counting theorem for radii of circles in the Apollonian circle packing which is analogous to that of Hadamard's famous proof of

the Prime Number Theorem [6]. (See "Comparison with the Prime Number Theorem".) We can consider the complex function

$$\eta(s) = \sum_n r_n^s$$

where the summation is over all the radii of circles in the Apollonian circle packing $\mathscr{A}$. This converges to a non-zero analytic function for $Re(s) > \delta$ and an analytic function with the following properties:

(1) $\eta(s)$ has a simple pole at $s = \delta$; and

(2) $\eta(s)$ has no zeros or poles on the line $Re(s) = \delta$.

In particular, we can write

$$\eta(s) = \frac{C}{s - \delta} + \psi(s)$$

where $\psi(s)$ is analytic in a neighbourhood of $Re(s) \geq \delta$. We can also write

$$\eta(s) = \int_0^\infty r^s \, dN(r)$$

and then the properties (1) and (2) above are enough to ensure that by the classical Ikehara–Wiener tauberian theorem we have that $N(r) \sim C r^{-\delta}$ as $r \to 0$ [3].

Comparison with the Prime Number Theorem

The original proof of the Prime Number Theorem used the Riemann Zeta function

$$\zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s}$$

which converges to a non-zero analytic function for $Re(s) > 1$ and the analytic extension has the following properties:

(1) $\zeta(s)$ has a simple pole at $s = 1$; and

(2) $\zeta(s)$ has no zeros or poles on the line $Re(s) = 1$.

The asymptotic formula for the $\pi(x)$ then follows from an application of the classical Ikehara-Wiener tauberian theorem.

Phillip Beecroft

There remains the non-minor detail of proving the properties (1) and (2) for $\eta(s)$.

A useful dynamical approach to generating circle packings was discovered by Philip Beecroft (1818–1862), a school teacher from Hyde, in Greater Manchester, who was the son of a miller and lived quietly with his two sisters. His approach was based on inversions in four complementary circles, a result he published in the pleasantly named *Lady's and Gentleman's diary* from 1842 [1, 2].

Theorem 5 (Beecroft). *Given 4 mutually tangent circles C_1, C_2, C_3, C_4 we can associate 4 new mutually tangent "dual" circles K_1, K_2, K_3, K_4 passing through the points $C_i \cap C_j$ $(1 \le i < j \le 4)$.*
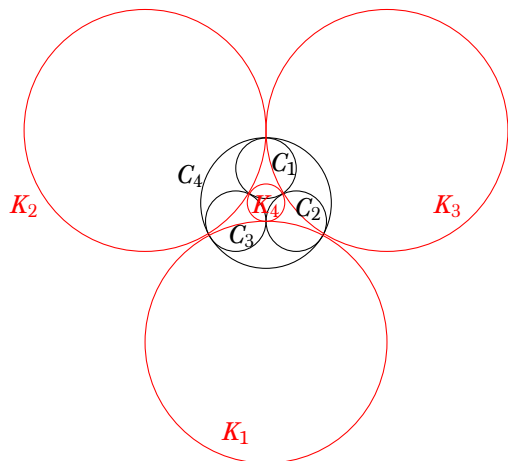
The 4 original circles C_1, C_2, C_3, C_4 and their 4 dual circles K_1, k_2, K_3, K_4

In the course of his work, Beecroft too rediscovered the formula of Descartes and Princess Elizabeth.

Apollonian groups

Considering the circles to lie in the complex plane, for each of the dual circles K_i, with center c_i and radius r_i, say, we can define a Möbius map $T_i(z) = \frac{r^2(z-c)}{|z-c|}$ $(i = 1, 2, 3, 4)$. These maps generate a group $\mathcal{G}$ called the *Apollonian group*.

The group $\mathcal{G}$ gives a systematic way to study the circles in $\mathcal{A}$. The images gC_i of the four initial circles

C_i under elements $g \in \mathcal{G}$ correspond to the circles in $\mathcal{A}$, with a few trivial exceptions. In particular, we can write

$$\eta(s) = \sum_{i=1}^{4} \sum_{g \in \Gamma} r(gC_i).$$

where $r(\cdot)$ just denotes the radius of a circle.

Of course, one might ask what we have gained through writing the function $\eta(s)$ in this way. In fact the group is closely related to an example of a Kleinian group and the function $\eta(s)$ is essentially what is called a Poincaré series. There are dynamical methods available to establish (1) and (2).

FURTHER READING

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