Perfect Powers that are Sums of Consecutive like Powers

Vandita Patel
University of Warwick
Number Theory Seminar,
University of Warwick

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A Diophantine Equation

\[(x + 1)^k + (x + 2)^k + \cdots + (x + d)^k = y^n.\]

**Question**

*Fix \( k \geq 2 \) and \( d \geq 2 \). Determine all of the integer solutions \((x, y, n)\).*
A Diophantine Equation

\[ x^k + (x + 1)^k + \cdots + (x + d - 1)^k = y^n. \]

**Question**

Fix \( k \geq 2 \) and \( d \geq 2 \). Determine all of the integer solutions \((x, y, n)\).

**Remark:** We can let \( n = p \) be a prime.
A Brief History: Sums of Consecutive Cubes

Euler:

\[ 6^3 = 3^3 + 4^3 + 5^3. \]

Dickson’s “History of the Theory of Numbers”:
Catalan, Cunningham, Lucas and Gennochi.

Later contributions from:

2. Uchiyama (1979): \( d = 3, n = 2 \) independently to Cassels.
3. Cassels (1985): \( y^2 = x^3 + (x + 1)^3 + (x + 2)^3. \)
4. Zhongfeng Zhang (2014): \( y^n = x^3 + (x + 1)^3 + (x + 2)^3. \)
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A Brief History

Well–Known:

\[
\sum_{i=0}^{d} i^3 = \sum_{i=1}^{d} i^3 = \left( \frac{d(d+1)}{2} \right)^2.
\]

Pagliani:

\[
\sum_{i=1}^{v^3} \left( \frac{v^4 - 3v^3 - 2v^2 - 2}{6} + i \right)^3 = \left( \frac{v^5 + v^3 - 2v}{6} \right)^3.
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where \( v \equiv 2, 4 \pmod{6} \).
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The Results

\[(x + 1)^k + (x + 2)^k + \cdots + (x + d)^k = y^n.\]

**Theorem (M. A. Bennett, V. Patel, S. Siksek)**

Let \(k = 3\) and \(2 \leq d \leq 50\). Then, any “non–trivial” integer solution \((x, y, n)\) must have \(n = 2\) or \(n = 3\).

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Cubes that are Sums of Consecutive Cubes

\[ 3^3 + 4^3 + 5^3 = 6^3, \text{ attributed to Lucas} \]
\[ 11^3 + 12^3 + 13^3 + 14^3 = 20^3, \]
\[ 3^3 + 4^3 + 5^3 + \cdots + 22^3 = 40^3, \]
\[ 15^3 + 16^3 + 17^3 + \cdots + 34^3 = 70^3, \]
\[ 6^3 + 7^3 + 8^3 + \cdots + 30^3 = 60^3, \]
\[ 291^3 + 292^3 + 293^3 + \cdots + 339^3 = 1155^3. \]
CUBES that are Sums of Consecutive Cubes

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2 Roughly speaking we have
   (Linear Factor in \(x\))(Quadratic Factor in \(x\)) = \(y^p\).

3 Linear Factor = \(\alpha y_1^p\).

4 Quadratic Factor = (Linear Factor)\(^2\) + Constant = \(\beta y_2^p\).

5 Substitution should give \(\alpha^2(y_1^2)^p + \text{Constant} \cdot 1^p = \beta(y_2)^p\)

Step 2.

1 \(p = 2\) solved by Stroeker (1995).

2 Integer points on Elliptic Curves.

3 Cubic in \(x = y^2\). Ask magma!
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The Magic of Sophie Germain

After Step 4. We have equations of the form:

\[ r y_2^p - s y_1^{2p} = t \]  (1)

where \( r, s, t \) are positive integers, and \( \gcd(r, s, t) = 1 \).

The linear forms in two logarithms bounds \( p \). For each tuple \( (r, s, t) \) we can apply the methods of Sophie Germain to eliminate equations/tuples for a fixed value of \( p \).
Lemma

Let $p \geq 3$ be prime. Let $r$, $s$ and $t$ be positive integers satisfying $\gcd(r, s, t) = 1$. Let $q = 2kp + 1$ be a prime that does not divide $r$. Define

$$\mu(p, q) = \{\eta^{2p} : \eta \in \mathbb{F}_q\} = \{0\} \cup \{\zeta \in \mathbb{F}_q^* : \zeta^k = 1\} \quad (2)$$

and

$$B(p, q) = \left\{\zeta \in \mu(p, q) : ((s\zeta + t)/r)^{2k} \in \{0, 1\}\right\}.$$

If $B(p, q) = \emptyset$, then equation (1) does not have integral solutions.
THE MAGIC OF SOPHIE GERMAINE

**Proof.**

Suppose \( B(p, q) = \emptyset \). Let \((y_1, y_2)\) be a solution to (1). Let \( \zeta = \overline{y_1}^{2p} \in \mu(p, q) \). From equation (1) we have

\[
(s\zeta + t)/r \equiv y_2^p \pmod{q}.
\]

Thus

\[
((s\zeta + t)/r)^{2k} \equiv y_2^{q-1} \equiv 0 \text{ or } 1 \pmod{q}.
\]

This shows that \( \zeta \in B(p, q) \) giving a contradiction. \( \Box \)
THE MAGIC OF SOPHIE GERMAIN - WHY DOES IT WORK?

1. If there are no solutions to $ry_2^{p} - sy_1^{2p} = t$,
2. and we take $p$ to be large, then
3. notice that $\#\mu(p, q) = k + 1$.
4. For $\zeta \in \mu(p, q)$, the element $((s\zeta + t)/r)^{2k} \in \mathbb{F}_q$ is either 0 or an $p$-th root of unity.
5. The “probability” that it belongs to the set $\{0, 1\}$ is $2/(p + 1)$.
6. The “expected size” of $B(p, q)$ is $2(k + 1)/(p + 1) \approx 2q/p^2$.
7. For large $p$ we expect to find a prime $q = 2kp + 1$ such that $2q/p^2$ is tiny and so we likewise expect that $\#B(p, q) = 0$. 

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**The Modular Way!** \((r = t)\)

\[ry_2^p - sy_1^{2p} = t\]

\[y_2^p - (s/r)y_1^{2p} = 1\]

Has solutions \((y_1, y_2) = (0, 1)\). This causes our previous lemma to fail.

However, the Modular Method does not see this solution. When constructing the Frey Curve, the discriminant is non-zero. Hence if \(y_1 = 0\) then the discriminant is zero. (Similar to Fermat’s Last Theorem).
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Pieter’s Parity Parrot: Designed by Pieter Moree, Drawn by Kate Kattegat
## The Case $k = 2$

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Vandita Patel  
University of Warwick  
Perfect Powers that are Sums of Consecutive like Powers
Dimensions of $S_2(N)$

When $k = 2$...

$$d = 22, \quad \dim = 5280$$

Dimension 200 is reasonable to compute with. We can push computations to dimension 2000 with some clever tricks.

When $k = 4$...

$$d = 21, \quad \dim \approx 1,500,000$$

$$d = 30, \quad \dim \approx 804,000,000$$
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If I try... naively

\[ \approx 10^{20} \]

If Mike Bennett tries... naively

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Pythagoras

\[ 3^2 + 4^2 = 5^2 \]

\[ 20^2 + 21^2 = 29^2 \]

An infinite family of solutions - can be given parametrically!
Even $k$ and Towards Densities

**Theorem (Zhang and Bai, 2013)**

Let $q$ be a prime such that $q \equiv 5, 7 \pmod{12}$. Suppose $q \parallel d$. Then the equation $x^2 + (x + 1)^2 + \cdots + (x + d - 1)^2 = y^n$ has no integer solutions.

**Corollary (Use Dirichlet’s Theorem)**

Let $A_2$ be the set of integers $d \geq 2$ such that the equation

$$x^2 + (x + 1)^2 + \cdots + (x + d - 1)^2 = y^n$$

has a solution $(x, y, n)$. Then $A_2$ has natural density zero.
**Theorem (Zhang and Bai, 2013)**

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**The Result**

**Theorem (V. Patel, S. Siksek)**

Let \( k \geq 2 \) be an even integer. Let \( A_k \) be the set of integers \( d \geq 2 \) such that the equation

\[
x^k + (x + 1)^k + \cdots (x + d - 1)^k = y^n, \quad x, y, n \in \mathbb{Z}, \quad n \geq 2
\]

has a solution \((x, y, n)\). Then \( A_k \) has natural density zero. In other words we have

\[
\lim_{X \to \infty} \frac{\#\{d \in A_k : d \leq X\}}{X} = 0.
\]
The Result

**Theorem (V. Patel, S. Siksek)**

Let $k \geq 2$ be an even integer and let $r$ be a non-zero integer. Let $A_{k,r}$ be the set of integers $d \geq 2$ such that the equation

$$x^k + (x + r)^k + \cdots (x + r(d - 1))^k = y^n, \quad x, y, n \in \mathbb{Z}, \quad n \geq 2$$

has a solution $(x, y, n)$. Then $A_{k,r}$ has natural density zero. In other words we have

$$\lim_{X \to \infty} \frac{\# \{ d \in A_{k,r} : d \leq X \}}{X} = 0.$$
Bernoulli polynomials and relation to sums of consecutive powers

**Definition (Bernoulli Numbers, \(b_k\))**

\[
x \frac{e^x - 1}{x} = \sum_{k=0}^{\infty} b_k \frac{x^k}{k!}.
\]

\(b_0 = 1, b_1 = -1/2, b_2 = 1/6, b_3 = 0, b_4 = -1/30, b_5 = 0, b_6 = 1/42.\)

**Lemma**

\(b_{2k+1} = 0 \text{ for } k \geq 1.\)
Bernoulli polynomials and relation to sums of consecutive powers

**Definition (Bernoulli Polynomial, $B_k$)**

$$B_k(x) := \sum_{m=0}^{k} \binom{k}{m} b_m x^{k-m}.$$ 

**Lemma**

$$x^k + (x+1)^k + \cdots + (x+d-1)^k = \frac{1}{k+1} (B_{k+1}(x+d) - B_k(x)).$$
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Apply Taylor’s Theorem and use \( B'_{k+1}(x) = (k + 1) \cdot B_k(x) \).

Lemma

Let \( q \geq k + 3 \) be a prime. Let \( d \geq 2 \). Suppose that \( q \mid d \). Then

\[ x^k + (x + 1)^k + \cdots + (x + d - 1)^k \equiv d \cdot B_k(x) \quad (\text{mod } q^2). \]
**BERNOULLI POLYNOMIALS AND RELATION TO SUMS OF CONSECUTIVE POWERS**

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Bernoulli polynomials and relation to sums of consecutive powers

\[ x^k + (x + 1)^k + \cdots + (x + d - 1)^k = y^n. \]

**Proposition (Criterion)**

Let \( k \geq 2 \). Let \( q \geq k + 3 \) be a prime such that the congruence

\[ B_k(x) \equiv 0 \pmod{q} \]

has no solutions. Let \( d \) be a positive integer such that \( \text{ord}_q(d) = 1 \). Then the equation has no solutions. (i.e. \( d \notin A_k \)).

**Remark:** Computationally we checked \( k \leq 75,000 \) and we could always find such a \( q \).
BERNOULLI POLYNOMIALS AND RELATION TO SUMS OF CONSECUTIVE POWERS

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**Relation to Densities?**

We need to use Chebotarev’s density theorem, which can be seen as “a generalisation of Dirichlet’s theorem” on primes in arithmetic progression.

**Proposition**

Let $k \geq 2$ be even and let $G$ be the Galois group of $B_k(x)$. Then there is an element $\mu \in G$ that acts freely on the roots of $B_k(x)$.

Assuming the proposition, we may then use Chebotarev’s density theorem to find a set of primes $q_i$ with positive Dirichlet density such that $\text{Frob}_{q_i} \in G$ is conjugate to $\mu$. Then we can apply Niven’s results to deduce our Theorem.
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**Niven’s Results (Flash!)**

The setup:

1. Let $A$ be a set of positive integers.
2. Define: $A(X) = \#\{d \in A : d \leq X\}$ for positive $X$.
3. Natural Density: $\delta(A) = \lim_{X \to \infty} A(X)/X$.
4. Given a prime $q$, define: $A^{(q)} = \{d \in A : \text{ord}_q(d) = 1\}$.

**Theorem (Niven)**

Let $\{q_i\}$ be a set of primes such that $\delta(A^{(q_i)}) = 0$ and $\sum q_i^{-1} = \infty$. Then $\delta(A) = 0$. 
**A Legendre Symbol analogue**

**Proposition**

Let \( k \geq 2 \) be even and let \( G \) be the Galois group \( B_k(x) \). Then there is an element \( \mu \in G \) that acts freely on the roots of \( B_k(x) \).

**Conjecture**

For any even integer \( k \), \( B_k(x) \) is irreducible over \( \mathbb{Q} \).

**Remark:** The conjecture implies the Proposition. This then proves our Theorem.
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Tough Stuff

A sketch of an unconditional proof!

**Proposition**

Let $k \geq 2$ be even and let $G$ be the Galois group $B_k(x)$. Then there is an element $\mu \in G$ that acts freely on the roots of $B_k(x)$.

**Theorem (von Staudt-Clausen)**

Let $n \geq 2$ be even. Then

\[ b_n + \sum_{(p-1)|n} \frac{1}{p} \in \mathbb{Z}. \]
2 is the Oddest Prime

The Newton Polygon of $B_k(x)$ for $k = 2^s \cdot t$, $s \geq 1$.

$$B_k(x) = \sum_{i=0}^{k} \binom{k}{k-i} b_{k-i} x^i = \sum_{i=0}^{k} a_i x^i$$

The Newton Polygon with points:
- $(0, 0)$
- $(0, -1)$
- $(k - 2^s, 0)$
- $(k - 2^s, -1)$
- $(k, 0)$

Slope: $1/2^s$
**Another nice result**

1. Sloping part corresponds to irreducible factor over $\mathbb{Q}_2$.
2. Root in $\mathbb{Q}_2$ must have valuation zero.
3. Root belongs to $\mathbb{Z}_2$ and is odd.
4. Symmetry $(-1)^kB_k(x) = B_k(1-x)$ gives a contradiction.

**Theorem (V. Patel, S. Siksek)**

Let $k \geq 2$ be an even integer. Then $B_k(x)$ has no roots in $\mathbb{Q}_2$.

**Theorem (K. Inkeri, 1959)**

Let $k \geq 2$ be an even integer. Then $B_k(x)$ has no roots in $\mathbb{Q}$. 
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What is Going On?

$L = \text{Splitting Field of } B_k(x)$

$G = \text{Galois Group}$

$H \subset G$

$\mathbb{Q}$

$C = \text{Cyclic}$

$\mathbb{Q}_2$

$\mathbb{F}_2 = \text{Residue Field}$
**What is Going On?**

\[ L = \text{Splitting Field of } B_k(x) \quad L_{\mathbb{Q}} \quad \mathbb{F}_{\mathbb{Q}} \]

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\[ \mathbb{Q} \quad \mathbb{Q}_2 \quad \mathbb{F}_2 = \text{Residue Field} \]

\[ \mu \text{ lives here!} \]
A sketch proof of the Proposition

The Setup:

- $k \geq 2$ is even.
- $L$ is the splitting field of $B_k(x)$.
- $G$ is the Galois group of $B_k(x)$.
- $\mathfrak{p}$ be a prime above 2.
- $\nu_2$ on $\mathbb{Q}_2$ which we extend uniquely to $L_\mathfrak{p}$ (also call it $\nu_2$).
- $H = \text{Gal}(L_\mathfrak{p}/\mathbb{Q}_2) \subset G$ be the decomposition subgroup corresponding to $\mathfrak{p}$. 
A sketch proof of the Proposition

\[ B_k(x) = g(x)h(x) \]

where \( g(x) \) has degree \( k - 2^s \). Label the roots \( \{\alpha_1, \ldots, \alpha_{k-2^s}\} \), and \( h(x) \) has degree \( 2^s \). Label the roots \( \{\beta_1, \ldots, \beta_{2^s}\} \).

- All roots \( \subset L_\beta \).
- \( h(x) \) is irreducible.
- Therefore \( H \) acts transitively on \( \beta_j \).
- Pick \( \mu \in H \) such that \( \mu \) acts freely on the roots of \( h(x) \).
- Check it doesn’t end up fixing a root of \( g(x) \).
“Bad Prime = Extremely Useful Prime!”

The Newton Polygon of $B_k(x)$ for $k = 2^s \cdot t$, $s \geq 1$.
**Finding $\mu$**

**Lemma**

Let $H$ be a finite group acting transitively on a finite set \{\(\beta_1, \ldots, \beta_n\)\}. Let $H_i \subset H$ be the stabiliser of $\beta_i$ and suppose $H_1 = H_2$. Let $\pi : H \rightarrow C$ be a surjective homomorphism from $H$ onto a cyclic group $C$. Then there exists some $\mu \in H$ acting freely on \{\(\beta_1, \ldots, \beta_n\)\} such that $\pi(\mu)$ is a generator of $C$.

1. Let $F_\mathfrak{p}$ be the residue field of $\mathfrak{p}$.
2. Let $C = \text{Gal} \left( F_\mathfrak{p} / F_2 \right)$.
3. $C$ is cyclic generated by the Frobenius map: $\bar{\gamma} \rightarrow \bar{\gamma}^2$.
4. Let $\pi : H \rightarrow C$ be the induced surjection.
5. Finally use the Lemma.
Finding $\mu$

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5. Finally use the Lemma.
Check $g(x)$

$$B_k(x) = g(x)h(x)$$

where $g(x)$ has degree $k - 2^s$. Label the roots $\{\alpha_1, \ldots, \alpha_{k-2^s}\}$, and $h(x)$ has degree $2^s$. Label the roots $\{\beta_1, \ldots, \beta_{2^s}\}$.

**Lemma**

$\mu$ acts freely on the $\alpha_i$.

1. Suppose not. Let $\alpha$ be a root that is fixed by $\mu$.
2. $\nu_2(\alpha) = 0$ so let $\bar{\alpha} = \alpha \pmod{\mathfrak{p}}$, $\bar{\alpha} \in \mathbb{F}_p$.
3. $\alpha$ fixed by $\mu$ hence $\bar{\alpha}$ fixed by $\langle \pi(\mu) \rangle = C$.
4. Hence $\bar{\alpha} \in \mathbb{F}_2$. $f(x) = 2B_k(x) \in \mathbb{Z}_2[x]$.
5. $f(\bar{1}) = f(\bar{0}) = \bar{1}$. A contradiction!
Thank you for listening!
SOLVING THE EQUATIONS FOR $k = 2$

$$d \left( \left( x + \frac{d + 1}{2} \right)^2 + \frac{(d - 1)(d + 1)}{12} \right) = y^p.$$

$$X^2 + C \cdot 1^p = \left( \frac{1}{d} \right) y^p$$
# Solving the Equations for $k = 2$

<table>
<thead>
<tr>
<th>d</th>
<th>Equation</th>
<th>Level</th>
<th>Dimension</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>$2y^p - 5 \times 7 = 3(2x + 7)^2$</td>
<td>$2^7 \times 3^2 \times 5 \times 7$</td>
<td>480</td>
</tr>
<tr>
<td>11</td>
<td>$11^{p-1}y^p - 2 \times 5 = (x + 6)^2$</td>
<td>$2^7 \times 5 \times 11$</td>
<td>160</td>
</tr>
<tr>
<td>13</td>
<td>$13^{p-1}y^p - 2 \times 7 = (x + 7)^2$</td>
<td>$2^7 \times 7 \times 13$</td>
<td>288</td>
</tr>
<tr>
<td>22</td>
<td>$2 \times 11^{p-1}y^p - 7 \times 23 = (2x + 23)^2$</td>
<td>$2^7 \times 7 \times 11 \times 23$</td>
<td>5,280</td>
</tr>
<tr>
<td>23</td>
<td>$23^{p-1}y^p - 2^2 \times 11 = (x + 12)^2$</td>
<td>$2^3 \times 11 \times 23$</td>
<td>54</td>
</tr>
<tr>
<td>26</td>
<td>$2 \times 13^{p-1}y^p - 3^2 \times 5^2 = (2x + 27)^2$</td>
<td>$2^7 \times 3 \times 5 \times 13$</td>
<td>384</td>
</tr>
<tr>
<td>33</td>
<td>$11^{p-1}y^p - 2^4 \times 17 = 3(x + 17)^2$</td>
<td>$2^3 \times 3^2 \times 11 \times 17$</td>
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<tr>
<td>37</td>
<td>$37^{p-1}y^p - 2 \times 3 \times 19 = (x + 19)^2$</td>
<td>$2^7 \times 3 \times 19 \times 37$</td>
<td>5,184</td>
</tr>
<tr>
<td>39</td>
<td>$13^{p-1}y^p - 2^2 \times 5 \times 19 = 3(x + 20)^2$</td>
<td>$2^3 \times 3^2 \times 5 \times 13 \times 19$</td>
<td>1,080</td>
</tr>
<tr>
<td>46</td>
<td>$2 \times 23^{p-1}y^p - 3^2 \times 5 \times 47 = (2x + 47)^2$</td>
<td>$2^7 \times 3 \times 5 \times 23 \times 47$</td>
<td>32,384</td>
</tr>
<tr>
<td>47</td>
<td>$47^{p-1}y^p - 2^3 \times 23 = (x + 24)^2$</td>
<td>$2^5 \times 23 \times 47$</td>
<td>1,012</td>
</tr>
<tr>
<td>59</td>
<td>$59^{p-1}y^p - 2 \times 5 \times 29 = (x + 30)^2$</td>
<td>$2^7 \times 5 \times 29 \times 59$</td>
<td>25,984</td>
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</table>
SOLVING THE EQUATIONS FOR $k = 4$

<table>
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<th>$d$</th>
<th>Equation</th>
<th>Level</th>
<th>Dimension</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>$y^p + 2 \times 73 = 5(X)^2$</td>
<td>$2^7 \times 5^2 \times 73$</td>
<td>5,472</td>
</tr>
<tr>
<td>6</td>
<td>$y^p + 7 \times 53 = 6(X)^2$</td>
<td>$2^8 \times 3^2 \times 7 \times 53$</td>
<td>12,480</td>
</tr>
<tr>
<td>7</td>
<td>$7^{p-1}y^p + 2^2 \times 29 = (X)^2$</td>
<td>$2^3 \times 7 \times 29$</td>
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</tr>
<tr>
<td>10</td>
<td>$y^p + 3 \times 11 \times 149 = 10(X)^2$</td>
<td>$2^8 \times 5^2 \times 3 \times 11 \times 149$</td>
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</tr>
<tr>
<td>13</td>
<td>$13^{p-1}y^p + 2 \times 7 \times 101 = (X)^2$</td>
<td>$2^7 \times 7 \times 13 \times 101$</td>
<td>28,800</td>
</tr>
<tr>
<td>14</td>
<td>$7^{p-1}y^p + 13 \times 293 = 2(X)^2$</td>
<td>$2^8 \times 7 \times 13 \times 293$</td>
<td>168,192</td>
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<tr>
<td>15</td>
<td>$y^p + 2^3 \times 7 \times 673 = 15(X)^2$</td>
<td>$2^5 \times 3^2 \times 5^2 \times 7 \times 673$</td>
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</tr>
<tr>
<td>17</td>
<td>$17^{p-1}y^p + 2^3 \times 3 \times 173 = (X)^2$</td>
<td>$2^5 \times 3 \times 17 \times 173$</td>
<td>5,504</td>
</tr>
<tr>
<td>19</td>
<td>$19^{p-1}y^p + 2 \times 3 \times 23 \times 47 = (X)^2$</td>
<td>$2^7 \times 3 \times 19 \times 23 \times 47$</td>
<td>145,728</td>
</tr>
<tr>
<td>21</td>
<td>$7^{p-1}y^p + 2 \times 11 \times 1321 = 3(X)^2$</td>
<td>$2^7 \times 3^2 \times 7 \times 11 \times 1321$</td>
<td>1,584,000</td>
</tr>
<tr>
<td>26</td>
<td>$13^{p-1}y^p + 3^2 \times 5 \times 1013 = 2(X)^2$</td>
<td>$2^8 \times 3 \times 5 \times 13 \times 1013$</td>
<td>777,216</td>
</tr>
<tr>
<td>29</td>
<td>$29^{p-1}y^p + 2 \times 7 \times 2521 = (X)^2$</td>
<td>$2^7 \times 7 \times 29 \times 2521$</td>
<td>1,693,440</td>
</tr>
<tr>
<td>30</td>
<td>$y^p + 19 \times 29 \times 31 \times 71 = 30(X)^2$</td>
<td>$2^8 \times 3^2 \times 5^2 \times 19 \times 29 \times 31 \times 71$</td>
<td>804,384,000</td>
</tr>
</tbody>
</table>

Where $X$ is a quadratic in the original variable $x$. 

Vandita Patel  
University of Warwick  
Perfect Powers that are Sums of Consecutive like Powers