# COUNTING IN HOMOLOGY: THIRTY YEARS AFTER

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This is intended as a gloss on my paper *Closed orbits in homology classes for Anosov flows* [10], published in 1993 (though written 1991-92), originally written fir one of my PhD students. When I wrote [10], I thought it was well written but now, 30 years later, it seems impenetrable[1]. Since this is not intended for publication, the style of writing is somewhat informal.

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## 1. Anosov Flows

Let $\phi^t : M \to M$ be a transitive Anosov flow generated by a vector field $X_\phi$. Assume $M$ has first Betti number $b \geq 1$, so $H_1(M,\mathbb{Z})$ is isomorphic to $\mathbb{Z}^b \oplus A$, where $A$ is a finite abelian group (referred to as torsion). The group $A$ is handled by bolting on the analysis from [5] or Chapter 12 of [6] to the analysis described below, so, for simplicity, we will assume that $A$ is trivial.

Let $\mathcal{P}_\phi$ denote the set of prime periodic $\phi$-orbits. For $\gamma \in \mathcal{P}_\phi$, write $l(\gamma)$ for its period and $[\gamma] \in H_1(M,\mathbb{Z})$ for its homology class. Write $\mathcal{P}_\phi(T) = \{\gamma \in \mathcal{P}_\phi : l(\gamma) \leq T\}$ and, for $\alpha \in H_1(M,\mathbb{Z})$,

$$\pi_\phi(T,\alpha) = \#\{\gamma \in \mathcal{P}_\phi(T) : [\gamma] = \alpha\}.$$

**Lemma 1.1** ([5])**.** *Let $\phi^t : M \to M$ be a transitive Anosov flow. Then $\{[\gamma] : \gamma \in \mathcal{P}_\phi\}$ generates $H_1(M,\mathbb{Z})$ as a group.*

*Proof.* Let $H$ be the subgroup of $H_1(M,\mathbb{Z})$ generated by $\{[\gamma] : \gamma \in \mathcal{P}(\phi)\}$. Suppose $H \neq H_1(M,\mathbb{Z})$. Then we can find a proper finite index subgroup $H'$ of $H_1(M,\mathbb{Z})$ which contains $H$. Let $G = H_1(M,\mathbb{Z})/H'$. By the Chebotarev theorem, the Frobenius elements (conjugacy classes are single elements since $G$ is abelian) of periodic orbits are equidistributed in $G$. But the $G$-Frobenius element of $\gamma$ is the image of $[\gamma]$ in $G = H_1(M,\mathbb{Z})/H'$, which is trivial since $[\gamma] \in H$. Contradiction. □

**Remark 1.2.** This can also be used in our situation of a 3-manifold with deleted knots since Theorem 1.1 of [4] gives equidistribition of Frobenius classes for finite groups.

We will make the following stronger assumption.

*Assumption A1.* $\phi^t$ is *homologically full*: the map $[\cdot] : \mathcal{P}_\phi \to H_1(M,\mathbb{Z})$ is a surjection.

**Remark 1.3.** If $\phi$ is homologically full then it is automatically topologically weak-mixing (equivalent to topological mixing). This follows from a theorem of Plante that a transitive Anosov flow only fails to be weak mixing if it is a constant time

[1]Though hopefully not as bad as the poetry of Robert Browning: "Mr. Browning himself, 'in the philosophic afternoon of life,' frankly confessed its difficulties, and referred to it with a grim smile as 'the entirely unintelligible Sordello.' And to an anxious admirer who asked him to explain its meaning he replied, 'When I wrote it, only God and I knew; now God alone knows!' ", *The Church Quarterly Review*, September 1890, following Browning's death in 1889. There is interesting information about the "God" quotation, which has been attributed to several people, on the invaluable online resource Quote Investigator

suspension of an Anosov diffeomorphism [7]. Such a suspension flow has no null-homologous periodic orbits. One way of characterising $\phi$ being topologically weak-mixing in that $\{l(\gamma) : \gamma \in \mathcal{P}_\phi\}$ generates $\mathbb{R}$ as a topological group. (Since $H_1(M,\mathbb{Z})$ is discrete, we could also formulate the conclusion of Lemma 1.1 as "$\{[\gamma] : \gamma \in \mathcal{P}_\phi\}$ generates $H_1(M,\mathbb{Z})$ as a topological group".)

We also need to notion of winding cycles (a.k.a. asymptotic cycles). See [8] (a great paper!) for the original definition. Let $m$ be a $\phi$-invariant probability measure. We define the winding cycle $\Phi_m \in H_1(M,\mathbb{R})$ as follows. Since $H_1(M,\mathbb{R})$ is the dual of $H^1(M,\mathbb{R})$, it suffices to say how $\Phi_m$ pairs with cohomology classes. This pairing is given by

$$\langle [\omega], \Phi_m \rangle = \int \omega(X_\phi)\, dm,$$

where $\omega$ is a closed 1-form and $[\omega]$ is its cohomology class.

**Lemma 1.4.** *The following statements are equivalent.*

1. $\phi^t$ *is homologically full.*
2. $0$ *lies in the interior of the convex hull of* $\{[\gamma] : \gamma \in \mathcal{P}_\phi\}$.
3. $0$ *lies in the interior of the convex hull of* $\{[\gamma]/l(\gamma) : \gamma \in \mathcal{P}_\phi\}$.
4. $0$ *lies in the interior of* $\{\Phi_m : m \in \mathcal{M}_\phi\}$*, where* $\mathcal{M}_\phi$ *is the set of* $\phi$*-invariant probability measures on* $M$.

**Remark 1.5.** In [1], (4) is made the definition of homologically full.

*Partial proof.* We will prove some of the implications at this point. Clearly, (1) implies (2) and (3), and (2) and (3) are equivalent. The set $\{\Phi_m : m \in \mathcal{M}_\phi\}$ is compact and convex and it contains $\{[\gamma]/l(\gamma) : \gamma \in \mathcal{P}_\phi\}$ as a dense subset (since periodic orbit measures are weak$^*$ dense in $\mathcal{M}_\phi$). Therefore, the convex hull of $\{[\gamma]/l(\gamma) : \gamma \in \mathcal{P}_\phi\}$ is equal to $\{\Phi_m : m \in \mathcal{M}_\phi\}$. Thus, (3) and (4) are equivalent. We will defer consideration of the fact that any of (2)-(4) imply (1) until later. (In fact, the logic will be that (1) implies (4), (4) implies Theorem 1.6 below, and the theorem trivially implies (1).) □

**Theorem 1.6** (Sharp [10]). *If $\phi$ is homologically full then, for all $\alpha \in \mathbb{Z}^b$, we have*

$$\pi(T,\alpha) \sim Ce^{-\langle \xi,\alpha\rangle} \frac{e^{h^*T}}{T^{1+b/2}}, \qquad \text{as } T \to \infty,$$

*where*

$$h^* = \sup\{h_\phi(m) : m \in \mathcal{M}_\phi \text{ and } \Phi_m = 0\}$$

*(and satisfies $0 < h^* \le h$, where $h$ is the topological entropy of $\phi$), $\xi \in H^1(M,\mathbb{R})$ minimizes the (well-defined) function $\beta : H^1(M,\mathbb{R}) \to \mathbb{R} : [\omega] \to P(\omega(X_\phi))$, and $C$ is a positive constant (independent of $\alpha$).*

**Remark 1.7.** I tend to (informally) call $\xi$ the *magic cohomology class.* In [1], it is called "Sharp's minimizer".

For the purposes of exposition, we will start by thinking about the proof of the following special case. Let $m_0$ be the measure of maximal entropy for $\phi$. (Note that, by Lemma 1.4, $\Phi_{m_0} = 0$ implies homologically full.)

**Theorem 1.8** (Katsuda and Sunada [2]). *If $\Phi_{m_0} = 0$ then, for all $\alpha \in \mathbb{Z}^b$, we have*

$$\pi(T,\alpha) \sim C\frac{e^{hT}}{T^{1+b/2}}, \qquad \text{as } T \to \infty,$$

*where $C$ is a positive constant (independent of $\alpha$).*

## 2. Suspension Flows

We'll now use symbolic dynamics to work at the symbolic level. We have the following objects:

- a mixing subshift of finite type $\sigma : \Sigma \to \Sigma$;
- a strictly positive Hölder continuous function $r : \Sigma \to \mathbb{R}$ giving a mixing suspension flow $\sigma_r^t : \Sigma^r \to \Sigma^r$;
- a Hölder continuous function $F : \Sigma^r \to \mathbb{R}^b$ such that, for every $\sigma^t$-periodic orbit $\gamma$,
$$[\gamma] = \int_\gamma F \in \mathbb{Z}^b.$$

*Notational conventions.* We'll use upper case letters for functions on $\Sigma^r$ and lower case letters for functions on $\Sigma$. Also, we'll use $m$ (with subscripts) for measures of $\Sigma^r$ and $\mu$ (with subscripts) for measures on $\Sigma$.

We can define $f : \Sigma \to \mathbb{R}^b$ by
$$f(x) = \int_0^{r(x)} F(\sigma_r^t(x, 0))\, dt = \int_0^{r(x)} F(x, t)\, dt.$$
By adding a coboundary, we may assume that $f : \Sigma \to \mathbb{Z}^b$. If $\sigma^n x = x$ corresponds to the $\phi$-periodic orbit $\gamma$ then
$$f^n(x) = [\gamma] \quad \text{and} \quad r^n(x) = l(\gamma).$$

The assumption that $\phi$ is homologically fully can be expressed translated into one of the following equivalent statements:

- 0 lies in the interior of $\{\int F\, dm : m \in \mathcal{M}_{\sigma_r}\}$,where $\mathcal{M}_{\sigma_r}$ is the set of $\sigma_r$-invariant probability measures on $\Sigma^r$.
- 0 lies in the interior of $\{\int f\, d\mu : \mu \in \mathcal{M}_\sigma\}$,where $\mathcal{M}_\sigma$ is the set of $\sigma$-invariant probability measures on $\Sigma$.

The stronger assumption that $\Phi_{m_0} = 0$ can be translated as
$$\int F\, dm_0 = 0,$$
where (abusing notation) $m_0$ is the measure of maximal entropy for $\sigma_r$. This is equivalent to
$$\int f\, d\mu_{-hr} = 0,$$
where $\mu_{-hr}$ is the equilibrium state for $-hr$.

**Spoiler 2.1.** In fact, $\phi$ homologically full and $\Phi_{m_0} = 0$ contain more information than the respective statements above.

## 3. $L$-functions

To prove Theorem 1.8, we introduce the dynamical $L$-functions (generalizing dynamical zeta functions)
$$L(s,t) = \prod_{\gamma \in \mathcal{P}} \left(1 - e^{-sl(\gamma) + 2\pi i \langle t, [\gamma]\rangle}\right)^{-1},$$
where $s \in \mathbb{C}$ and $t \in \mathbb{R}^b/\mathbb{Z}^b$, defined wherever the product converges. ($\langle \cdot, \cdot \rangle$ is the standard inner product on $\mathbb{R}^b$.) We also have the following formulae:
$$L(s,t) = \exp \sum_{\gamma \in \mathcal{P}} \sum_{m=1}^\infty \frac{1}{m} e^{-sml(\gamma) + 2\pi i m \langle t, [\gamma]\rangle} = \exp \sum_{n=1}^\infty \frac{1}{n} \sum_{\sigma^n x = x} e^{-sr^n(x) + 2\pi i \langle t, f^n(x)\rangle}.$$

**Lemma 3.1** ([6])**.** *$L(s,t)$ converges for* $\mathrm{Re}(s) > h$ *and all* $t \in \mathbb{R}^b/\mathbb{Z}^b$.

**Remark 3.2.** There is also the issue that the symbolic dynamics does not induce a bijection between $\mathcal{P}$ and $\mathcal{P}_\phi$. This can be dealt with using the Bowen–Manning trick, although this becomes more involved in the general homologically full case. However, on 3-manifolds the discrepancy only involves a finite number of periodic orbits, so we'll ignore it here.

We should also recall the orthogonality relation

$$\int_{\mathbb{R}^b/\mathbb{Z}^b} e^{2\pi i\langle t,y\rangle} = \begin{cases} 1 \text{ if } y=0 \\ 0 \text{ if } y\in\mathbb{Z}^b\setminus\{0\} \end{cases}.$$

As in the proof of the Prime Orbit Theorem in [6], we might be tempted to proceed by studying the logarithmic derivative $L'/L = (\log L)'$ (where the prime is $\partial/\partial s$), noting that

$$\int_{\mathbb{R}^b/\mathbb{Z}^b} e^{-2\pi i\alpha}\frac{L'(s,t)}{L(s,t)}\,dt = -\sum_{m=1}^{\infty}\sum_{\substack{\gamma\in\mathcal{P}:\\ m[\gamma]=\alpha}} l(\gamma)\,e^{-sml(\gamma)}.$$

Assuming we can drop the terms with $m\geq 2$, we might hope to know enough about the LHS (as a function of $s$) be able to apply some Tauberian theorem to get an asymptotic for

$$\sum_{\substack{\gamma\in\mathcal{P}:\\ [\gamma]=\alpha}} l(\gamma)$$

and thence for $\pi(T,\alpha)$. This nearly works except that it is technically more convenient to consider the functions

$$\eta(s,t) = \frac{\partial^{\nu+1}}{\partial s^{\nu+1}}\log L(s,t),$$

where $\nu=[b/2]$. (Other choices are possible but this choice of $\nu$ gives the simplest singularities, boiling down to the two cases $b$ even and $b$ odd.)

To carry out this programme, we need to look at $L(s,t)$ is a bit more detail. We want

- $L(s,t)$ is non-zero and analytic in $(\{s : \mathrm{Re}(s)\geq h\}\times\mathbb{R}^b/\mathbb{Z}^b)\setminus(h,0)$;
- $L(s,t)$ has a form we can analyse for $(s,t)$ close to $(h,0)$.

Let us state the first point as a lemma and then try to prove it.

**Lemma 3.3.** *$L(s,t)$ is non-zero and analytic in a neighbourhood of $(h+i\tau,t)$ unless $(\tau,t)=(0,0)\in\mathbb{R}\times\mathbb{R}^b/\mathbb{Z}^b$.*

From [6], $L(s,t)$ is non-zero and analytic in a neighbourhood of $(h+i\tau,t)$ except when $-\tau r+2\pi\langle t,f\rangle$ is cohomologous to a function $\psi\in C(\Sigma,2\pi\mathbb{Z})$:

$$-\tau r+2\pi\langle t,f\rangle = \psi+u-u\circ\sigma. \tag{$*$}$$

By Livsic's theorem, the latter statement is equivalent to

$$-\tau r^n(x)+2\pi\langle t,f^n(x)\rangle = \psi^n(x) \tag{$**$}$$

whenever $\sigma^n x=x$. This implies

$$e^{-i\tau r^n(x)+2\pi i\langle t,f^n(x)\rangle}=1 \tag{$\dagger$}$$

whenever $\sigma^n x = x$. Or, in terms of $\phi$,

$$\exp\left(-\tau l(\gamma)+2\pi\int_\gamma \omega(X_\phi)\right)=1 \quad \forall\gamma\in\mathcal{P}_\phi, \tag{$\ddagger$}$$

where $\omega(X_\phi): M\to\mathbb{R}$ pulls back to $\langle t,F\rangle$ by the symbolic coding. We want to show that one of these implies $(\tau,t)=(0,0)$. It is here that the conditions $\int F\,dm_0=0$

or $\int f \, d\mu_{-hr} = 0$ are not by themselves sufficient. For example, integrating $(*)$ with respect to $\mu_{-hr}$ gives

$$-\tau \int r \, d\mu_{-hr} = \int \psi \, d\mu_{-hr}.$$

If we knew the RHS were zero then we could conclude that (since $r > 0$) $\tau = 0$ and substitute back into (†) to get $e^{2\pi i \langle t, f^n(x) \rangle} = 1$ whenever $\sigma^n x = x$. Since (by Lemma 1.1) $\{f^n(x) : \sigma^n x = x\}$ generates $\mathbb{Z}^b$, we have $e^{2\pi i \langle t, y \rangle} = 1$ for all $y \in \mathbb{Z}^b$ and so $t = 0 \in \mathbb{R}^b/\mathbb{Z}^b$. But we know nothing about $\int \psi \, d\mu_{-hr}$. Let us see how homological fullness helps us.

*Proof of Lemma 3.3.* This is the argument from [10]. A function $U : M \to \mathbb{C}$ is called continuously differentiable in the flow direction if

$$U'(x) := \lim_{t \to 0} \frac{U(\phi^t x) - U(x)}{t}$$

exists everywhere and is continuous. By a multiplicative version of Livsic's theorem for flows, (‡) implies that there is a function $U : M \to \{z \in \mathbb{C} : |z| = 1\}$ which is continuously differentiable in the flow direction such that

$$-\tau + \omega(X_\phi) = \frac{1}{2\pi i} \frac{U'}{U}.$$

Integrating by $m_0$ we get

$$-\tau + \langle [\omega], \Phi_{m_0} \rangle = \int \frac{1}{2\pi i} \frac{U'}{U} \, dm_0 = \left\langle \left[ \frac{1}{2\pi i} \frac{U'}{U} \right], \Phi_{m_0} \right\rangle,$$

where the last expression in square brackets in the Bruschlinsky cohomology class represented by $U$. (See [8] for an explanation of Bruschlinsky cohomology.) Since $\Phi_{m_0} = 0$, we see that $\tau = 0$ and then argue as above to get $t = 0 \in \mathbb{R}^b/\mathbb{Z}^b$. □

This might be a sticky point for our envisaged result, where we can't use homological fullness. If we have the set-up from the start of the section: $\sigma_r$ weak-mixing and $\{f^n(x) : \sigma^n x = x\}$ generates $\mathbb{Z}^b$, then we need to make the following assumptions.

*Assumption B1.* 0 lies in the interior of $\{\int f \, d\mu : \mu \in \mathcal{M}_\sigma\}$ (or an equivalent formulation as discussed above).

*Assumption B2.* $\{(l(\gamma), [\gamma]) : \gamma \in \mathcal{P}\}$ (or $\{(r^n(x), f^n(x)) : \sigma^n x = x, \ n \geq 1\}$) generates $\mathbb{R} \times \mathbb{Z}^b$ as a topological group.

In line with, for the moment, restricting to the special case $\Phi_{m_0} = 0$, we can strengthen B1 to the following.

*Assumption B1*.* $\int f \, d\mu_{-hr} = 0$.

*Proof of Lemma 3.3 subject to Assumptions B1* and B2.* Think of

$$\chi(u, v) = e^{-i\tau u + 2\pi i \langle t, v \rangle}$$

as a character on $\mathbb{R} \times \mathbb{Z}^b$. (†) says that the character is trivial on a generating set and so it is trivial everywhere. Thus $(\tau, t) = (0, 0) \in \mathbb{R} \times \mathbb{R}^b/\mathbb{Z}^b$. □

After all that, we should start a new section ...

## 4. The Singularity

Recall that we need to understand $L(s, t)$ for $(s, t)$ close to $(h, 0)$. From [6], for $(s, t)$ in a neighbourhood $\mathcal{U}$ of $(h, 0)$ we have

$$L(s, t) = \frac{A_1(s, t)}{1 - e^{P(-sr + 2\pi i \langle t, f \rangle)}},$$

with $A_1(s, t)$ non-zero and analytic. Using the implicit function theorem, we can define an analytic function $s : \mathcal{U} \to \mathbb{C}$ by $P(-s(t)r + 2\pi i \langle t, f \rangle) = 0$. (We can think of $s(t)$ as $P(2\pi i\langle t, F\rangle)$, an extension of the pressure function for $\sigma_r$ to complex functions.) We get

$$L(s,t) = \frac{A_2(s,t)}{s - s(t)},$$

with $A_2(s,t)$ non-zero and analytic.

The function $s(t)$ has the following properties.

**Lemma 4.1** ([2],[9]).

(1) $\mathrm{Re}(s(t))$ *is and even function and* $\mathrm{Im}(s(t))$ *is an odd function;*
(2) $\nabla \mathrm{Re}(s(0)) = 0$*;*
(3) $\nabla \mathrm{Im}(s(0)) = 2\pi \int F \, dm_0 = 0$
(4) $\nabla^2 \mathrm{Re}(s(0))$ *is negative definite;*
(5) $\nabla^2 \mathrm{Im}(s(0)) = 0$.

Note that we have used Assumption B1* to get $\nabla \mathrm{Im}(s(0)) = 0$. The only difficult point is that $\nabla^2 \mathrm{Re}(s(0))$ is negative definite. The justification for this will appear in the proof of Lemma 6.3 below.

Recall the function

$$\eta(s,t) = \frac{\partial^{\nu+1}}{\partial s^{\nu+1}} \log L(s,t)$$
$$= \sum_{\gamma \in \mathcal{P}} \sum_{m=1}^{\infty} m^{\nu} (-l(\gamma))^{\nu+1} e^{-sml(\gamma) + 2\pi i \langle t, m[\gamma]\rangle}$$

where $\nu = [b/2]$. In view if the analysis above, we have

$$\eta(s,t) = \frac{(-1)^{\nu+1}\nu!}{(s - s(t))^{\nu+1}} + A_3(s,t),$$

with $A_3(s,t)$ analytic.

Now, for $\alpha \in \mathbb{Z}^b$, we introduce a function

$$\eta_\alpha(s) = \int_{\mathbb{R}^b/\mathbb{Z}^b} e^{-2\pi i\langle t, \alpha\rangle} \eta(s,t)\, dt.$$

By Lemma 3.3, $\eta_\alpha(s)$ is analytic in a neighbourhood of $\{s \in \mathbb{C} : \mathrm{Re}(s) \geq h\} \setminus \{h\}$. We need to know what happens close to $h$. Again by Lemma 3.3, we have

$$\eta_\alpha(s) = \int_{\mathcal{U}} e^{-2\pi i\langle t, \alpha\rangle} \eta(s,t)\, dt + A_4(s),$$

with $A_4(s)$ analytic. Now the key idea is that, since $\mathrm{Re}(s(t))$ has a non-degenerate critical point at 0, we can apply the Morse lemma to change co-ordinates on $\mathcal{U}$, to give new co-ordinates $\theta$ with $\mathrm{Re}(s(t(\theta))) = h - \|\theta\|^2$. The long calculation is carried out in [2]; the following lemma is a bowdlerized version.

**Lemma 4.2.**

(1) *If $b$ is even then $\eta_\alpha(s)$ has a singularity of the form $1/(s-h)$ near $s = h$.*
(2) *If $b$ is odd then $\eta_\alpha(s)$ has a singularity of the form $1/\sqrt{s-h}$ near $s = h$.*

Given the more precise version of this lemma proved in [2], we can apply appropriate Tauberian theorems to conclude that

$$\sum_{\gamma \in \mathcal{P}} \sum_{\substack{m=1 \\ ml(\gamma) \leq T \\ m[\gamma] = \alpha}}^{\infty} l(\gamma)^{1+b/2} \sim Ce^{hT}.$$

From there, it is straightforward to deduce Theorem 1.8.

## 5. The Homologically Full Case: Heuristics

Now suppose that $\phi : M \to M$ is homologically full. Let's try the same approach as before and consider the $L$-function

$$L(s,t) = \prod_{\gamma\in\mathcal{P}} \left(1 - e^{-sl(\gamma)+2\pi i\langle t,[\gamma]\rangle}\right)^{-1} = \exp \sum_{\gamma\in\mathcal{P}} \sum_{m=1}^{\infty} \frac{1}{m} e^{-sml(\gamma)+2\pi i m\langle t,[\gamma]\rangle}$$
$$= \exp \sum_{n=1}^{\infty} \frac{1}{n} \sum_{\sigma^n x = x} e^{-sr^n(x)+2\pi i\langle t, f^n(x)\rangle}.$$

**Lemma 5.1.** *$L(s,t)$ is non-zero and analytic on $\{s \in \mathbb{C} : \mathrm{Re}(s) > h\} \times \mathbb{R}^b/\mathbb{Z}^b$. Furthermore, $L(s,t)$ is non-zero and analytic in a neighbourhood of $(h+i\tau, t)$ unless $(\tau, t) = (0,0) \in \mathbb{R}\times\mathbb{R}^b/\mathbb{Z}^b$.*

The first statement is standard from [6]. The second statement is proved in the same way as Lemma 3.3, except $\Phi_{m_0} = 0$ is replaced by $\Phi_{m_G} = 0$, where $m_G$ is the equilibrium state for some Hölder continuous function $G : M \to \mathbb{R}$ (the existence of such a measure $m_G$ being guaranteed by $\phi$ being homologically full).

The problem lies in the singularity near $(h, 0)$. From Lemma 4.1, we have that

$$\nabla \mathrm{Im}(s(0)) = 2\pi \int F\, dm_0.$$

We are no longer assuming $\int F\, dm_0 = 0$ and if it is non-zero then $s(t)$ does not have critical point at 0. This leads the calculations leading to Lemma 4.2 to break down. So we need another idea.

The following is by nature of a thought experiment. Recall that we want to get asymptotics for

$$S(T,\alpha) := \sum_{\gamma\in\mathcal{P}} \sum_{\substack{m=1 \\ ml(\gamma)\le T \\ m[\gamma]=\alpha}}^{\infty} l(\gamma)^{1+b/2}$$

(which is just a hop and a step away from having asymptotics for $\pi(T,\alpha)$). Observe that if we take $w \in H^1(N,\mathbb{R})$ (which we can also abuse notation by thinking of as $w \in \mathbb{R}^b$) then

$$S_w(T,\alpha) := \sum_{\gamma\in\mathcal{P}} \sum_{\substack{m=1 \\ ml(\gamma)\le T \\ m[\gamma]=\alpha}}^{\infty} l(\gamma)^{1+b/2} e^{\langle w, m[\gamma]\rangle}$$

is an equally good object to look at, since

$$S_w(T,\alpha) = \sum_{\gamma\in\mathcal{P}} \sum_{\substack{m=1 \\ ml(\gamma)\le T \\ m[\gamma]=\alpha}}^{\infty} l(\gamma)^{1+b/2} e^{\langle w, \alpha\rangle} = e^{\langle w,\alpha\rangle} S(T,\alpha).$$

The "right" $L$-function to study $S_w(T,\alpha)$ is

$$L_w(s,t) := \prod_{\gamma\in\mathcal{P}} \left(1 - e^{-sl(\gamma)+\langle w,[\gamma]\rangle+2\pi i\langle t,[\gamma]\rangle}\right)^{-1}$$
$$= \exp \sum_{n=1}^{\infty} \frac{1}{n} \sum_{\sigma^n x = x} e^{-sr^n(x)+\langle w, f^n(x)\rangle+2\pi i\langle t, f^n(x)\rangle}.$$

Applying the results of [6] again, this converges to an analytic function (for all $t \in \mathbb{R}^b/\mathbb{Z}^b$) provided

$$P(-\mathrm{Re}(s) r + \langle w, f\rangle) < 0.$$

Recalling that $P(-P(\langle w, F\rangle) r + \langle w, f\rangle) = 0$, this maybe rephrased as

$$\mathrm{Re}(s) > P(\langle w, F\rangle) = P(\omega(X_\phi)) = \beta(w),$$

as defined in Theorem 1.6, where $\omega$ is a closed 1-form with $[\omega] = w$.

So, is there a value of $w$ for which everything works? As we saw above, the issue is the nature of the singularity of $L_w$. Let us look at this more closely. From [6], for $(s,t)$ in a neighbourhood $\mathcal{U}_w$ of $(\beta(w), 0)$ we have

$$L_w(s,t) = \frac{A_1(s,t)}{1 - e^{P(-sr + \langle w,f\rangle + 2\pi i\langle t,f\rangle)}},$$

with $A_1(s,t)$ non-zero and analytic. Using the implicit function theorem, we can define an analytic function $s_w : \mathcal{U}_w \to \mathbb{C}$ by $P(-s(t)r + \langle w,f\rangle + 2\pi i\langle t,f\rangle) = 0$. (We can think of $s_w(t)$ as $P(\langle w + 2\pi it, F\rangle)$, an extension of the pressure function for $\sigma_r$ to complex functions.) We get

$$L_w(s,t) = \frac{A_2(s,t)}{s - s_w(t)},$$

with $A_2(s,t)$ non-zero and analytic.

We see that

$$\nabla s_w(0) = 2\pi i \nabla\beta(w).$$

(Note that the derivative of $s_w$ is with respect to $t$ while the derivative of $\beta$ is with respect to $w + 2\pi it$). So $s_w$ has a critical point at 0 if and only if $\beta$ has a critical point at $w$. Now $\beta$ (which is a pressure function) is convex and we might hope it to be strictly convex (this needs proof). Let us suppose that it is strictly convex. Then $\beta$ has a critical point if and only if it has a finite minimum and, if it exists, this finite minimum is unique. A reasonable strategy therefore seems to be

- show that $\phi$ homologically full implies that $\beta$ is strictly convex and has a finite minimum, which we'll call $\xi$;
- use the $L$-function $L_\xi$ to attack the counting problem.

## 6. The Function $\beta$

As before, define $\beta : H^1(M,\mathbb{R})$ by $\beta(w) = P(\omega(X_\phi))$, with $[\omega] = w$. We can identify $H^1(M,\mathbb{R})$ with $\mathbb{R}^b$ be choosing a basis $[\omega_1], \ldots, [\omega_b]$ for $H^1(M,\mathbb{R})$, with $w = (w_1, \ldots, w_b) \in \mathbb{R}^n$ identified with $w = w_1[\omega_1] + \cdots + w_b[\omega_b] \in H^1(M,\mathbb{R})$. We will first to choose a free generating set $\alpha_1, \ldots, \alpha_b$ for the $\mathbb{Z}$-module $H_1(M,\mathbb{Z})/(\text{torsion})$ (which also provides a basis for $H_1(M,\mathbb{R})$) and then choose the $[\omega_i]$ according to $\langle [\omega_i], \alpha_j\rangle = \delta_{ij}$.) With this identification,

$$\beta(w) = P(\langle w, F\rangle) = P(w_1F_1 + \cdots + w_bF_b).$$

The following result is just standard differentiation of pressure. We will use it shortly.

**Lemma 6.1.** *We have*

$$\nabla\beta(w) = \left(\int F_1\, dm_{\langle w,F\rangle}, \ldots, \int F_b\, dm_{\langle w,F\rangle}\right) = \int F\, dm_{\langle w,F\rangle},$$

*where $m_{\langle w,F\rangle}$ is the equilibrium state for $\langle w, F\rangle$.*

Let us now see what homological fullness tells us. As a warm-up, it is easy to show that if $\phi$ is homologically full then $\beta$ is bounded below by zero.

**Lemma 6.2.** *If $\phi$ is homologically full then $\beta$ is bounded below by zero.*

*Proof.* By (4) in Lemma 1.4, we have that, in particular, $\Phi_m = 0$ for some $m \in \mathcal{M}_\phi$. Then

$$\begin{aligned}\beta(w) &= \sup_{\nu\in\mathcal{M}_\phi} h_\phi(\nu) + \langle w, \Phi_\nu\rangle \\ &\geq h_\phi(m) + \langle w, \Phi_m\rangle = h_\phi(m) \geq 0.\end{aligned}$$

□

However, this is weaker than $\beta$ having a finite minimum, which we still have to prove. As it stands, $\beta$ could have an asymptote. (For example $e^{-x}$ is strictly convex, bounded below, but has no finite minimum.)

**Lemma 6.3.** *If $\phi$ is homologically full then $\beta$ is strictly convex.*

*Proof.* For $\beta$ to be strictly convex, it is sufficient that $\nabla^2\beta$ be positive definite everywhere. The latter holds unless there is a $w \neq 0$ such that $\langle w, F\rangle$ is cohomologous to a constant, $c$ say. Since $\phi$ is homologically full, there is an $m \in \mathcal{M}_\phi$ for which $\int F\,dm = 0$, we have $c = 0$. Then $\langle w, [\gamma]\rangle = \int_\gamma \langle w, F\rangle = 0$ for all $\gamma \in \mathcal{P}(\phi)$. By Lemma 1.1, this gives $\langle w, \alpha\rangle = 0$ for all $\alpha \in \mathbb{Z}^b$ and so $w = 0$. (The end of the proof could be slightly shortened by using homological fullness again but the argument I wrote shows that this is not necessary.) □

**Lemma 6.4.** *If $\phi$ is homologically full then $\beta$ has a finite minimum.*

*Proof.* For $x = \sum_{i=1}^b x_i\alpha_i \in H_1(M,\mathbb{R})$, we set $x^* = \sum_{i=1}^b x_i[\omega_i] \in H^1(M,\mathbb{R})$. We can then define norms on $H_1(M,\mathbb{R})$ and $H^1(M,\mathbb{R})$ by $\|x\| = \|x^*\| = \langle x^*, x\rangle$. (Since $\langle[\omega_i],\alpha_j\rangle = \delta_{ij}$, these are just the Euclidean norms with respect to the bases $\alpha_1,\ldots,\alpha_b$ and $[\omega_1],\ldots,[\omega_b]$, respectively.)

Write $\mathcal{C}$ for the convex hull of $\{[\gamma]/l(\gamma) : \gamma \in \mathcal{P}(\phi)\}$. If $\phi$ is homologically full then $0 \in \mathrm{int}(\mathcal{C})$ (statement (3) in Lemma 1.4). Then we can choose $\epsilon > 0$ such that

$$\{x \in H_1(M,\mathbb{R}) : \|x\| < 2\epsilon\} \subset \mathcal{C}.$$

It follows that $\epsilon x/\|x\| \in \mathcal{C}$ for all $x \neq 0$, so

$$\epsilon\|x^*\| = \epsilon\|x\| = \epsilon\frac{\langle x^*, x\rangle}{\|x\|} \le \max_{y\in\mathcal{C}}\langle x^*, y\rangle \tag{†}$$

(the inequality holding trivially for $x = 0$). A general element $y \in \mathcal{C}$ has the form

$$\sum_{\gamma\in\mathcal{P}(\phi)} a_\gamma \frac{[\gamma]}{l(\gamma)},$$

with $0 \le a_\gamma \le 1$ and $\sum_{\gamma\in\mathcal{P}(\phi)} a_\gamma = 1$. Let $m_\gamma \in \mathcal{M}_\phi$ be the normalized Lebesgue measure around $\gamma$. Then, for any $x^* \in H^1(M,\mathbb{R})$,

$$\begin{aligned}
\beta(x^*) &= \sup_{\nu\in\mathcal{M}_\phi} h_\nu(\phi) + \int \langle x^*, F\rangle\,d\nu \\
&\ge h_\phi\left(\sum_{\gamma\in\mathcal{P}(\phi)} a_\gamma m_\gamma\right) + \sum_{\gamma\in\mathcal{P}(\phi)} a_\gamma \int \langle x^*, F\rangle\,dm_\gamma \\
&= h_\phi\left(\sum_{\gamma\in\mathcal{P}(\phi)} a_\gamma m_\gamma\right) + \sum_{\gamma\in\mathcal{P}(\phi)} a_\gamma \frac{\langle x^*, [\gamma]\rangle}{l(\gamma)} \\
&= h_\phi\left(\sum_{\gamma\in\mathcal{P}(\phi)} a_\gamma m_\gamma\right) + \langle x^*, y\rangle \ge \langle x^*, y\rangle,
\end{aligned}$$

so

$$\beta(x^*) \ge \max_{y\in\mathcal{C}}\langle x^*, y\rangle. \tag{‡}$$

Combining (†) and (‡) gives

$$\beta(x^*) \ge \epsilon\|x^*\|.$$

Now choose $r \in \mathbb{R}$ in the image of $\beta$. For any $w \in H^1(M,\mathbb{R})$ with $\|w\| > r/\epsilon$, we have $\beta(w) \ge \epsilon\|w\| > r$. It follows that $\beta$ has a unique finite minimum at some $\xi$ (satisfying $\|\xi\| \le r/\epsilon$). □

This argument was shamelessly stolen from [3][2]. To end the section, we tie a few other things together.

**Lemma 6.5.** *Let $\xi$ be as above. We have*

$$\beta(\xi) = h_\phi(m_{\langle \xi, F\rangle}) = \sup\{h_\phi(m) : m \in \mathcal{M}_\phi \text{ and } \Phi_m = 0\} > 0$$

*and*

$$\nabla\beta(\xi) = \int F\, dm_{\langle \xi, F\rangle} = 0.$$

*Proof.* First we note that $\nabla\beta(\xi) = 0$ since $\xi$ is a global minimum and the formula $\nabla\beta(\xi) = \int F\, dm_{\langle \xi, F\rangle}$ is from Lemma 6.1. Now,

$$\begin{aligned}\beta(\xi) &= h_\phi(m_{\langle \xi, F\rangle}) + \int \langle \xi, F\rangle\, dm_{\langle \xi, F\rangle} \\ &= h_\phi(m_{\langle \xi, F\rangle}) + \left\langle \xi, \int F\, dm_{\langle \xi, F\rangle}\right\rangle = h_\phi(m_{\langle \xi, F\rangle}) > 0,\end{aligned}$$

where the last term is positive because equilibrium states for Hölder continuous functions have positive entropy. Finally, if $m \neq m_{\langle \xi, F\rangle}$ satisfies $\Phi_m = 0$ (equivalently, $\int F\, dm = 0$), then

$$\beta(\xi) > h_\phi(m) + \int \langle \xi, F\rangle\, dm = h_\phi(m) + \left\langle \xi, \int F\, dm \right\rangle = h_\phi(m),$$

so the characterisation of $\beta(\xi)$ as a supremum also holds. ☐

## 7. Completing the Proof

As Nike's slogan says, Just Do It. Work with the function $L_\xi(s,t)$, which converges for $\mathrm{Re}(s) > h^* := \beta(\xi)$. We have:

**Lemma 7.1.** *$L_\xi(s,t)$ is non-zero and analytic in a neighbourhood of $(h^* + i\tau, t)$ unless $(\tau, t) = (0,0) \in \mathbb{R} \times \mathbb{R}^b/\mathbb{Z}^b$.*

*Proof.* The proof is the same as the proof of Lemma 3.3 except that $\Phi_{m_{\langle \xi, F\rangle}} = 0$ is used instead of $\Phi_{m_0} = 0$. ☐

We it comes to the singularity, we have set things up so that close to $(h^*, 0)$,

$$L_\xi(s,t) = \frac{A(s,t)}{s - s_\xi(t)},$$

where $A(s,t)$ is analytic and $s_\xi$ satisfied the following.

**Lemma 7.2** ([2],[9])**.**

1. $\mathrm{Re}(s_\xi(t))$ *is and even function and* $\mathrm{Im}(s_\xi(t))$ *is an odd function;*
2. $\nabla\mathrm{Re}(s_\xi(0)) = 0$;
3. $\nabla\mathrm{Im}(s_\xi(0)) = -2\pi \int F\, dm_{\langle \xi, F\rangle} = 0$
4. $\nabla^2\mathrm{Re}(s_\xi(0)) = -\nabla^2\beta(\xi)$ *is negative definite;*
5. $\nabla^2\mathrm{Im}(s_\xi(0)) = 0$.

Writing

$$\eta_\xi(s,t) = \frac{\partial^{\nu+1}}{\partial s^{\nu+1}} \log L_\xi(s,t)$$

and

$$\eta_{\alpha,\xi}(s) = \int_{\mathbb{R}^b/\mathbb{Z}^b} e^{-2\pi i\langle t, \alpha\rangle} \eta_\xi(s,t)\, dt,$$

we continue as in section 4 to get:

[2] *Bad artists copy, great artists steal.* Variants of this saying have been attributed to T. S. Eliot, Picasso and Stravinsky, among others. See Quote Investigator again.

**Lemma 7.3.**

(1) *If $b$ is even then $\eta_{\alpha,\xi}(s)$ has a singularity of the form $1/(s-h^*)$ near $s = h^*$.*
(2) *If $b$ is odd then $\eta_{\alpha,\xi}(s)$ has a singularity of the form $1/\sqrt{s-h^*}$ near $s = h^*$.*

From the precise version of this lemma, it is plain sailing to complete the proof of Theorem 1.6.

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