Computing modular forms: sheet 1
Ariel Pacetti - a.pacetti@warwick.ac.uk

The problems include two different topics, students should hand in (at least) 2/3 of the problems of each part.

Generalities on modular forms. Let \( \mathfrak{h} \) denote the upper half plane.

1. Let \( \text{GL}_2(\mathbb{R})^+ \) denote the set of \( 2 \times 2 \) matrices with real entries and positive determinant.
   
   (a) Prove that the action of \( \text{GL}_2(\mathbb{R})^+ \) in \( \mathfrak{h} \) is transitive.
   
   (b) Compute the stabiliser of the point \( \sqrt{-1} \in \mathfrak{h} \).
   
   (c) Express \( \mathfrak{h} \) as the quotient of two real groups.

2. Let \( D = \{ z \in \mathfrak{h} : |z| \geq 1 \text{ and } |\text{Re}(z)| \leq 1/2 \} \). Prove the following facts:
   
   (a) Given \( z \in \mathfrak{h} \), there exists \( \gamma \in \text{SL}_2(\mathbb{Z}) \) such that \( \gamma \cdot z \in D \).
       (Hint: prove that there exists \( g \in \text{SL}_2(\mathbb{Z}) \) such that \( \text{Im}(g \cdot z) \) is maximum, and then prove that a translate of it lies in \( D \)).

   (b) Prove that if \( z_1, z_2 \in D \) are equivalent under the action of \( \text{SL}_2(\mathbb{Z}) \) then either \( \text{Re}(z_1) = \pm 1 \) and \( z_2 = z_1 \mp 1 \) or \( |z_1| = 1 \) and \( z_2 = -1/z_1 \).

3. Let \( \Gamma(2) \subset \text{SL}_2(\mathbb{Z}) \) be the normal subgroup of matrices congruent to the identity modulo 2. Prove that \( [\text{SL}_2(\mathbb{Z}) : \Gamma(2)] = 6 \) and compute a connected fundamental domain for it. How many cusps has \( \Gamma(2) \)?

4. Let \( k > 1 \) be a positive integer. Let

\[
G_k(z) = \sum_{(m,n) \in \mathbb{Z}^2 \setminus (0,0)} \frac{1}{(mz+n)^{2k}}.
\]

Prove that \( G_k(z) \) converges absolutely and uniformly on compact sets of \( \mathfrak{h} \) and defines a modular form of weight \( 2k \) for \( \text{SL}_2(\mathbb{Z}) \).

(Hint: to prove uniform convergence in compact sets, suppose first that \( z \in D \) and prove the lower bound \( |mz+n|^2 \leq |m\rho+n|^2 \), where \( \rho = \exp(2\pi i/3) \)).

Generalities on quaternion algebras. Let \( K \) denote a field.

1. Let \( B/K \) be a quaternion algebra. Prove that the map

\[
B \to \text{End}_B(B)^{\text{op}}, \quad \alpha \to \phi_\alpha(x) = x\alpha,
\]

is an isomorphism, where \( \text{End}_B(B) \) are endomorphisms of \( B \)-modules, and the superscript \( \text{op} \) means with the opposite operation, i.e. \( \phi^{\text{op}} \psi = \psi \circ \phi \).
2. Fill in the details to prove that if \( B/K \) is a quaternion algebra, then either \( B \) is a division algebra, or \( B \simeq M_2(K) \).

3. Let \( B/K \) be a quaternion algebra, where \( \text{char}(K) \neq 2 \). Let \( b(x, y) := \frac{1}{2} \text{Tr}(x\epsilon(y)) \) be the bilinear form attached to the reduced norm quadratic form. Prove that \( b(x, y) \) is non-degenerate, and that \( \{1, i, j, k\} \) is an orthogonal basis for it.

4. Recall that if \((V_1, q_1)\) and \((V_2, q_2)\) are quadratic spaces, and isometry between is a vector space isomorphism which preserves the quadratic forms (and so the bilinear forms attached to them). Recall that if \( B/K \) is a quaternion algebra, we defined \( B^0 \) as the pure quaternions (those of zero trace). In this problem we will prove the following result (for \( \text{char}(K) \neq 2 \)):

**Theorem:** Let \( B \) and \( \tilde{B} \) be quaternion algebras over \( K \). Then \( B \) and \( B' \) are isomorphic if and only if \( B^0 \) and \( \tilde{B}^0 \) are isometric (as quadratic spaces with the reduced norm).

(a) Prove that the nonzero elements in \( B^0 \) are the elements in \( B \) which are not in \( K \), but whose square is in \( K \). With this characterisation prove that if \( \phi : B \to \tilde{B} \) is an isomorphism, then it induces an isomorphism between \( B^0 \) and \( \tilde{B}^0 \) which is an isometry.

(b) Reciprocally, prove that if \( \sigma : B^0 \to \tilde{B}^0 \) is an isometry, then the set \( \{\sigma(i), \sigma(j), \sigma(k)\} \) is a basis of \( \tilde{B}^0 \) and \( \{1, \sigma(i), \sigma(j), \sigma(k)\} \) is a basis for \( \tilde{B} \) satisfying the axioms of a quaternion algebra.