Effective intinsic ergodicity for expanding interval maps

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1 Introduction

The variational principle is one of the central pillars of smooth ergodic theory and thermodynamic formalism. It was originally formulated for hyperbolic systems by Ruelle [11] and proved in full generality by Walters [16]. It relates the thermodynamic pressure function to entropies and integrals with respect to invariant measures.

We will consider the particular case where $T : I \to I$ is a piecewise C^2 mixing expanding map of the interval $I = [0, 1)$ and where $\phi : I \to R$ is a Hölder continuous function.

Definition 1.1. *We can denote by m_ϕ the unique equilibrium state associated to ϕ, i.e., m is the unique probability measure realising the following supremum*

$$P(\phi) := \sup \left\{ h(\mu) + \int \phi d\mu \, : \, \mu = T\text{-invariant probability} \right\},$$

where $h(\mu)$ is the entropy (i.e., the variational principle).

For definiteness, we will consider the following well known examples.

Example 1.2. *Let $\beta > 1$ and consider $T : [0, 1) \to [0, 1)$ defined by*

$$T(x) = \beta x \pmod 1$$

then T is called a β-transformation. This is piecewise affine on the intervals

$$\left[0, \frac{1}{\beta}\right), \left[\frac{1}{\beta}, \frac{2}{\beta}\right), \cdots \left[\frac{[1/\beta] - 1}{\beta}, \frac{[1/\beta]}{\beta}\right), \left[\frac{[1/\beta]}{\beta}, 1\right)$$

where $[\cdot]$ denotes the integer part of a real number. example

We claim the following analogue of the Einseidler-Kaydev-Polo inequality (originally established for Markov expanding maps, subshifts of finite type and Anosov diffeomorphisms [2], [12]) also holds in this context.

Theorem 1.3. *There exists a constant $C_0 = C_0(\phi)$ such that for any T-invariant probability μ we have*

$$\left| \int f d\mu - \int f dm_\phi \right| \le C_0 \|f\| \sqrt{P(\phi) - \left(h(\mu) + \int \phi d\mu \right)} \tag{1.1}$$

where $\|\cdot\|$ is the norm of bounded variation.

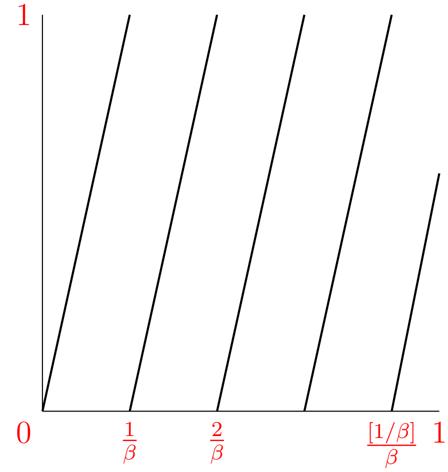

Figure 1: Graph of a β-transformation

In the special case that $\beta \in \mathbb{N}$ (or more generally the orbit of $1/\beta$ is finite) the map is Markov. In this case Theorem 1.3 would be a consequence of Kaydev's theorem [2]..

Example 1.4 (Parry measure)**.** *If we take $\phi = 0$ then $P(0) = h(T)$ is the topological entropy and the equilibrium state is the unique measure m_0 which maximizes the the entropy. In particular, Parry showed that m_0 is absolutely continuous with density $\rho : [0,1] \to \mathbb{R}^+$ given by*

$$\rho(x) = \frac{\sum_{n:T^n(1)>x} \beta^{-n}}{\int \left(\sum_{n:T^n(1)>x} \beta^{-n} \right) dx}$$

with normalization constant $K > 0$ [9].

Theorem 1.3 now has the following corollary (when $\phi = 0$).

Corollary 1.5. *There exists a constant $C_0 > 0$ such that for any T-invariant measure μ we have*

$$\left| \int f d\mu - \int f dm_0 \right| \le C_0 \|f\| \sqrt{h(T) - h(\mu)}. \tag{1.2}$$

A little history. A version of this result was apparently first proved in the thesis of Polo for doubling maps [10], where it was attributed to Einseidler. The above theorem was proved by Kadyrov for finite state shift spaces when $\phi = 0$ (which was called effective intrinsic ergodicity) [2]. This was extended to Hölder potentials and infinite state shift spaces by Ruhr [12]. Subsequently, Ruhr-Sarig gave an alternative proof and a local version where the upper bound has the variance replacing the norm [13].

2 Proof of Theorem 1.2

We begin by recalling the definition of the bounded variation semi-norm of a function $\psi : I \to \mathbb{R}$ which takes the form

$$\|\psi\|_{BV} = \sup\left\{\left|\sum_{i=0}^{n} \psi(x_i) - \psi(x_{i+1})\right| \; : \; 0 = x_0 < x_1 \cdots < x_n < x_{n+1} = 1\right\}$$

and let $\|\psi\|_{L^1} = \int_I |\psi(x)|dx$ denote the L^1-norm. We let $BV(I)$ denote the Banach space of measurable functions $\psi : I \to \mathbb{C}$ with norm $\|\psi\| := \|\psi\|_{BV} + \|\psi\|_{L^1} < +\infty$.

Definition 2.1. *Let $T : I \to I$ be a monotone piecewise continuous map. Given $\phi \in BV(I)$ we can define the transfer operator $\mathcal{L}_\phi : BV(I) \to BV(I)$ by*

$$\mathcal{L}_\phi w(x) = \sum_{Ty=x} e^{\phi(y)} w(y).$$

Under additional assumptions on the function ϕ the operator $\mathcal{L}_\phi : BV(I) \to BV(I)$ has a number of useful properties. For simplicity we first assume that ϕ is Lipschitz so that we can use more of the specific methods form [16]. In §4 we will consider more general results using the subsequent analysis in [3], [7].

Lemma 2.2. *Let $\phi : I \to \mathbb{R}$ be Lipschitz.*

1. *There exists a maximal eigenfunction $h \in BV(I)$ with $h > 0$ such that $\mathcal{L}_\phi h = e^{P(\phi)} h$ and, in particular,*
$$e^{P(\phi)} = \lim_{n \to +\infty} \left\| \mathcal{L}_\phi^n 1(x) \right\|_\infty^{\frac{1}{n}} .$$
2. *There exists a non-atomic probability measure ν_ϕ such that $\mathcal{L}_\phi^* \nu_\phi = e^{P(\phi)} \nu_\phi$.*
3. *There exists $a > 0$ such that $h(x) \geq a > 0$*
4. *h is continuous except at the points $\{T^n(\beta) : n \geq 0\}$.*
5. *There exists $C > 0$ and $0 < \rho < 1$ such that $\|\mathcal{L}_\phi^n(w) - e^{nP(\phi)} h\nu(w)\| \leq C(\rho e^{P(\phi)})^n \|w\|$ for any $w \in BV(I)$ and $n \geq 1$.*

Proof. The existence of the eigenfunction h in part 1 follows from ([7], Remark 6.8) (as observed in [1] on p. 460, where the authors also observe that it follows from their own Theorem 2). The existence of the measure in ν_ϕ in Part 2 follows from Proposition 6.10 of [7] (as also observed in [1], p.460.) [1]

Part 3 appears as Part (iii) in Lemma 9 in [16] (which in turn is based on Lemma 1 (i) in [16]) and we briefly recall the proof. Let $\mathcal{P}$ be the partition of I into intervals of the form

$$\mathcal{P} = \left\{ \left[0, \frac{1}{\beta}\right), \left[\frac{1}{\beta}, \frac{2}{\beta}\right), \cdots \left[\frac{[1/\beta] - 1}{\beta}, \frac{[1/\beta]}{\beta}\right), \left[\frac{[1/\beta]}{\beta}, 1\right) \right\}. \tag{2.1}$$

[1] See ([16], Lemma 3 and Lemma 9) for a proof of the first 2 parts for a different Banach space and under the additional assumption that ϕ is Lipschitz

An interval $J \in \vee_{i=0}^{n-1} T^{-i}\mathcal{P}$ is called *a full interval of rank* n if $T^n : J \to I$ is a bijection. Let $N > 0$ then I is covered by full intervals of rank at least N (by Lemma 1 (i) in [16]). Thus for any x there is a sequence $k_j \to +\infty$ of full intervals $J_j \in \vee_{i=0}^{k_j-1} T^{-i}\mathcal{P}$ with rank at least k_j with $\{x\} = \cap_{j=1}^{\infty} J_j$. We claim that there exists some $N > 0$ and some interval J with $T^N : J \to I$ a bijection (i.e., J is of full rank N) and for which $b := \inf_{y \in J} h(y) > 0$. If this was not the case then for all N we could choose x with $h(x) = 0$ and since $\mathcal{L}_\phi^N h(x) = e^{NP(\phi)} h(x)$ we have that $h(y) = 0$ for all preimages $y \in T^{-N}x$, which in turn would imply h is identically zero giving the contradiction. Thus for J as in this claim, since by assumption $T^N J = I$ we have that for $x \in I$:

$$h(x) = e^{-P(\phi)N} \mathcal{L}_\phi^N h(x) \geq e^{-P(\phi)N} b e^{-N\|\phi\|_\infty} =: a.$$

Part 4 follows the same lines as the proof of Lemma 9 (iv) in [16], and we briefly recall the idea. We observe that $e^{-nP(\phi)} \mathcal{L}_\phi^n 1(x)$ is continuous at points not in $\cup_{i=1}^{n} T^i(\{\beta\})$ (where $\{\beta\} = \beta - [\beta]$ denotes the fractional part of β) and continuous from the right at these points. The result then follows from $h(x) = \lim_{n\to+\infty} e^{-nP(\phi)} \mathcal{L}_\phi^n 1(x)$.

For Part 5, we first observe that replacing ϕ by $\psi = \phi - \log h \circ T + \log h \in BV(I)$ (by virtue of Part 3) we have that the associated operator satisfies $\mathcal{L}_\psi 1 = 1$ and has spectral radius 1. Moreover

$$\theta_\psi := \limsup_{n\to+\infty} \left\| \exp\left(\sum_{k=0}^{n-1} \psi(T^k x) \right) \right\|_\infty^{\frac{1}{n}} < 1.$$

This corresponds to $\theta_\phi < e^{P(\phi)}$. By Theorem 1 in [1] we have that $\mathcal{L}_\phi$ is quasi-compact and the essential spectral radius. is at most θ_ϕ. Therefore, it suffices to observe that $e^{P(\phi)}$ is a simple eigenvalue and that there are no other eigenvalues of absolute value $e^{P(\phi)}$. □

It is convenient to consider coboundaries $u \circ T - u$ where $u \in BV(I)$. The following result follows easily from the definitions (and Part (iii) of Lemma 2.2).

Lemma 2.3. *We can add constants and coboundaries in $BV(I)$ to ϕ without changing the equilibrium state m_ϕ.* [2]

In particular, we can consider $h \in BV(I)$ as in Lemma 2.2 (1) and observe that since Lemma 2.2 (3) we have $h \geq a > 0$ we have that $\log h \in BV(I)$. In particular, we can replace ϕ by $\psi = \phi - P(\phi) + \log h - \log h \circ T$ and then we can assume can assume that the associated transfer operator $\mathcal{L}_\psi : BV(I) \to BV(I)$ satisfies $\mathcal{L}_\psi 1 = 1$. Thus, without loss of generality we can assume $P(\phi) = 0$.

We now consider a simple lemma, which is a special case of the well known Pinsker inequality.

Lemma 2.4 (Pinsker Inequality)**.** *Given probability vectors $\underline{q} = (q_1, \cdots, q_k)$ and $\underline{p} = (p_1, \cdots, p_k)$ we then have the basic inequality*

$$-\sum_{i=1}^{k} q_i \log q_i + \sum_{i=1}^{k} q_i \log p_i \leq -\frac{1}{2} \sum_{i=1}^{k} |p_i - q_i|^2. \tag{2.2}$$

[2] The new function ϕ may no longer be Lipschitz since h was not necessarily Lipschitz

We can now follow the lines of the standard proof of the variational principle (cf. [16]). Let $\mathcal{P} = \{P_i\}_{i=1}^k$ be the partition of I into intervals given in (2.1) where $k = k(x) = [1/\beta]$ or $[1/\beta]+1$, as appropriate. Since this is clearly a generating partition we have that the entropy satisfies $h(\mu) = H_\mu(\mathcal{P}|T^{-1}\mathcal{P})$ [16]. We can also make the following choices:.

1. Given $x \in I$ we can let $p_1(x), \cdots p_{k(x)}(x)$ take the values $\{e^{\psi(y)} : Ty = x\}$; and
2. For any T-invariant probability measure μ we let $q_1(x), \cdots, q_k(x)$ $(1 \le i \le k(x))$ take the values $\mu(\mathcal{P}|T^{-1}\mathcal{B})(y)$ where $T(y) = x$ for a.e. (μ) $x \in X$ where $\mathcal{B}$ is the Borel sigma algebra.

We can substitute these choices into (2.2) and integrate with respect to μ to get:

$$
\begin{aligned}
&h(T,\mu) + \int \psi(x) d\mu(x) \\
&= -\int \left(\sum_{y \in T^{-1}} \mu(\mathcal{P}|T^{-1}\mathcal{B})(y) \log \mu(\mathcal{P}|T^{-1}\mathcal{B})(y) + \sum_{y \in T^{-1}x} \mu(\mathcal{P}|T^{-1}\mathcal{B})(y)\psi(y) \right) d\mu(x) \\
&= -\int \left(\sum_{i=1}^k q_i(x) \log q_i(x) + \sum_{i=1}^k q_i(x)\psi(x) \right) d\mu(x) \\
&\le -\frac{1}{2} \int \sum_{i=1}^k |p_i(x) - q_i(x)|^2 d\mu(x).
\end{aligned}
\tag{2.3}
$$

We can get a slightly weaker, but more useful, lower bound by using the Cauchy-Schwartz inequality to write

$$
\left(\int \sum_{i=1}^k |p_i(x) - q_i(x)| d\mu(x) \right)^2 \le \int \sum_{i=1}^k |p_i(x) - q_i(x)|^2 d\mu(x). \tag{2.4}
$$

Moreover, we can define the usual norm on the dual space $BV(I)^*$ of $BV(I)$ by

$$
\|\nu\| = \sup \left\{ \left| \int g d\nu \right| : g \in BV(I) \text{ with } \|g\| \le 1 \right\}
$$

for $\nu \in BV(I)^*$. This leads to the following.

Lemma 2.5. $\|\mathcal{L}_\psi^* \mu - \mu\| \le \int \sum_{i=1}^k |p_i(x) - q_i(x)| d\mu(x)$.

Proof. Given $g \in B_{BV}$ with $\|g\|_\infty \le 1$ we have

$$
\begin{aligned}
\left| \int (\mathcal{L}_\psi g - g) d\mu \right| &\le \int \sum_{i=1}^k g(y) \left| p_i(x) - q_i(x) \right| d\mu(x) \\
&\le \int \sum_i \left| p_i(x) - q_i(x) \right| d\mu(x)
\end{aligned}
$$

as required. □

Finally, we have the following lemma.

Lemma 2.6. *There exists $C_1 > 0$ such that $\|\mu - m_\phi\| \le C_1 \|\mathcal{L}_\psi^* \mu - \mu\|$.*

Proof. It is at this point that we use the result from part 5 of Lemma 2.2 that there exists $0 < \rho < 1$ such that $\mathcal{L}^{*n}\mu = m + U^{*n}\mu$ where $\|U^{*n}\| = O(\rho^n)$. From this we conclude that the series $Q = \sum_{n=0}^{\infty} U^n$ converges. We can then write

$$m_\phi = \lim_{n\to+\infty} \mathcal{L}_\psi^{*n}\mu = \mu + \sum_{n=0}^{\infty} \mathcal{L}_\psi^{*n}(\mathcal{L}_\psi^* - I)\mu.$$

Finally, we can write $\|\mu - m_\phi\| \le \|Q\|.|\mathcal{L}_\psi^* - I)\mu\|$. □

Combining (2.3), (2.4) and the inequalities in Lemma 2.6 and 2.7 completes the proof of Theorem 1.2.

3 A generalization of Theorem 1.3

Theorem 1.3 for β-transformations is a special case of a more general result for piecewise monotonic transformation where there exist $b_0 = 0 < b_1 < \cdots < b_N = 1$ such that the restriction $T|_{(b_i,b_{i+1})} : (b_i, b_{i+1}) \to I$ is continuous and strictly monotone.

We recall the following property for T

Definition 3.1. *We will say that T is topologically exact if the any $\epsilon > 0$ there exists n such that for any $x \in I$ we have $T^n(B(x,\epsilon) = I$.*

Definition 3.2. *We say that a function $\phi : I \to \mathbb{R}$ has summable variation if*

$$\sum_n var_n(\phi) < +\infty$$

where $var_n(\phi) = \sup\{|\phi(x) - \phi(y)| \,:\, x, y$ are in same monotonicity interval of $T^n\}$, for $n \ge 1$.

The generalization of Theorem 1.3 takes the following form:

Theorem 3.3. *Let T be a piecewise monotonic transformation which is topologically exact. Let$\phi : I \to \mathbb{R}$ be a (continuous) function of bounded variation such that either*

a) ψ has summable variation; or

b) ψ is Hölder continuous.

Then exists a constant $C_0 = C_0(\phi)$ such that for any T-invariant probability μ we have

$$\left| \int f d\mu - \int f dm_\phi \right| \le C_0 \|f\| \sqrt{P(\phi) - \left(h(\mu) + \int \phi d\mu \right)} \tag{3.1}$$

where $\|\cdot\|$ is the norm of bounded variation.

The definitiion of bounded variation can be generalized as follows

Definition 3.4. *For $p \geq 1$ we can define the bounded p-variation semi-norm of a function $\psi : I \to \mathbb{R}$ which takes the form*

$$\|\psi\|_{BV} = \left(\sup \left\{ \sum_{i=0}^{n} |\psi(x_i) - \psi(x_{i+1})|^p \ : 0 = x_0 < x_1 \cdots < x_n < x_{n+1} = 1 \right\} \right)^{1/p}$$

and let $\|\psi\|_{L^1} = \int_I |\psi(x)|dx$ denote the L^1-norm. We let $BV_p(I)$ denote the Banach space of measurable functions $\psi : I \to \mathbb{C}$ with norm $\|\psi\| := \|\psi\|_{BV} + \|\psi\|_{L^1} < +\infty$.

An even larger space of functions in [6] are the following:

Definition 3.5. *Given $\epsilon > 0$ we denote*

$$\mathrm{osc}_1(x, \phi, \epsilon) = \mathrm{esssup}(\phi | B(x, \epsilon)) - \mathrm{essinf}(\phi | B(x, \epsilon))$$

and then we denote $\mathrm{osc}_1(\phi, \epsilon) := \int_I \mathrm{osc}_1(x, \phi, \epsilon)dx$. Fix $\alpha > 0$ and then for $\epsilon_0 > 0$ we can then write

$$\|\phi\|_{\alpha,1} := \sup_{0<\epsilon\leq\epsilon_0} \frac{\mathrm{osc}_1(\phi, \epsilon)}{\epsilon^\alpha}.$$

We can then define a norm $\|\phi\| = \|\phi\|_{\alpha,1} + \|\phi\|_{L^1}$ and let $H^{\alpha,1}$ be the associated Banach space (see [6], Theorem 1.13,b).

The following relationships between these spaces come from [6] and [8]

Lemma 3.6. . *Let $p = \frac{1}{\alpha}$.*

1. *If $\phi : I \to \mathbb{R}$ is α-Hölder continuous then $\psi \in BV_p(I)$.*
2. *$BV_p \subset H^{\alpha,1}$.*

The proof of Theorem 3.3 requires a suitable generalization of Lemma 2.2.

Lemma 3.7. *Let T be a piecewise monotonic transformation which is topologically exact. Let $\phi : I \to \mathbb{R}$ be a (continuous) function of bounded variation such that either*

a) ψ has p-summable variation; or

b) ψ is α-Hölder continuous.

Let $\mathcal{L}_\phi$ be the associated transfer operator on BV_p and $H^{\alpha,1}$, respectively.

1. *There exists a maximal eigenfunction $h \in BV(I)$ with $h > 0$ such that $\mathcal{L}_\phi h = e^{P(\phi)}h$ and, in particular,*
$$e^{P(\phi)} = \lim_{n\to+\infty} \left\| \mathcal{L}_\phi^n 1(x) \right\|_\infty^{\frac{1}{n}}.$$
2. *There exists a non-atomic probability measure ν_ϕ such that $\mathcal{L}_\phi^* \nu_\phi = e^{P(\phi)}\nu_\phi$.*
3. *There exists $a > 0$ such that $h(x) \geq a > 0$*

4. *There exists $C > 0$ and $0 < \rho < 1$ such that $\|\mathcal{L}_\phi^n(w) - e^{nP(\phi)}h\nu(w)\| \le C(\rho e^{P(\phi)})^n\|w\|$ for any $w \in BV(I)$ and $n \ge 1$.*

Proof. Under hypothesis a) the results follow from the results in [3]. Under hypothesis b) the results follow from the results in [6]. We briefly recall the main ideas.

The measure ν_ϕ in Part 2 occurs as a fixed point for the map $\nu \mapsto \mathcal{L}_\phi\nu/\nu(1)$ on the space of probability measures, i.e., $\mathcal{L}_\phi\nu_\phi = \lambda\nu_\phi$, where $\lambda = \nu_\phi(1)$ (see p.135 of [3]). Later one can identify $\lambda = e^{P(\phi)}$.

Let $g(x) := e^{\phi(x)}/\log\lambda$ and then in each of the two cases one show that for large enough n we have

$$\|\prod_{i=0}^{n-1} g(T^i x)\|_\infty < 1$$

(see [3], pp. 135-136). This implies that the operator $P = \mathcal{L}_{\log g}$ satisfies a Lasota-Yorke inequality (see [3], Lemma 7), i.e., there exists $0 < \rho < 1$ and $\beta > 0$ such that

a) under hypothesis a)

$$\|P^n f\|_{BV} \le \beta\|f\|_{L^1} + \rho\|f\|_{BV}, \quad n \ge 0;$$

b) under hypothesis b)

$$\|P^n f\|_{1,\alpha} \le \beta\|f\|_{L^1} + \rho\|f\|_{1,\alpha}, \quad n \ge 0.$$

Moreover, the unit balls in $BV(I)$ and $H^{1,\alpha}$ are compact in the L^1-norm (by [3], Lemma 5 and [6] respectively). This leads to the quasi-compactness of the transfer operator $\mathcal{L}_\phi$ on the respective spaces, i.e., $\mathcal{L}_\phi$ has spectral radius $e^{P(\phi)}$ and essential spectral radius at most $0 < \rho < 1$. The hypothesis of topological exactness implies that $e^{P(\phi)}$ is a simple eigenvalue and there are no other eigenvalues of modulus $e^{P(\phi)}$. This is nicely explained in the proof of Corollary 4.4 in [8]. If π_ϕ is the one dimensional eigenprojection associated to $e^{P(\phi)}$ then we can let $h = \pi_\phi(1)$, the image of the constant function 1, in Part 1. Part 4 is a standard application of the Ionescu-Tulcea and Marinescu theorem [4].

□

Remark 3.8. Similar results will hold for transformations $T : I \to I$ with a finite number of monotone branches providing there are additional hypotheses which ensures part 3 of the lemma.

Remark 3.9. More generally, it would be sufficient to assume that $g : I \to \mathbb{R}$ has bounded p-variation, which would include the case of the g being Hölder continuous.

4 A Ruhr-Sarig type local result

In the case that in (1.1) that $P(\phi) - (h(\mu) + \int \phi d\mu)$ is sufficiently small then a sightly different bound can be given by modifying the proof in [13].

We can consider the pressure

$$\begin{aligned} P(\phi) &= \sup\{h(m) + \int \phi dm : m = T\text{-invariant }\} \\ &= \log \rho(\mathcal{L}_\phi) \end{aligned}$$

(where $\rho(\cdot)$ is the spectral radius) and for $t \in (-\epsilon, \epsilon)$

1. The function $t \mapsto p(t) = P(\phi + t\psi)$ is analytic.
2. $\frac{dP(\phi+t\psi)}{dt}|_{t=0} = \int \psi d\mu_\phi =: a_0$
3. If ψ is not cohomologous to a coboundary plus a constant $\frac{dP^2(\phi+t\psi)}{dt^2}|_{t=0} < 0$ and $P(\phi + t\psi)$ is convex in a neighbourhood of 0.

Provided that a_1 is sufficiently close to a_0 we can use the above properties to choose t (close to 0) such that $\frac{dP(\phi+t\psi)}{dt}|_{t=0} = \int \psi d\mu_\phi =: a_1$.

We can now introduce the following restricted pressure function.

Definition 4.1. *For $a \in \mathbb{R}$ we define*

$$Q(a) = \sup \left\{ h(\mu) + \int \phi d\mu \, : \, \int \psi d\mu = a \right\}$$

which is well defined provided $\inf_m\{\int \psi dm\} \leq a \leq \sup_m\{\int \psi dm\}$

In particular, we observe $q(a) \leq P(\phi)$. Since the function $a \mapsto Q(a)$ is analytic we can deduce $\frac{dQ(a)}{da}|_{a_0} = 0$

We can use the Taylor expansion at $a = a_0$ to write

$$Q(a_0) - Q(a_1) = Q'(a_0) + Q''(a_0)(a_1 - a_0)^2(1 + o(1)).$$

The function Q is actually the Legendre transform of P. More precisely,

$$\begin{aligned} P(t) &= h(m_t) + \int (\phi + t\psi) dm_t \\ &= \underbrace{h(m_t) + \int \phi dm_t}_{=:Q(t)} + t \int \psi dm_t \end{aligned}$$

where m_t is the equilibrium state of $\phi + t\psi$. This allows us to deduce that $\frac{dQ^2(a)}{dt^2}|_{a=a_0} = \frac{dP^2(t)}{dt^2}|_{t=t_0}$.

Since

$$Q(a_1) \geq h(\nu) + \int \phi d\nu$$

since $\int \psi d\nu = a_1$ this implies

$$\begin{aligned} P(\phi) - \left(h(\nu) + \int \phi d\nu \right) &\geq Q(a_0) - Q(a_1) \\ &= \frac{dQ^2(a)}{dt^2}|_{a=a_0}(a_1 - a_0)^2 \,(1 + o(1)) \end{aligned}$$

Finally, we conclude that for $\int \psi d\nu$ is sufficiently close to $\int \psi d\mu_\phi$ then we can bound

$$\left|\int f d\mu - \int f dm_\phi\right| \leq (1+o(1))\sqrt{|\frac{dP^2(t)}{dt^2}|_{t=0}|}\sqrt{P(\phi) - \left(h(\mu) + \int \phi d\mu\right)} \tag{1.1}$$

5 Miscellaneous Comments

(a) The original applications of these pressure results was to subshifts of finite type and Axiom A diffeomorphisms [12].[2] However, by using a simple model by suspension flows [?] the corresponding result also extends to Axiom A flows. More precisely, assume that $\phi_t : \Lambda \to \Lambda$ is a C^1 Axiom A flow restricted to a basic set, m_ϕ is a ϕ-invariant equilibrium state for a Hölder continuous potential $\phi : \Lambda \to \mathbb{R}$ and $F : \Lambda \to \mathbb{R}$ is Hölder continuous then

$$\left|\int F d\mu - \int F dm_\phi\right| \leq C\|F\|\sqrt{P(\phi) - \left(h(\mu) + \int \phi d\mu\right)}$$

(b) The proof used the strong estimate in Part 5 of Lemma 2.2 to define Q in the proof of Lemma 2.6. However, under any weaker bounds on $\|U^n\| \to 0$ such that the series $Q = \sum_{n=0}^{\infty} U^n$ converges the same argument will hold.

(c) It may be possible to extend the result to higher dimensional transformations with singularities. In light of [13] one might ask if $\|f\|$ can be replaced by the variance $\sigma^2(f)$.

(d) Ruhr and Sarig have a corresponding result for subshifts where $\|f\|$ is replaced by an expression involving the variance $\sigma^2(f)$ which gives a more refined estimate. It is a natural question to ask if this is also true for (1.1).

References

[1] V. Baladi and G. Keller, Zeta functions and transfer operators for piecewise monotone transformations, Commun. Math. Phys. 127 (1990) 459-477.

[2] S. Kadyrov. Effective uniqueness of Parry measure and exceptional sets in ergodic theory. Monatshefte für Mathematik, 178(2):237–249, 2015

[3] F. Hofbauer and G. Keller, Ergodic properties of invariant measures for piecewise monotonic transformations

[4] C. T. Ionescu Tulcea and G. Marinescu, Theorie Ergodique Pour Des Classes D'Operations Non Completement Continues, Annals of Mathematics, 52 (1950) 140-147

[5] G. Keller, Generalized Bounded Variation and Applications to Piecewise Monotonic Transformations, Z. Wahrscheinlichkeitstheorie verw. Gebiete 69 (1985) 461-478

[6] G. Keller. Generalized bounded variation and applications to piecewise monotonic transformations. Z. Wahrsch. Verw. Gebiete, 69(3):461–478, 1985.

[7] G. Keller, Markov extensions, zeta functions, and Fredholm theory for piecewise invertible dynamical systems, Trans. Amer. Math. Soc. 314 (1989), 433-497

[8] H. Li and J. Rivera-Letelier, Equilibrium states of interval maps for hyperbolic potentials, Nonlinearity 27 (2014) 1779–1804

[9] W. Parry, On the β-expansion of real numbers, Acta math. Acad. Sci. Hungar. 11 (1960) 401-416

[10] F. Polo. Equidistribution in chaotic dynamical systems. ProQuest LLC, Ann Arbor, MI, 2011. Thesis (Ph.D.), The Ohio State University

[11] D. Ruelle, Statistical mechanics on a compact set with $\mathbb{Z}^v$ action satisfying expansiveness and specification, Trans. Amer. Math. Soc., 185 (1973), 237-252

[12] R. Rühr, Pressure Inequalities for Gibbs Measures of Countable Markov Shifts (arXiv:2012.13226)

[13] R. Rühr and O. Sarig, Effective intrinsic ergodicity for countable state Markov shifts, to appear in Israel J. Math. (arXiv:2112.01186)

[14] M. Rychlik, Bounded variation and invariant measures. Studia math. 76 (1983) 69-80

[15] P. Walters, A Variational Principle for the Pressure of Continuous Transformations, Trans. Amer. Math. Soc., 97 (1975) 937-971

[16] P. Walters, Equilibrium states for β-transformations and related transforms, Math. Z. 159 (1978), no. 1, 65–88.