

Scaling of Navier-Stokes trefoil reconnection

Robert M. Kerr, Warwick University, Coventry, UK

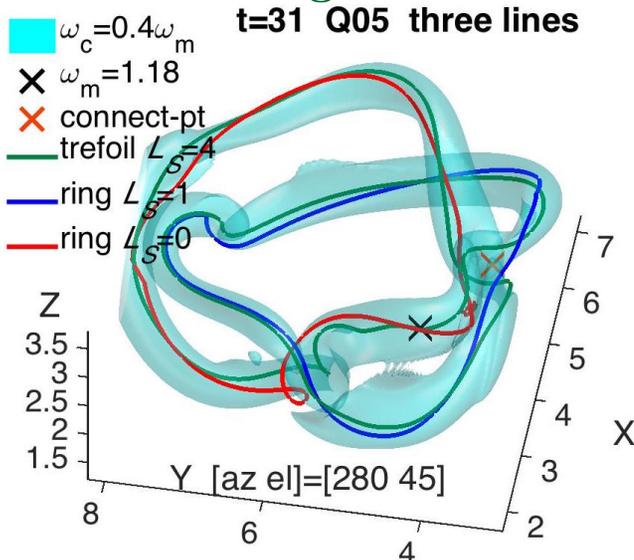
September 27, 2016

Trefoil Experiment



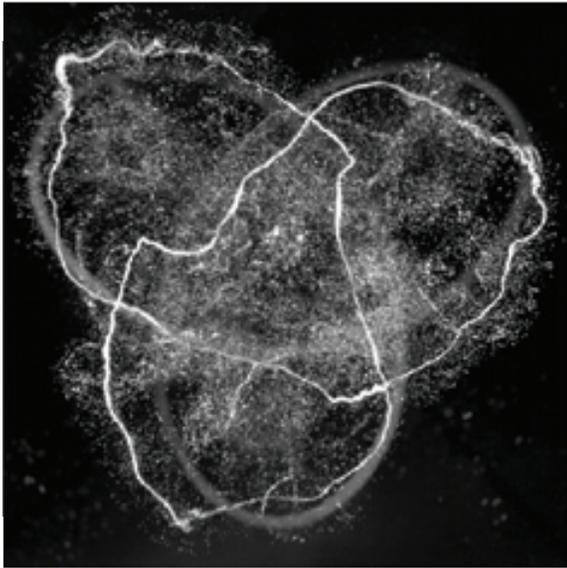
- The reconnection of a trefoil vortex knot is examined numerically to determine how its helicity and two vorticity norms behave.
- Helicity is remarkably preserved as in experiment.
- Self-similar growth where $\sqrt{\nu}Z(t_x)$ cross at $t_x = 41$, independent of the viscosity and radius r_e for vortex.
- Z is volume-integrated vorticity squared or enstrophy. t_x is the end of the first reconnection.
- $1/(\sqrt{\nu}Z)^{1/2}$ is linear for $t \lesssim t_x$. Collapse onto one curve if time rescaled: $\delta t_\nu = (t - t_x)/(T_c(\nu) - t_x)$.

Simulation during reconnection



- Navier-Stokes $\|\omega\|_\infty$ is bounded by the Euler values for all ν .
- Euler $\|\omega\|_\infty$ growth is at most exponential of exponential Kerr (2013b).
- Nonetheless, at early times very small viscosity Navier-Stokes Z greater than Euler Z . Which allows:
- Viscosity independent dissipation rate $\epsilon = \nu Z$ at $t \approx t_x$.
- *Dissipation anomaly* (finite dissipation in a finite time) without singularities or roughness as $\nu \rightarrow 0$

Trefoil Experiment



Scaling of Navier-Stokes trefoil reconnection

← **Inspiration:** Helicity conservation by flow across scales in reconnecting vortex links and knots Proc. Nat. Acac. Sci. **111** (2014). Scheeler, D. K., D. P., G. L. K., W. T. M. Irvine. **Helicity preservation is claimed**

Euler $\omega \times u$ **Navier-Stokes** ℓ^3 periodic domains

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u = -\nabla p + \nu \Delta u \quad (??)$$

$$\frac{\partial u}{\partial t} + (\omega \times u) = -\nabla p_h + \nu \Delta u \quad (??)$$

- **Address this paradox:**

Real world: There is:

Finite energy dissipation in a finite time.

Mathematics says not, unless the solution under Euler (inviscid) has a singularity or $\nu > \nu_0(\ell)$. (Const1986)

Euler numerics are **super-exponential: NOT** singular.

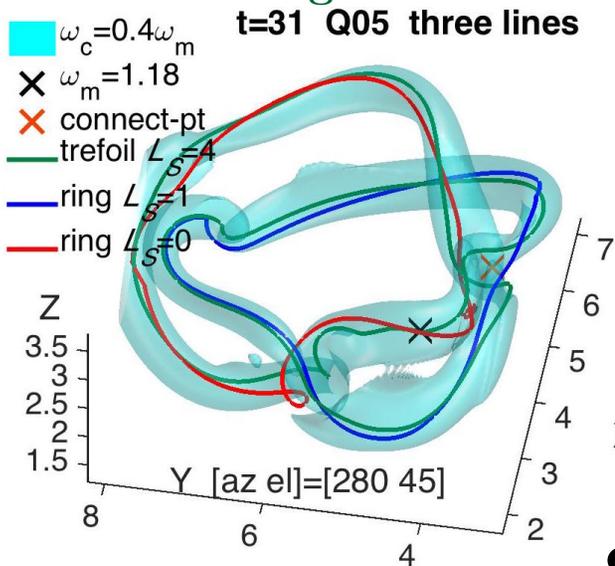
- If proven $\forall T$, step to solve **Clay Prize** for Navier-Stokes.
- **But the Clay Prize is not the full story.**

← **Tools: Asymmetric numerical trefoil.**

- **As $\nu \rightarrow 0$:** Follow two classes of norms in time.

- Vorticity-like: enstrophy $Z(t) = \int_V \omega^2$.
- Helicity-like: helicity $\mathcal{H}(t) = \int_V u \cdot \omega$.

Simulation during reconnection



What trefoil reconnection says about Navier-Stokes regularity R. M Kerr, U. Warwick, UK

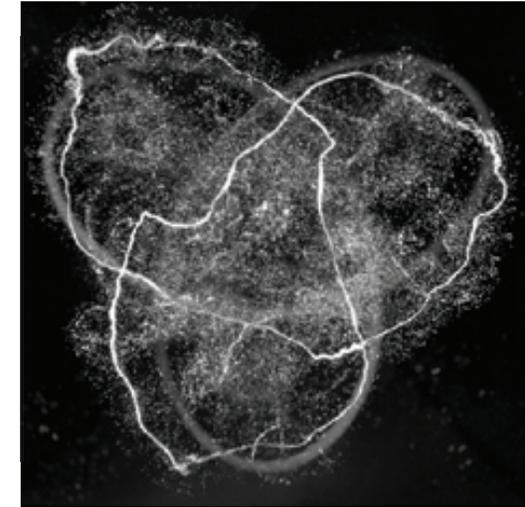
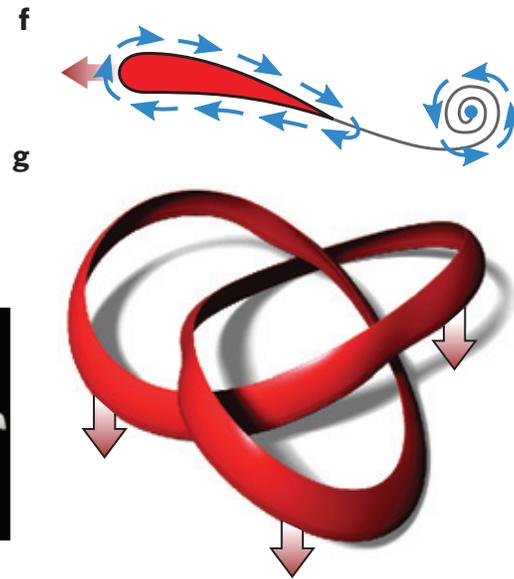
Experimental 3D PRINTED

knot made of Teflon.

Immerse in water saturated with hydrogen bubbles. The bubbles stick to the Teflon.

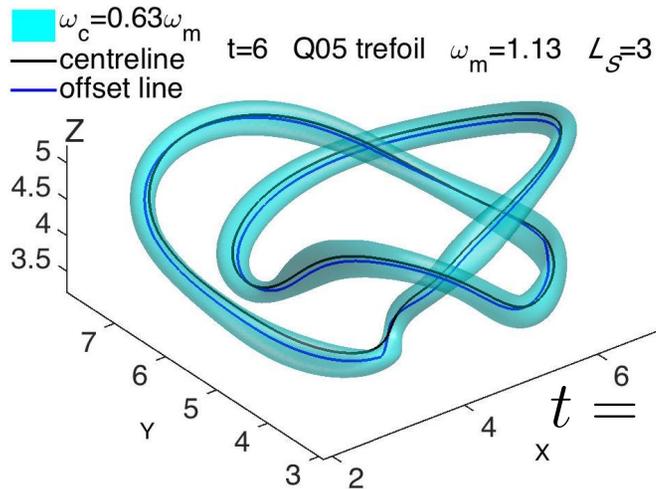


Yank out of water rapidly.



Helicity conservation by flow across scales in reconnecting vortex links and knots

Proc. Nat. Acac. Sci. **111** (2014). Martin W. Scheeler, Dustin Kleckner, Davide Promente, Gordon L. Kindlmann, and William T. M. Irvine. **Helicity preservation is claimed**

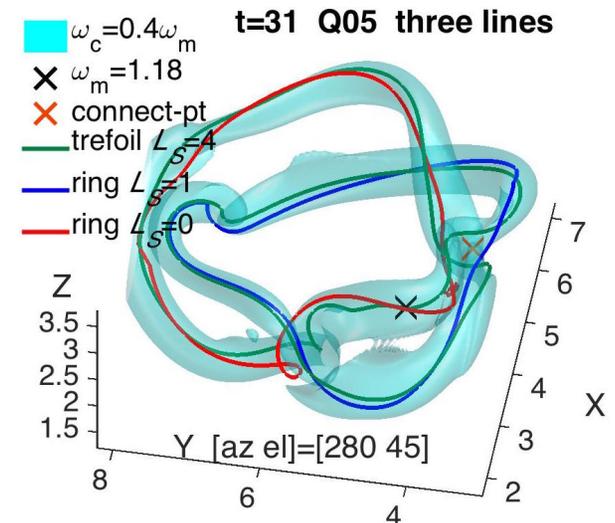


Up to 2048^3 numerical self-reconnecting trefoil.

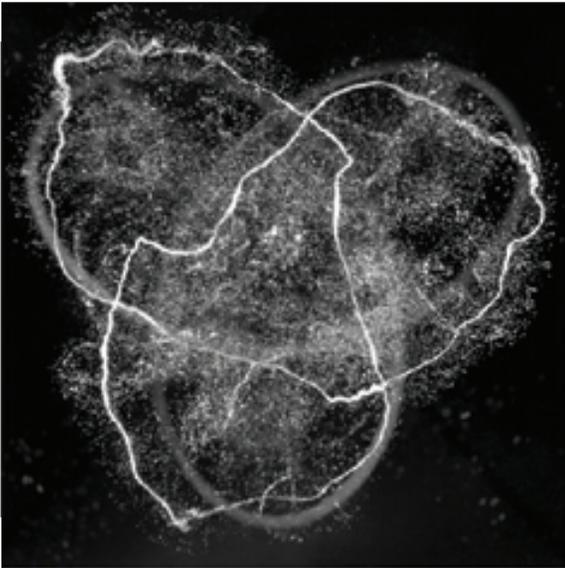
In both experiments and simulations

HELICITY IS REMARKABLY PRESERVED. But..

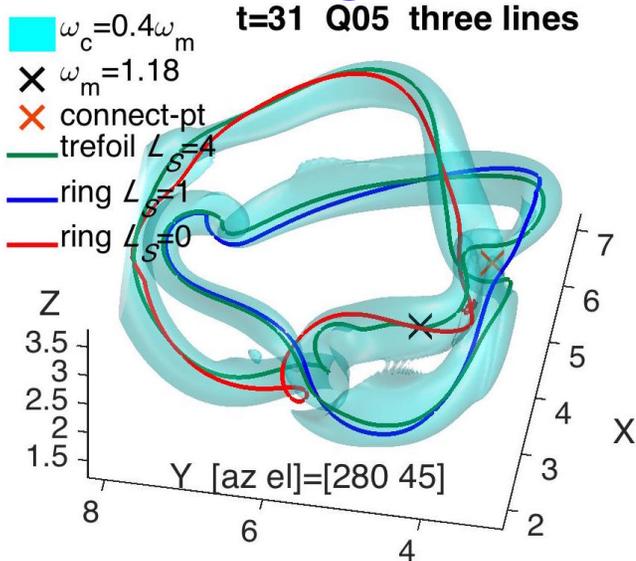
IS HELICITY PRESERVED FOREVER?



Trefoil Experiment



Simulation during reconnection



⇐ **Inspiration:** Helicity conservation by flow across scales in reconnecting vortex links and knots. Proc. Nat. Acac. Sci. **111** (2014). Scheeler, D. K., D. P., G. L. K., W. T. M. Irvine. **Helicity preservation is claimed**

⇐ **To address Navier-Stokes regularity:**

$$\frac{\partial \mathbf{u}}{\partial t} + (\boldsymbol{\omega} \times \mathbf{u}) = -\nabla p_{\text{h}} + \nu \Delta \mathbf{u} \quad (\text{board}) \quad (??)$$

• **As $\nu \rightarrow 0$:**

Follow two classes of local and volume integrated norms in time.

• $Z(t)$: Enstrophy = $\int_V \omega^2$. **All grow significantly.**

– Leray scaled: $\sqrt{\nu} Z(t)$, **(board)**

– dissipation $\epsilon(t) = \nu Z(t)$, **(board)**

– $\omega_m = \max |\omega| = \|\omega\|_{\infty}$. **(board)**

• $\mathcal{H}(t)$: Helicity = $\int_V \mathbf{u} \cdot \boldsymbol{\omega}$. **All barely change.**

– Suppresses $\boldsymbol{\omega} \times \mathbf{u}$ nonlinearity.

Drop in \mathcal{H} is especially small when compared with the dramatic changes in the vorticity norms during the first time period.

$(\mathcal{H})^{1/2}$ is related to these regularity norms:

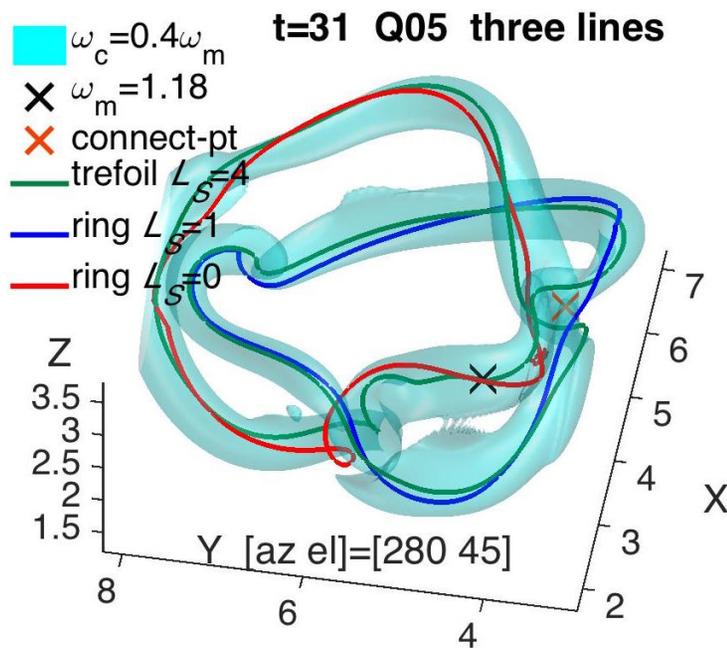
– $L_3(t) = \ell \|u\|_{L^3} = \left(\int_V |u|^3 \right)^{1/3}$ **(board)**

– $H^{(1/2)}(t) = \ell^{3/2} \|u\|_{\dot{H}^{1/2}} = \ell^{3/2} \left(\int d^3 k |\mathbf{k}|^1 |\mathbf{u}(\mathbf{k})|^2 \right)^{1/2}$ **b**

Simulations of vortex reconnection can address the mathematics of intense events if:

GOALS TODAY:

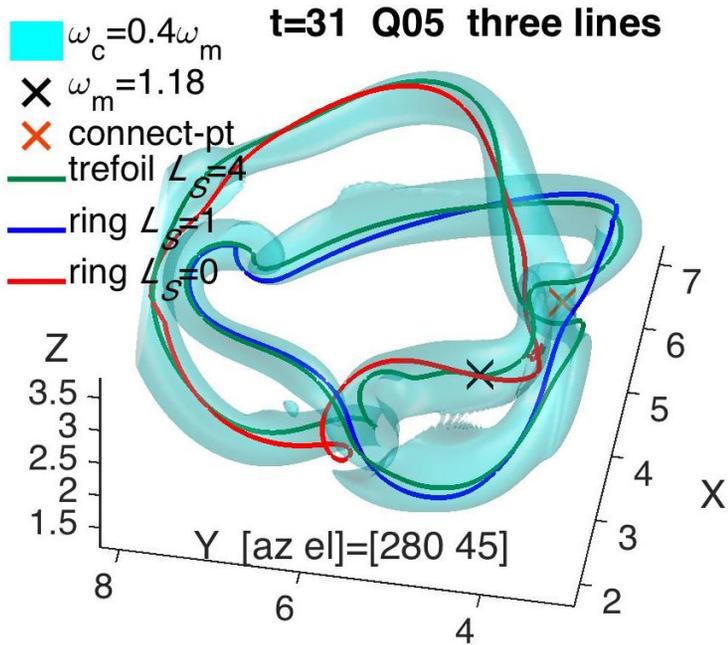
- Use helicity \mathcal{H} decay and enstrophy Z growth to:
- * Characterise $\nu \rightarrow 0$ scaling of:



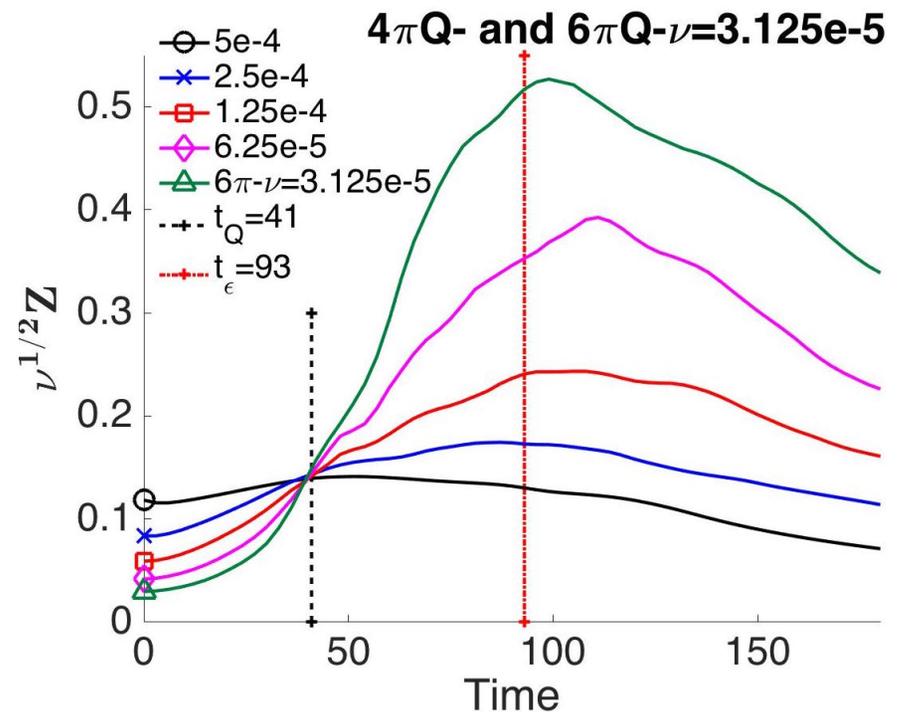
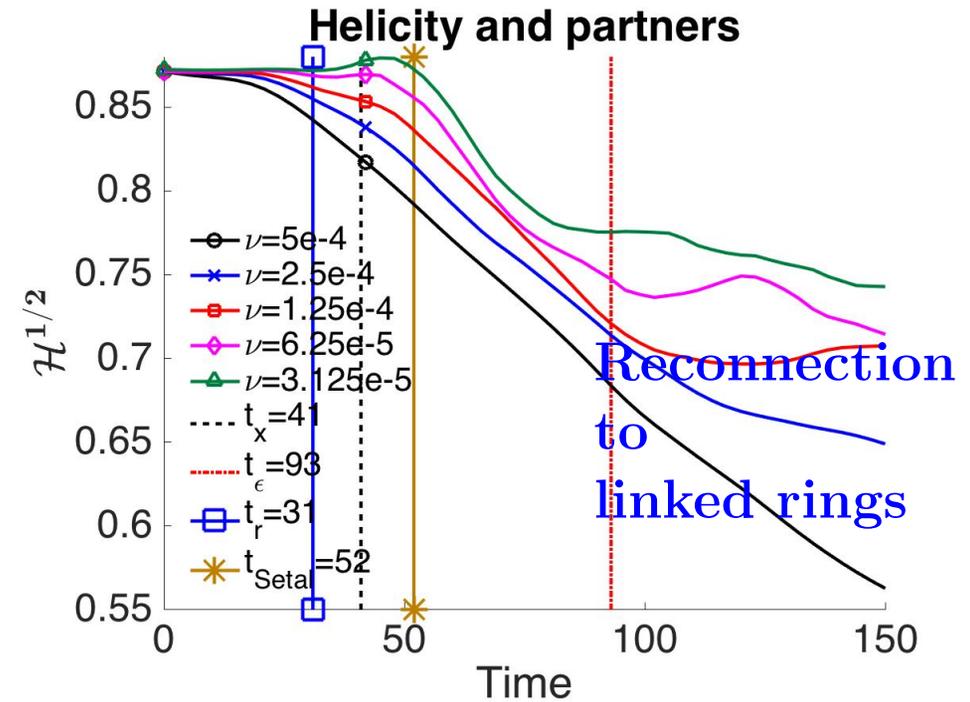
- **One has a robust initial condition that is stable against instabilities within the vortex core.**
- All post-Melander/Hussain (1989) anti-parallel calculations including mine (pre-2013) and ones still being published have unstable cores.
- **One has mathematics that allows one to isolate, in increments, the diagnostics that are accessible to the numerics.**
- That is relationships between moments, including how they bound each other's time derivatives and production terms.
- **Huge amounts of computational power.**
- Example of doing it right: Kerr, RM 2013 .. *J. Fluid Mech.* **729**, R2. Bounds for Euler from vorticity moments and line divergence. By comparing $\Omega_m = \|\omega\|_{2m}$ using directly computed $d\Omega_m/dt$ all Ω_m grow as exponentials of exponentials and the coefficients can be determined.

GOALS TODAY:

- Use helicity \mathcal{H} decay and enstrophy Z growth to:
- * Characterise $\nu \rightarrow 0$ scaling of:



- Helicity does decay, but not until first reconnection completes.
- Indicated by amazing scaling where all $\sqrt{\nu}Z$ cross at $t = t_x = 41$.



- Governing equations.

Navier-Stokes

$$\frac{\partial \mathbf{u}}{\partial t} - \underbrace{\nu \Delta \mathbf{u}}_{\text{dissipation}} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla p. \quad (1)$$

Using $\boldsymbol{\omega} = \nabla \times \mathbf{u}$ as the vorticity vector, there are two quadratic, inviscid ($\nu = 0$) invariants:

- Energy $E = \int \frac{1}{2} \mathbf{u}^2 dV$ and helicity $\mathcal{H} = \int \mathbf{u} \cdot \boldsymbol{\omega} dV$ (Moffatt , 1969).
- But not the enstrophy $Z = \int \boldsymbol{\omega}^2 dV$ Defining enstrophy and helicity densities, $|\boldsymbol{\omega}|^2$ and $h = \mathbf{u} \cdot \boldsymbol{\omega}$, their equations and volume-integrated norms are:

$$\frac{\partial |\boldsymbol{\omega}|^2}{\partial t} + (\mathbf{u} \cdot \nabla) |\boldsymbol{\omega}|^2 = \underbrace{2\boldsymbol{\omega} \mathbf{S} \boldsymbol{\omega}}_{Z_p = \text{production}} + \nu \Delta |\boldsymbol{\omega}|^2 - \underbrace{2\nu (\nabla \boldsymbol{\omega})^2}_{\epsilon_\omega = Z\text{-dissipation}}, \quad Z = \int \boldsymbol{\omega}^2 dV. \quad (2)$$

$$\frac{\partial h}{\partial t} + (\mathbf{u} \cdot \nabla) h = \underbrace{-\boldsymbol{\omega} \cdot \nabla \Pi}_{\omega\text{-transport}} + \underbrace{\nu \Delta h}_{\nu\text{-transport}} - \underbrace{2\nu \text{tr}(\nabla \boldsymbol{\omega} \cdot \nabla \mathbf{u}^T)}_{\epsilon_h = \mathcal{H}\text{-dissipation}}, \quad \mathcal{H} = \int \mathbf{u} \cdot \boldsymbol{\omega} dV, \quad (3)$$

- Unlike the energy, helicity can be of either sign, is not Galilean invariant and can grow due to its viscous terms (Biferale & Kerr , 1995).

The enstrophy Z can grow due to its production term Z_p .

Norms that a spectral code can calculate

One can also determine spectra, the spectral transfer of energy and a variety of higher-order Lebesgue $\|u\|_{L^p}$ and Sobolev $\|u\|_{\dot{H}^s}$ norms:

$$\|u\|_{L^p} = \left(\frac{1}{V} \int dV |u|^p \right)^{1/p} \quad \text{and} \quad \|u\|_{\dot{H}^s} = \left(\int d^3k |\mathbf{k}|^{2s} |u(\mathbf{k})|^2 \right)^{1/2}. \quad (4)$$

Hölder ineq: $\|u\|_{L^p} = \|u\|_p \leq \|u\|_{p+1}$ specifically $\|\omega\|_2 \leq \|\omega\|_\infty$ **board** (5)

The vorticity diagnostics used will be:

$$\omega_m = \sup |\boldsymbol{\omega}| = \|\boldsymbol{\omega}\|_\infty = \|\boldsymbol{\omega}\|_{L^\infty} \quad \text{and} \quad Z = \ell^3 \|\boldsymbol{\omega}\|_2^2 = \left(H^{(1)} \right)^2 \quad \text{where} \quad H^{(1)} = \ell^{3/2} \|u\|_{\dot{H}^1}. \quad (6)$$

The following two diagnostics, with the same dimensional scaling as the helicity are significant in our understanding of Navier-Stokes regularity:

$$L_3 = \ell \|u\|_{L^3} \quad \text{and} \quad H^{(1/2)} = \ell^{3/2} \|u\|_{\dot{H}^{1/2}}. \quad (7)$$

Escauriaza, L., Seregin, G., & Sverák, V. (2003) shows that L_3 controls Navier-Stokes regularity. The proof by contradiction used **Leray similarity**:

$$\mathbf{u}(\mathbf{x}, t) = \frac{1}{\sqrt{2a(T-t)}} \mathbf{U} \left(\frac{\mathbf{x}}{\sqrt{2a(T-t)}} \right) \quad (8)$$

Resulting in: ($\mathbf{y} = \mathbf{x} / \sqrt{2a(T-t)}$, a same units as ν .)

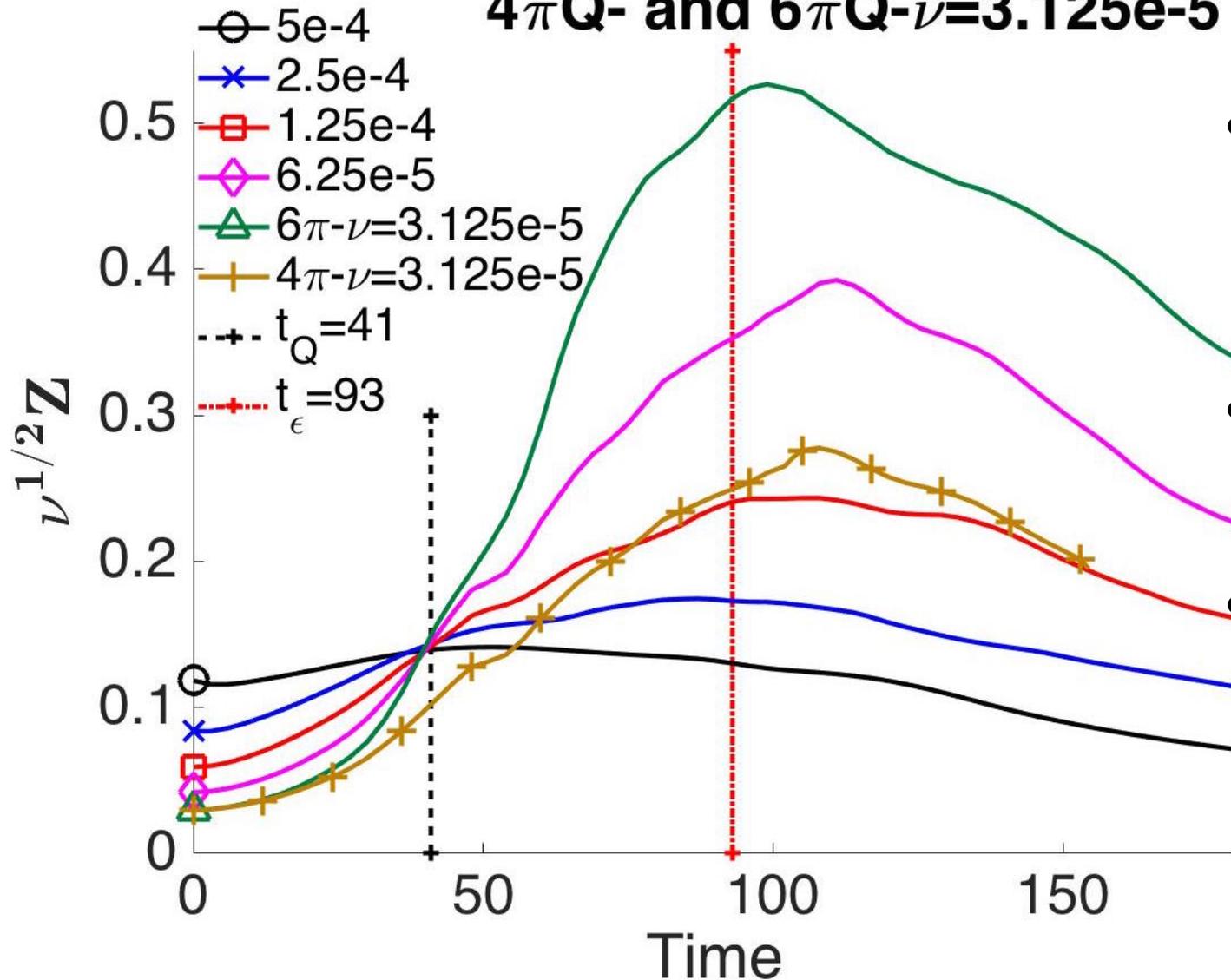
$$\left. \begin{aligned} -\nu \Delta \mathbf{U} + a \mathbf{U} + a(\mathbf{y} \nabla \mathbf{U}) + (\mathbf{U} \cdot \nabla) \mathbf{U} + \nabla P &= 0 \\ \nabla \cdot \mathbf{U} &= 0 \end{aligned} \right\} \quad \text{in } \mathbb{R}^3 \quad (9)$$

Simulation parameters

- Same $\sqrt{\nu}Z$ as before, with an extra **brown +** curve.

- The radius of the trefoil is $r_t = 2$, loop separation is $\delta_a = 1$.

$4\pi Q$ - and $6\pi Q$ - $\nu=3.125e-5$



- Viscosities from $\nu = 0.00025$ to $\nu = 3.125e-5$ **Resolved**. (plus to $\nu = 2e-6$ underresolved)

- The four radii calculated are $r_e(SRQP) = 0.24, 0.33, 0.47, 0.66$.

- $\sqrt{\nu}Z$ due to Leray?

$\sqrt{\nu}Z$ for the smallest ν , highest Reynolds number, in a smaller box does not cross the others at $t = t_x = 41$.

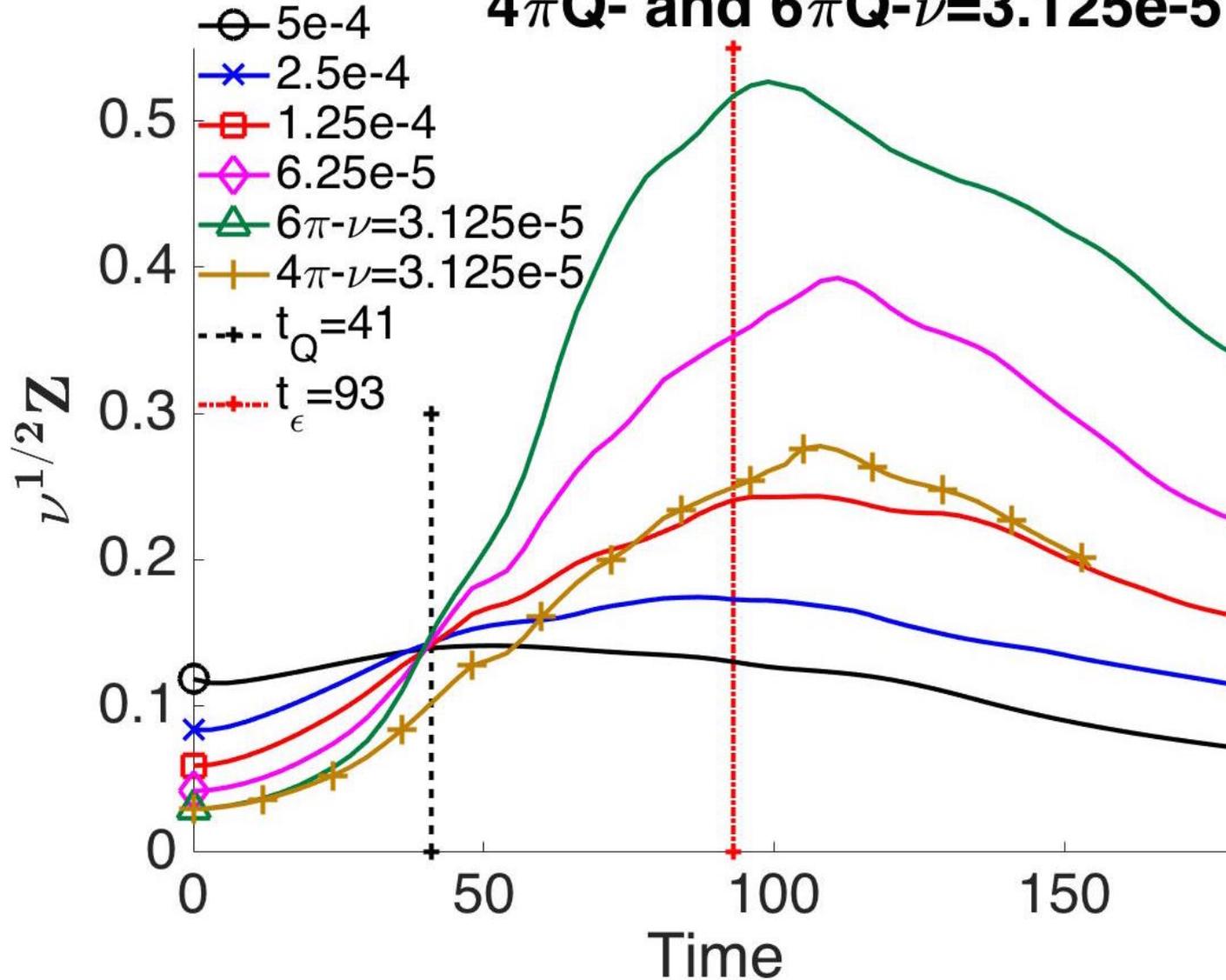
- **Brown +** calculation uses the same smallest viscosity $\nu = 3.125e-5$ as the **green curve**, but the domain is smaller, $(4\pi)^3$ not $(6\pi)^3$.

- **But $\sqrt{\nu}Z$ CROSSES** in a larger $(6\pi)^3$ domain.

WHY in $(6\pi)^3$ but not $(4\pi)^3$?

- Same $\sqrt{\nu}Z$ as before, with an extra **brown +** curve.

$4\pi Q$ - and $6\pi Q$ - $\nu=3.125e-5$



- **Brown +** calculation uses the same smallest viscosity $\nu = 3.125e - 5$ as the **green curve**, but the domain is smaller, $(4\pi)^3$ not $(6\pi)^3$.

* A partial answer comes from **Constantin CMP (1986)**

“Note on Loss of Regularity for Solutions of the 3D Incompressible Euler and Related Equations”.

If the Euler solution for an initial condition in a Sobolev space is regular up to a certain time.

Then there is a critical viscosity below which as $\nu \rightarrow 0$ Navier-Stokes calculation will be bounded by a function of the regular Euler solution.

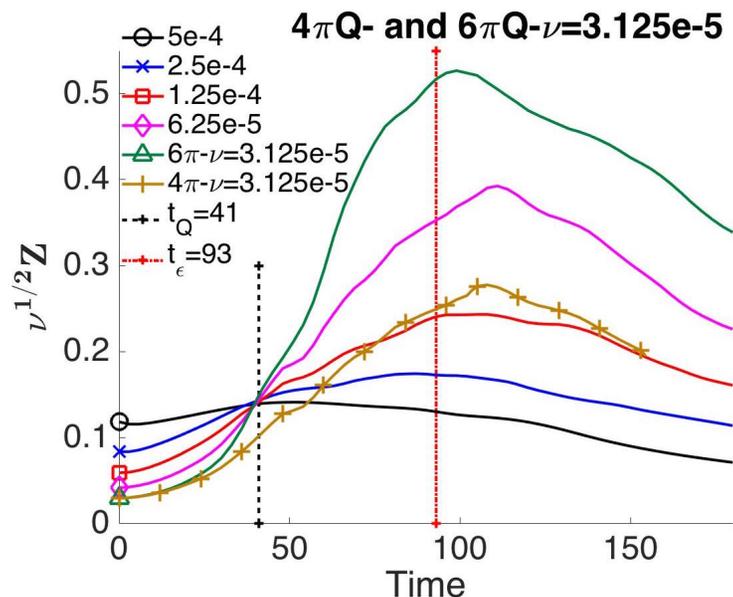
NO $\nu \rightarrow 0$ finite dissipation rate $\epsilon = \nu Z$.

- A partial answer comes from the **Constantin (1986) “Note on Loss of Regularity for Solutions of the 3D Incompressible Euler and Related Equations”**.
- In a Sobolev space, when the Euler solution for an initial condition is regular:
- **Then for very small viscosities: $\nu \leq \nu_0(\ell)$.**

Regular Euler functions will bound Navier-Stokes.

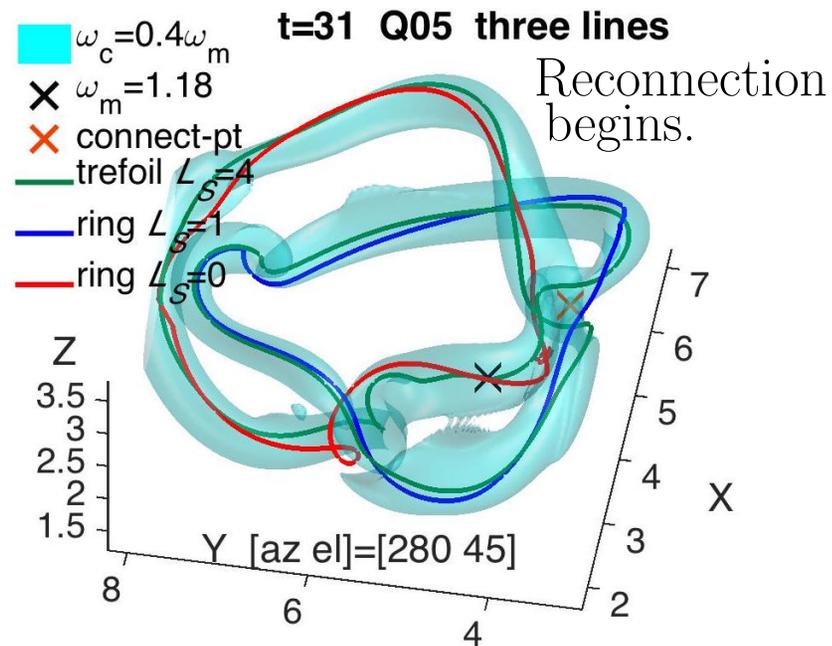
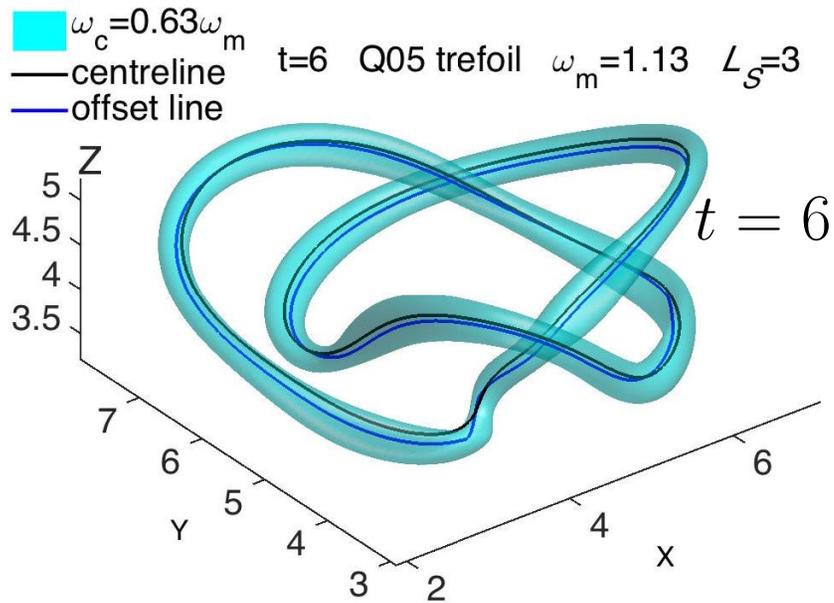
There CANNOT be a non-Euler $\nu \rightarrow 0$ limit.

- **Solution:** This critical viscosity decreases with ℓ as the outer boundary of the ℓ^3 periodic domain is increased.
- **So increasing ℓ relaxes the $\nu \leq \nu_0(\ell)$ bound.**



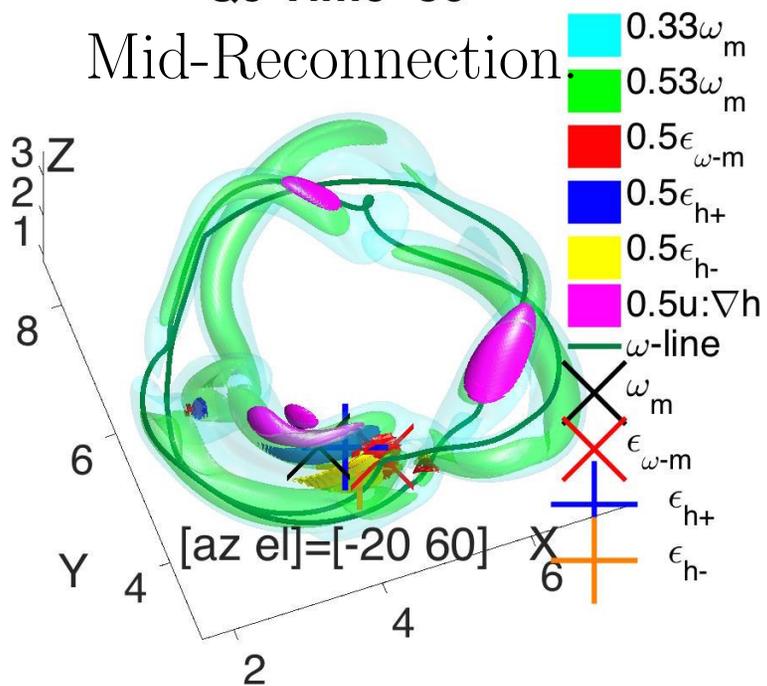
But there must be more than this. Must also address:

- There is a sketch of a proof by Masmoudi (2007) for Whole Space that will be addressed at the end.

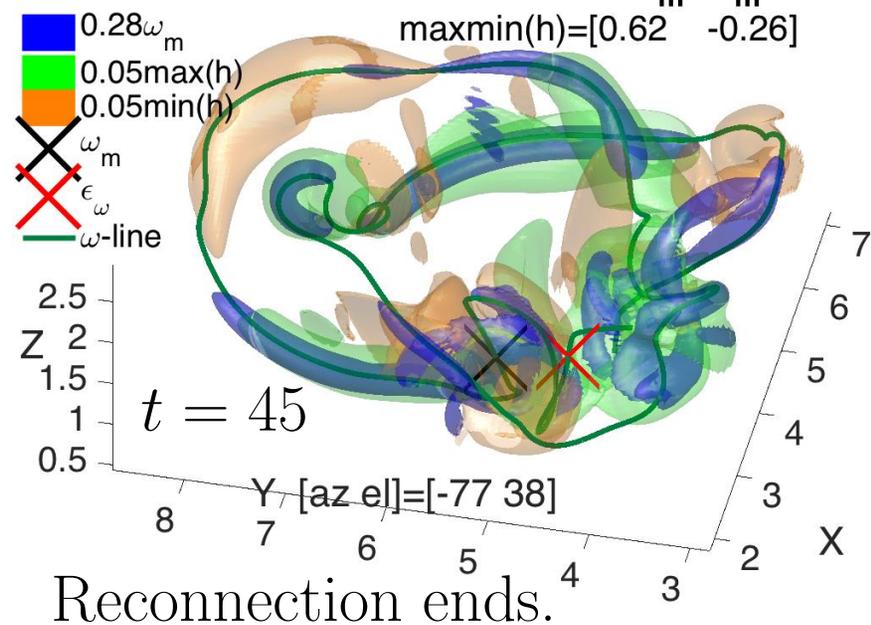


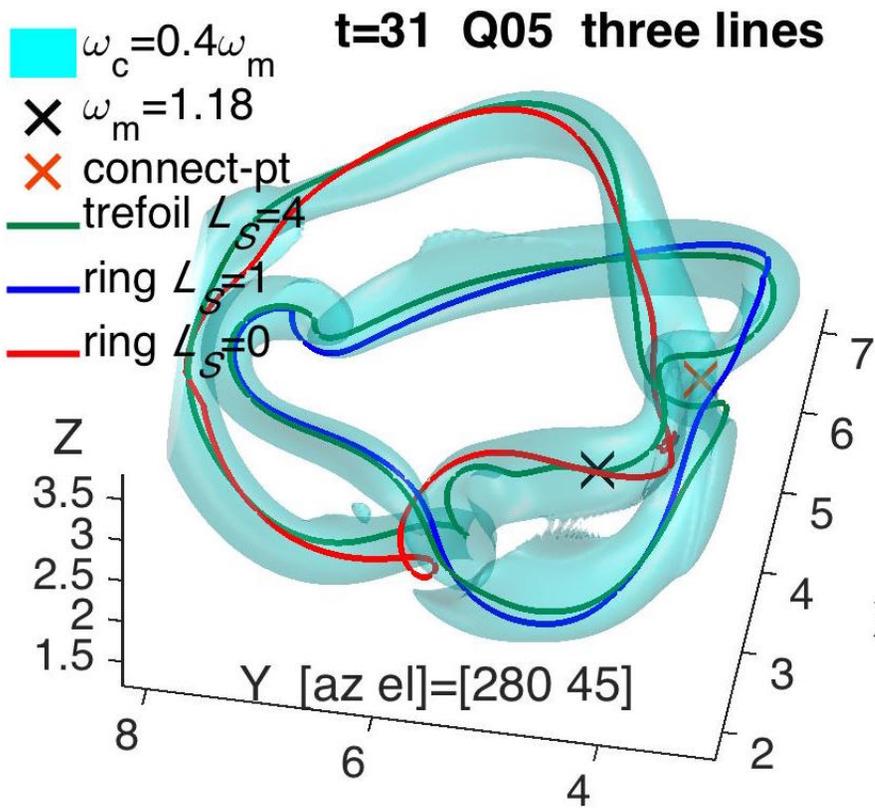
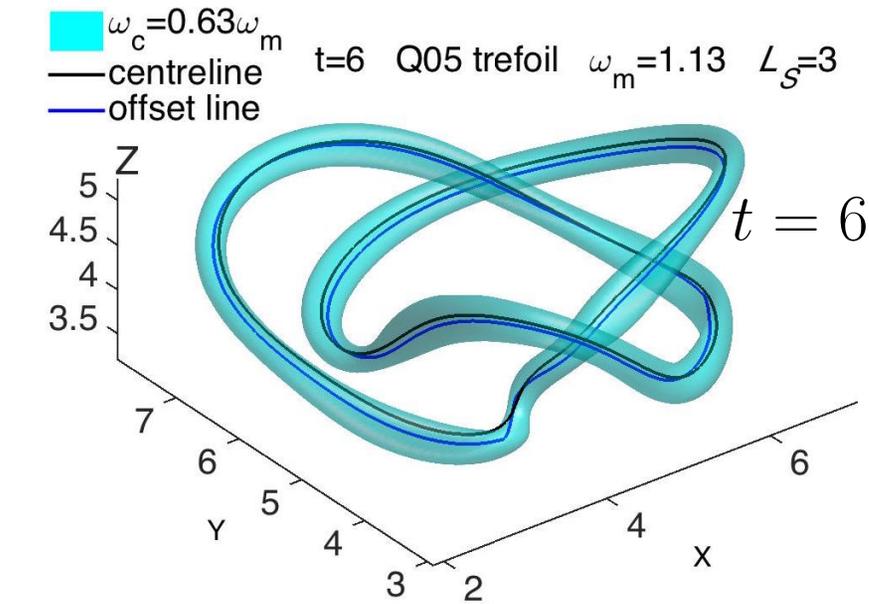
Q5 Time=36

Mid-Reconnection



t=45 X3q trefoil $\nu = 0.00025$ $\omega = 0.28\omega_m$ $\omega_m = 3.4$
 maxmin(h)=[0.62 -0.26]

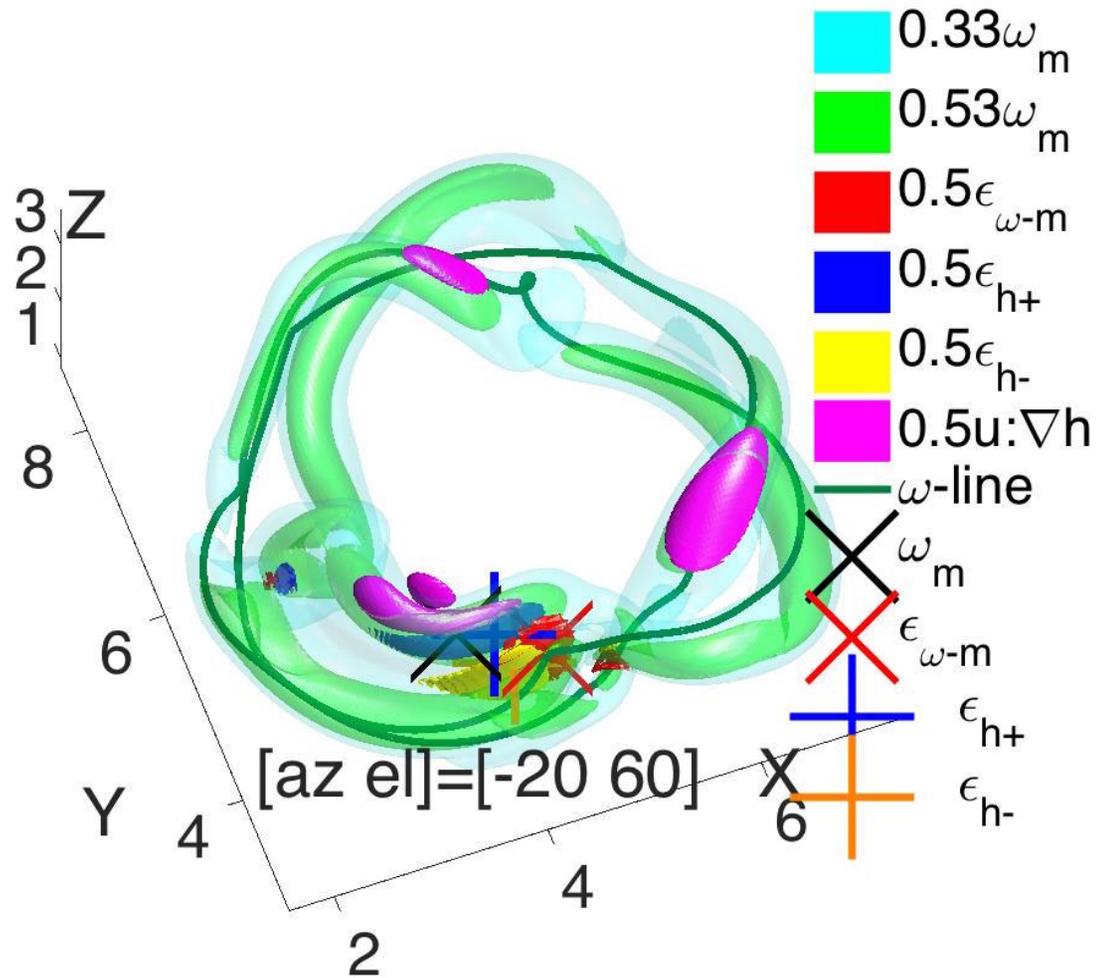




Vorticity isosurface plus two closed vortex lines of the perturbed trefoil vortex at $t = 6$, not long after initialization. Its self-linking is $\mathcal{L}_S = 3$, which can be split into writhe $\mathcal{W} = 3.15$ and twist $\mathcal{T} = -0.1$.

A single vorticity isosurface plus three closed vortex lines at $t = 31$. The green line follows a remaining trefoil trajectories seeded near ω_m , indicated by **X**. Its $\mathcal{L}_S = 4$, which can be split into $\mathcal{W} + \mathcal{T} = 2.85 + 1.15 = 4$. The orange cross is the “reconnection point”, the point between the closest approach of the trefoil’s two loops and where, due to an extra twist, the loops are locally anti-parallel. The Red $\mathcal{L}_S = 0$ and blue $\mathcal{L}_S = 1$ lines originate on either side of the reconnection point and are linked, which gives a total linking of $\mathcal{L}_t = 2\mathcal{L}_{rb} + \mathcal{L}_{Sb} + \mathcal{L}_{Sr} = 2 + 1 + 0 = 3$, the linking of the original trefoil.

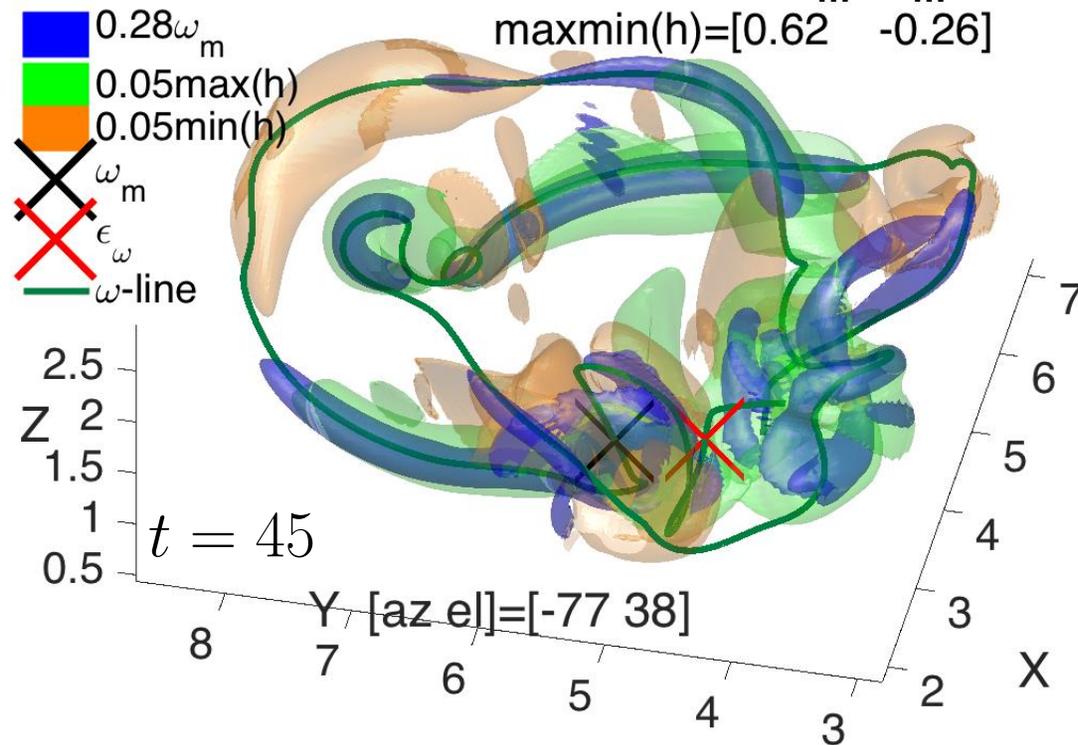
Q5 Time=36



The perspective for $t = 36$ is rotated 90° clockwise from the $t = 31$ figure so that the reconnection gap to the right of the **red** cross and between the **green** $0.53\omega_m$ and **red** enstrophy dissipation $0.5\epsilon_{\omega-m}$ isosurfaces can be seen directly, where $\omega_m = \max(|\omega|) = 1.24$ and $\epsilon_{\omega-m} = \max(\epsilon_\omega) = 185$. What is left of the trefoil is outlined with the $0.33\omega_m$ **cyan** vorticity isosurface and the unique closed trefoil line in **green**. Strong positive helicity is essentially collocated with the $0.53\omega_m$ vorticity isosurface.

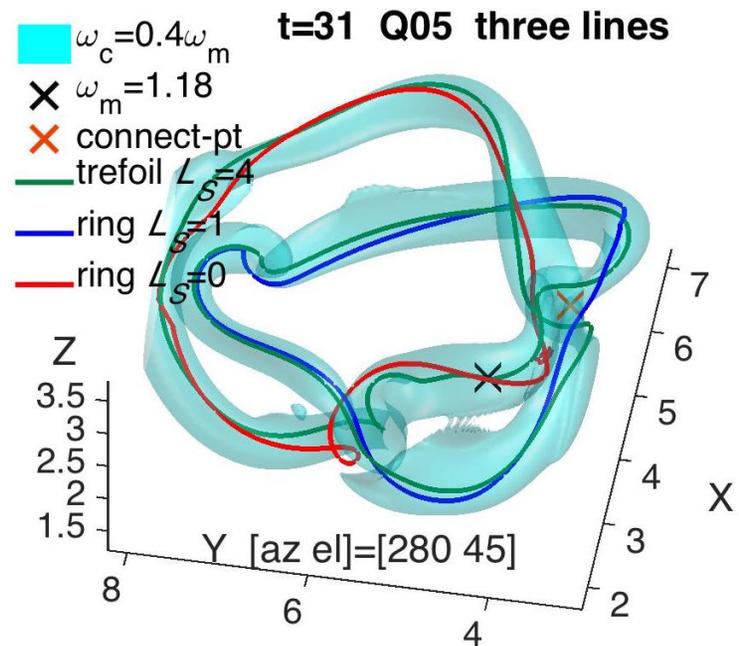
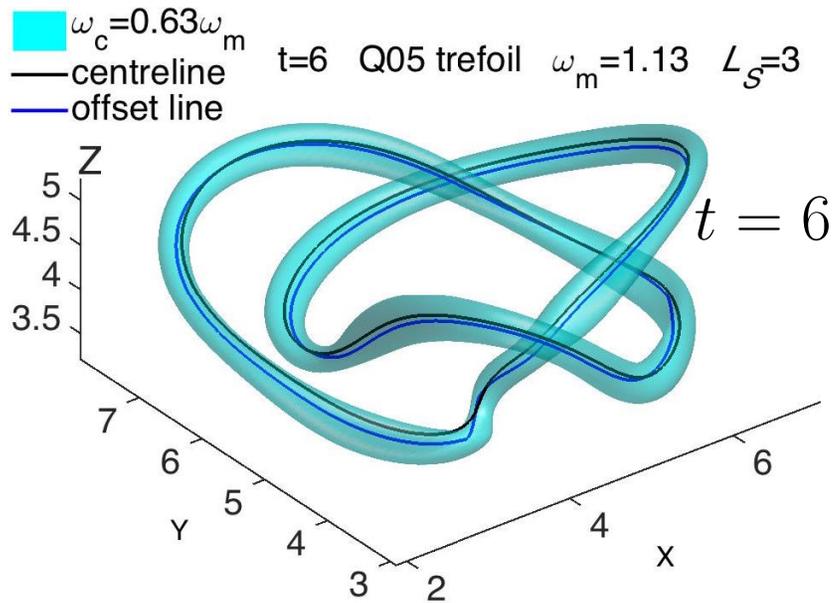
Besides enstrophy dissipation, helicity dissipation (of both signs) and helicity transport have been rendered. For $\max(\epsilon_h) = \epsilon_{h+} = 4.9$, $\min(\epsilon_h) = \epsilon_{h-} = -3.2$ and $\max(\mathbf{u} \cdot \nabla h) = 0.06$ and the surfaces are at $\epsilon_h = 0.5\epsilon_{h+}$ **blue**, $\epsilon_h = 0.5\epsilon_{h-}$ **yellow-orange**, and $\mathbf{u} \cdot \nabla h = 0.5 \max(\mathbf{u} \cdot \nabla h)$ **magenta**. The three dissipation surfaces for, ϵ_ω , $\epsilon_h > 0$, $\epsilon_h < 0$, and $\mathbf{u} \cdot \nabla h$ to the left of the gap are sheets and layered on top of one another, so extra X's and +'s have been added to indicate where their maxima (or minima) are. Note how helicity is advected from both sides into the reconnection zone.

t=45 X3q trefoil $\nu=0.00025$ $\omega=0.28\omega_m$ $\omega_m=3.4$

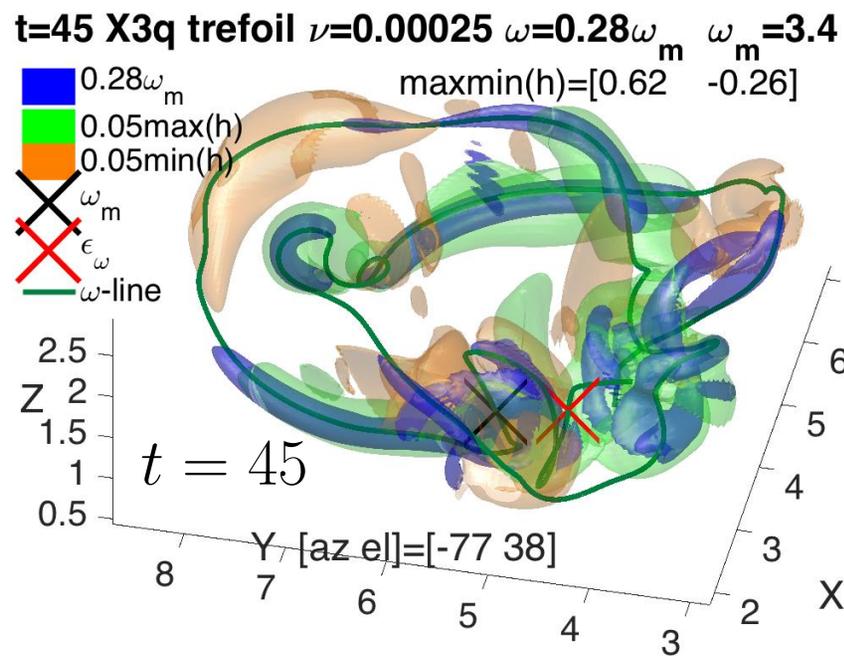
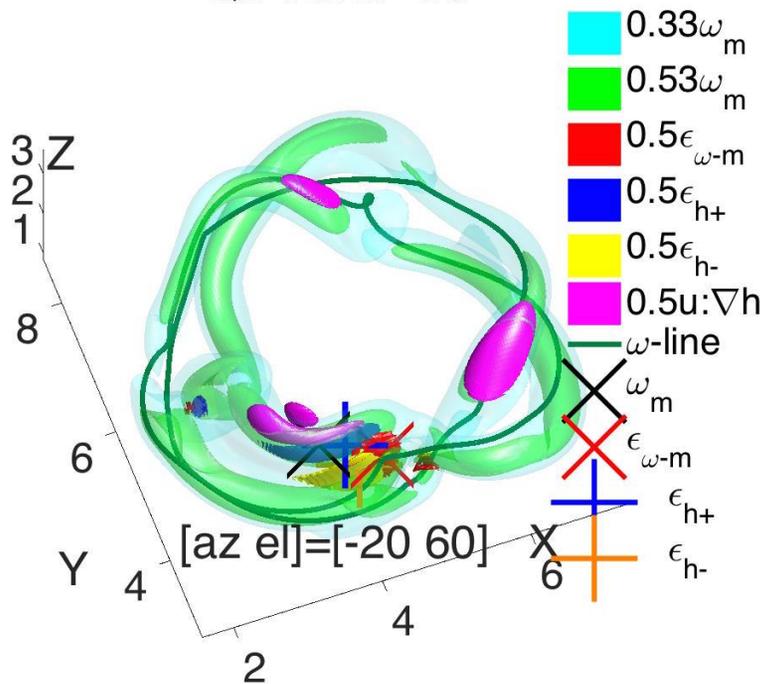


Isosurfaces and one vortex line at $t = 45$ just as the first reconnection has ending. Isosurfaces of vorticity isosurfaces in **blue** and helicity isosurfaces at $0.05 \max(h)$ in **green** and $0.05 \min(h)$ in **yellow** where $\max(h) = 0.62$ and $\min(h) = -0.26$. The position where reconnection began is the red X.

The vortex line seeded at the point of maximum vorticity at X still has the flavour of the original trefoil as it circumnavigates the centre twice and passes through both the regions with large vorticity and large helicity of both signs. The region between the X's that saw a sandwiching of dissipation surfaces at $t = 36$ now is dominated by negative helicity. The region to the right is now a twisted vortex. Note in particular the region of large negative helicity on the other side of the reconnection zone in the upper left.

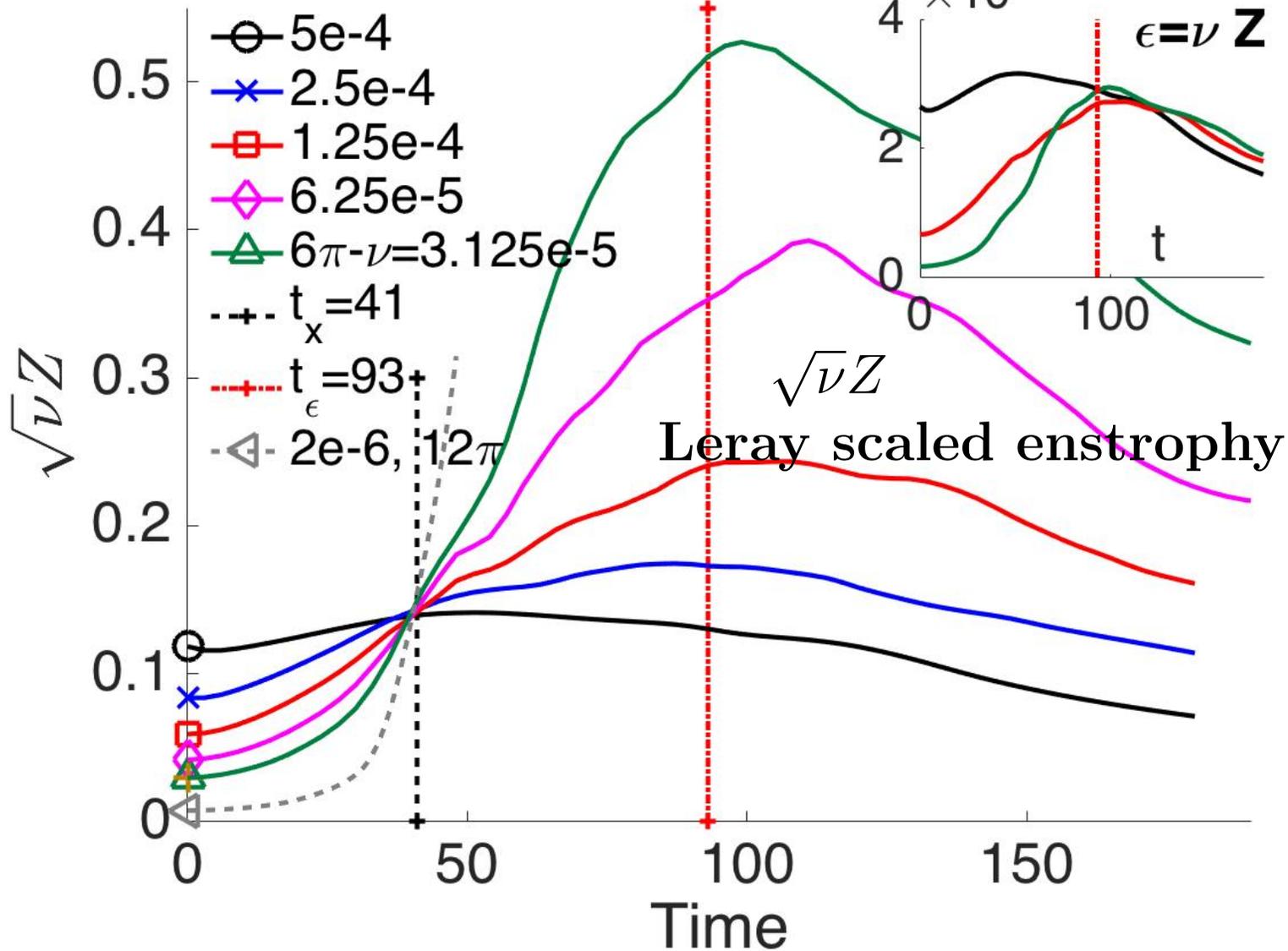


Q5 Time=36

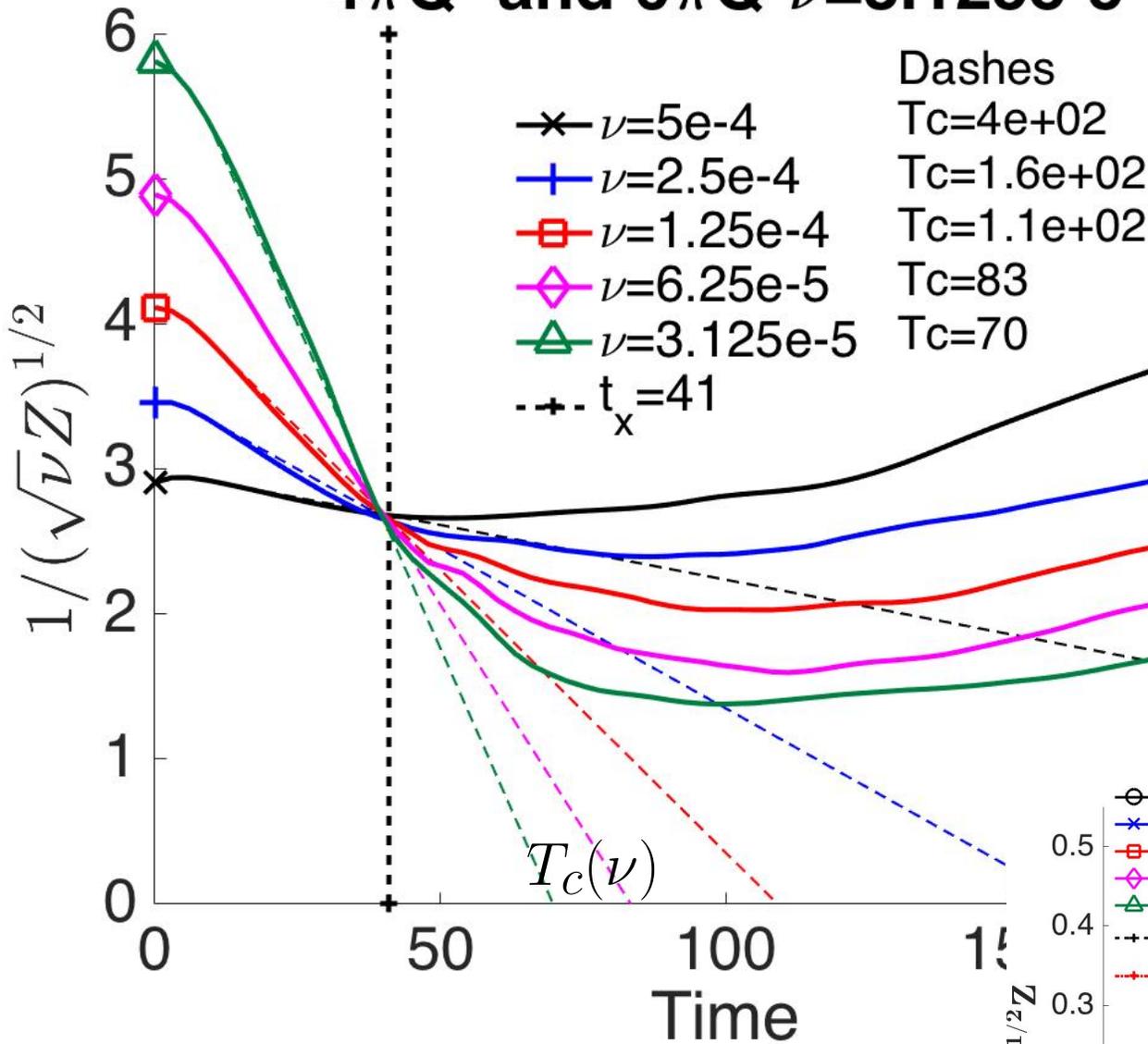


Getting self-similar scaling

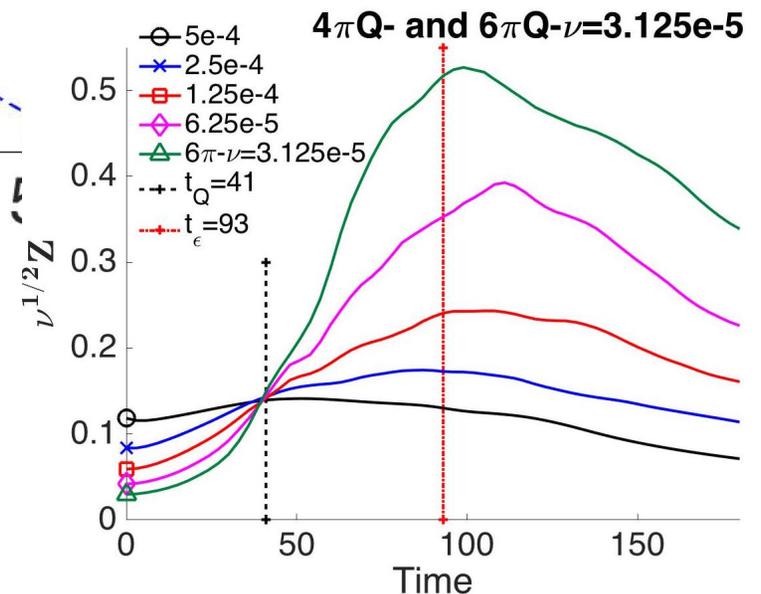
$4\pi Q$ - and $6\pi Q$ - $\nu=3.125e-5$



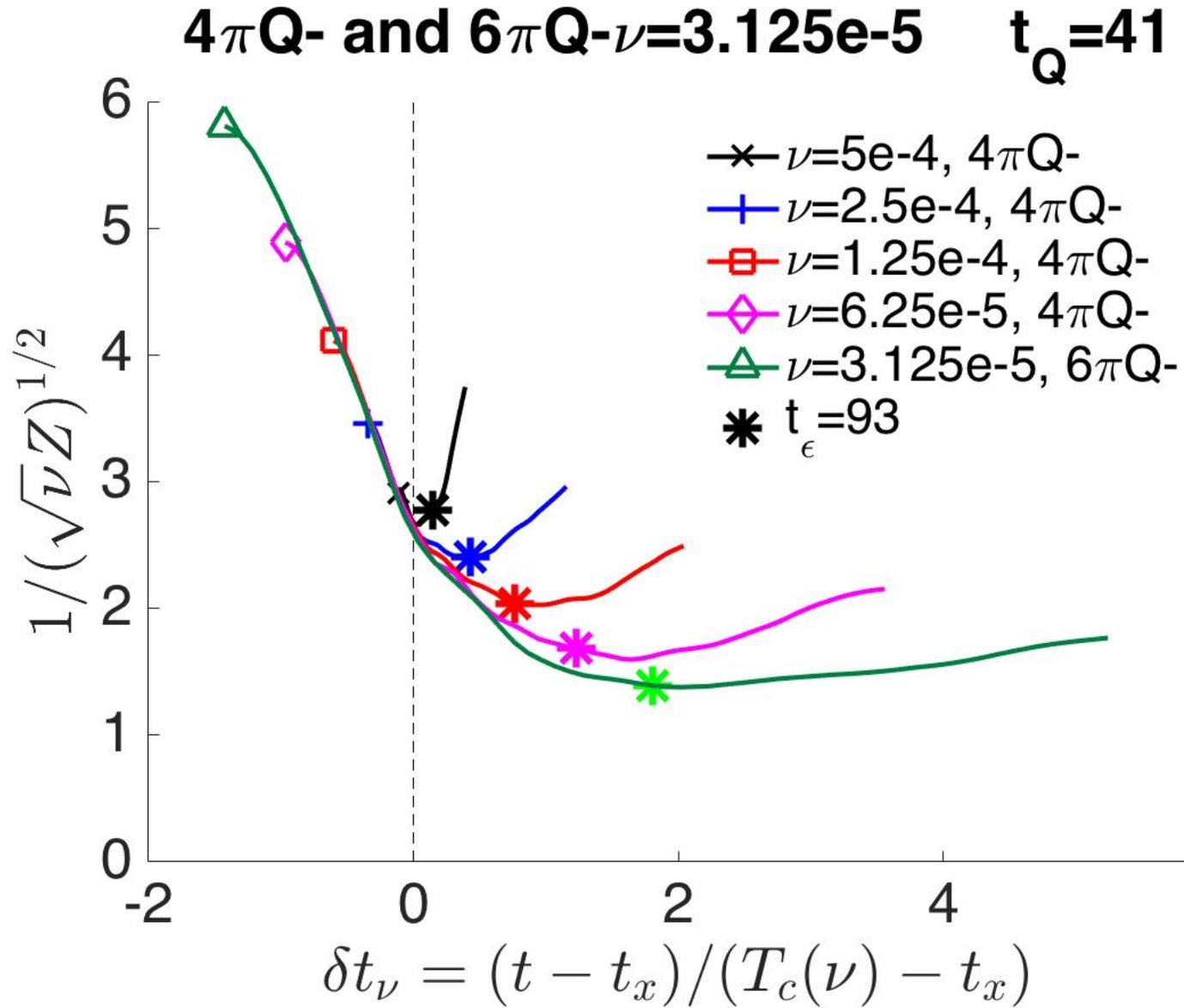
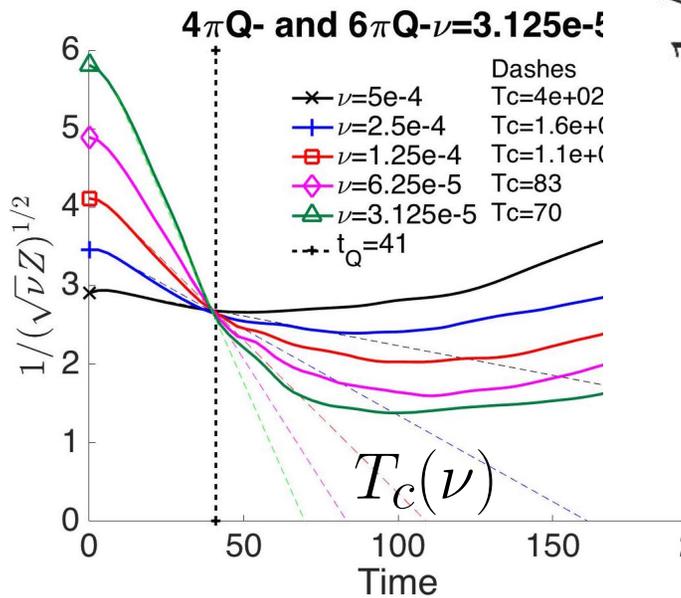
4πQ- and 6πQ-ν=3.125e-5



- Apply Hölder-like scaling to enstrophy.
- That is, if $\|\omega\|_\infty \sim 1/(T_c - t)$
- then: $H^{(1)} = Z^{1/2} \sim 1/(T_c - t)$
- Then estimate projected critical times by extending the fits to $1/(\sqrt{\nu}Z)^{1/2} = 0$.
- Draw extensions.

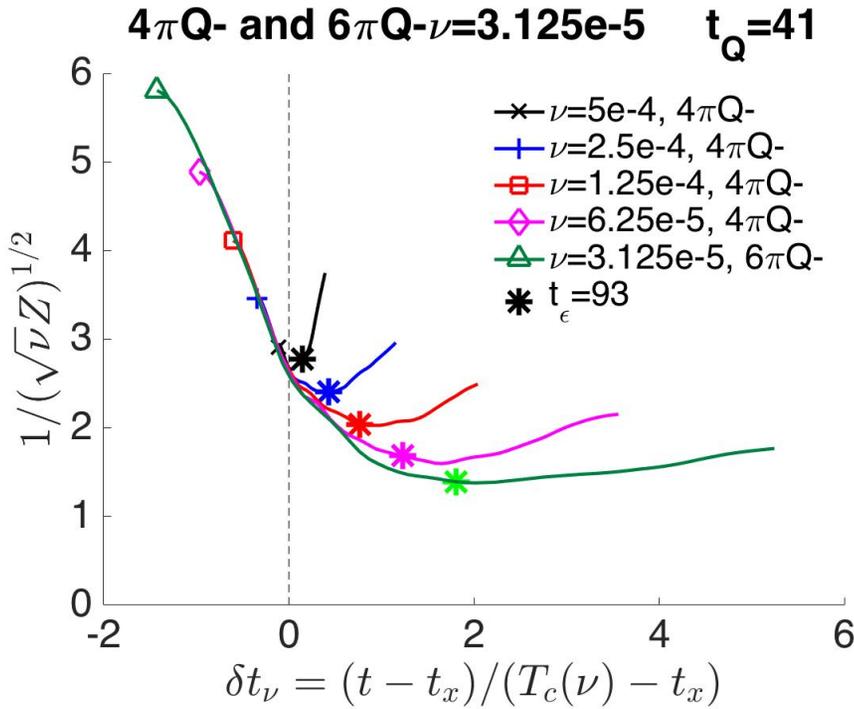


- Apply Hölder scaling to enstrophy.
- Then estimate projected critical times by extending the fits to $1/(\sqrt{\nu}Z)^{1/2} = 0$.
- Draw extensions.



- $\Delta t(\nu) = T_c(\nu) - t_x$
- Left-end symbols are $t = 0$

Path to the scaling:



- If $\nu^{1/2}Z(t_x) = B_{\sqrt{\nu}Z} \approx 2.5 \forall \nu$
then:

$$\frac{1}{\sqrt{\nu^{1/2}Z(t)}} = \frac{T_c - t}{B_{\sqrt{\nu}Z}^{1/2}(T_c - t_x)}$$

- $\Delta t(\nu) = T_c(\nu) - t_x$

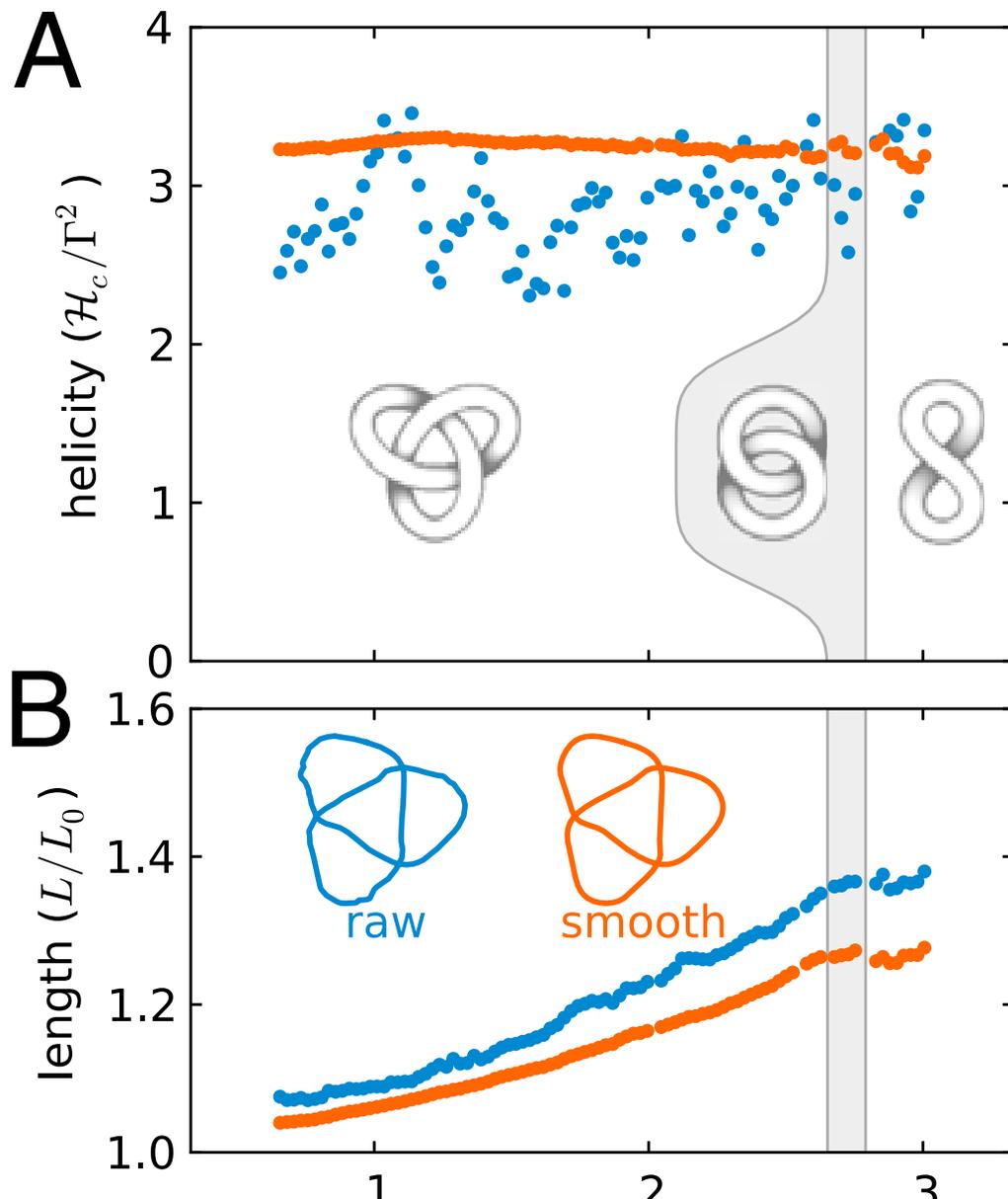
- Left-end symbols are $t = 0$
 $T_c(\nu) = t_x / \left(1 - \sqrt{\nu^{1/2}Z/B_{\sqrt{\nu}Z}}\right)$

(Very crudely) $\|\omega\|_\infty \sim 1/(\nu(T_c - t))$, the Leray prediction, for consistency from the prediction of Leray scaling, one would expect to see $Z \sim 1/(\nu(T_c - t))^{1/2}$, which would imply linear behaviour for $1/Z^2 \sim (T_c - t)$.

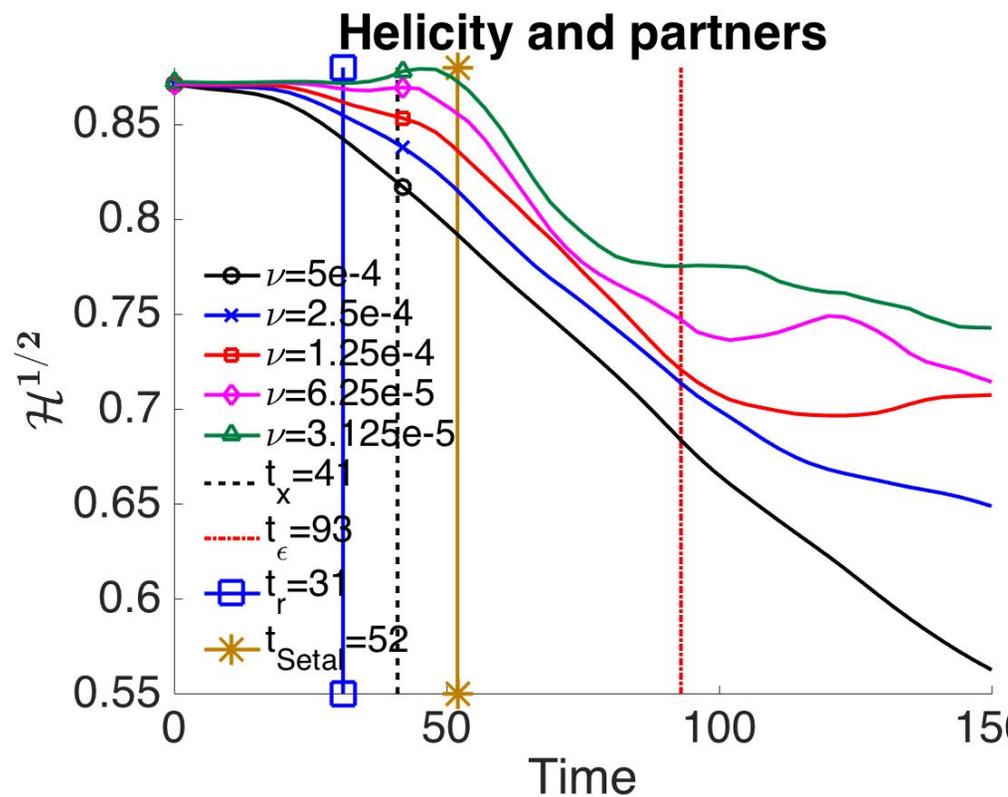
SKIP

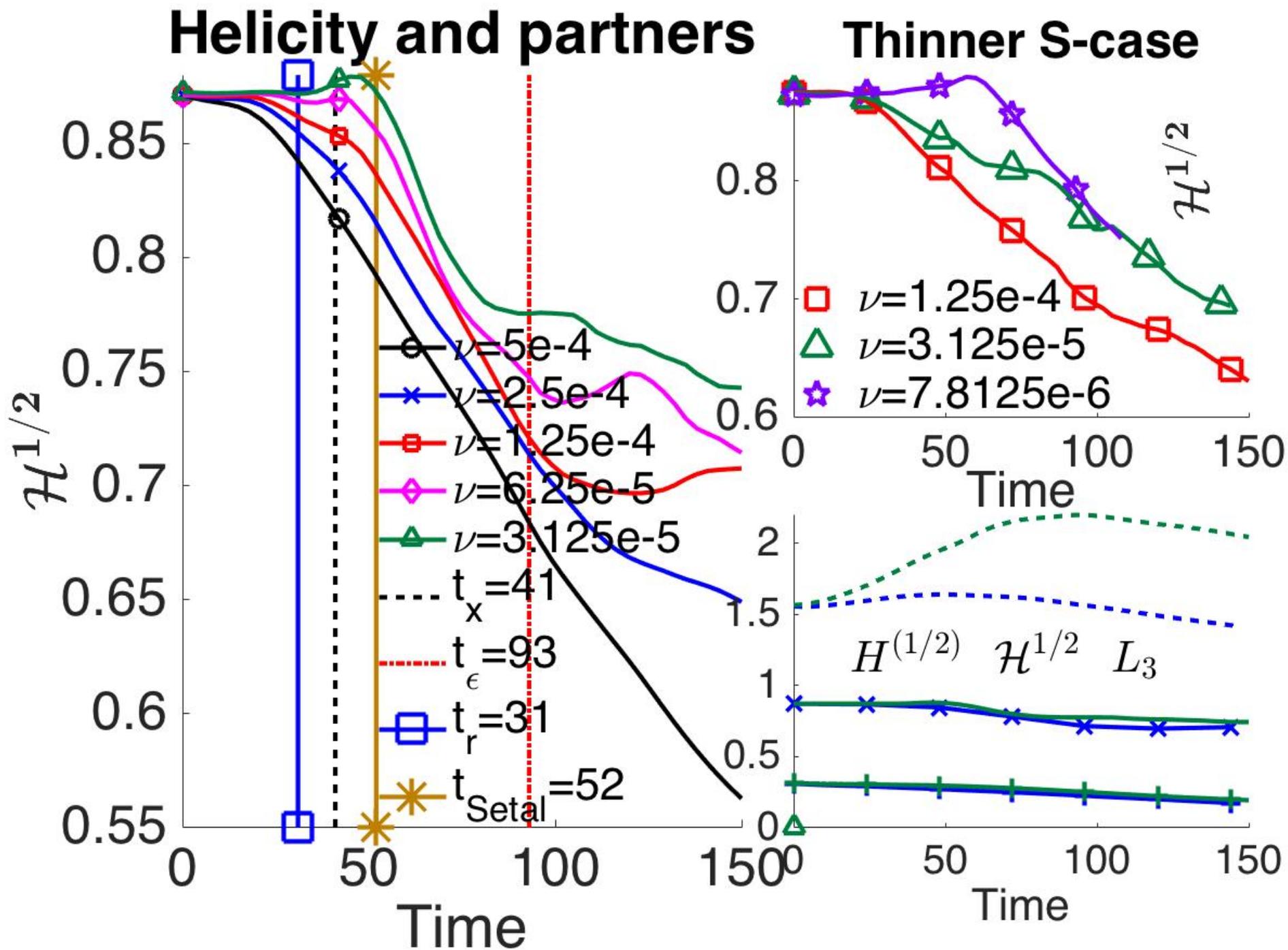
From the shape of all the $\nu^{1/2}Z(t)$ curves, the enstrophy growth seems to be some inverse power law. The Hölder inequality would suggest that $Z \sim 1/(T_c - t)^2$. To be linear, $1/\sqrt{\nu^{1/2}Z}$ is plotted. Since these are linear-linear, the lines can then be extrapolated to where they would cross $1/\sqrt{\nu^{1/2}Z} = 0$, giving effective singular times $T_c(\nu)$. Next, times with respect to t_x are rescaled using $\delta t_\nu = T_c(\nu) - t_x$, which gives the next figure where both the enstrophy and time have been rescaled. These curves collapse on top of each other up to t_x .

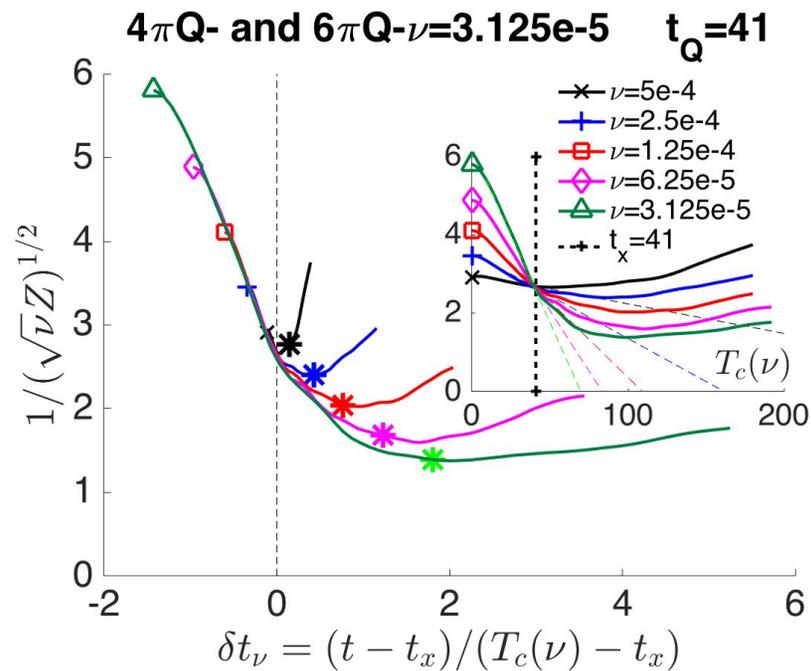
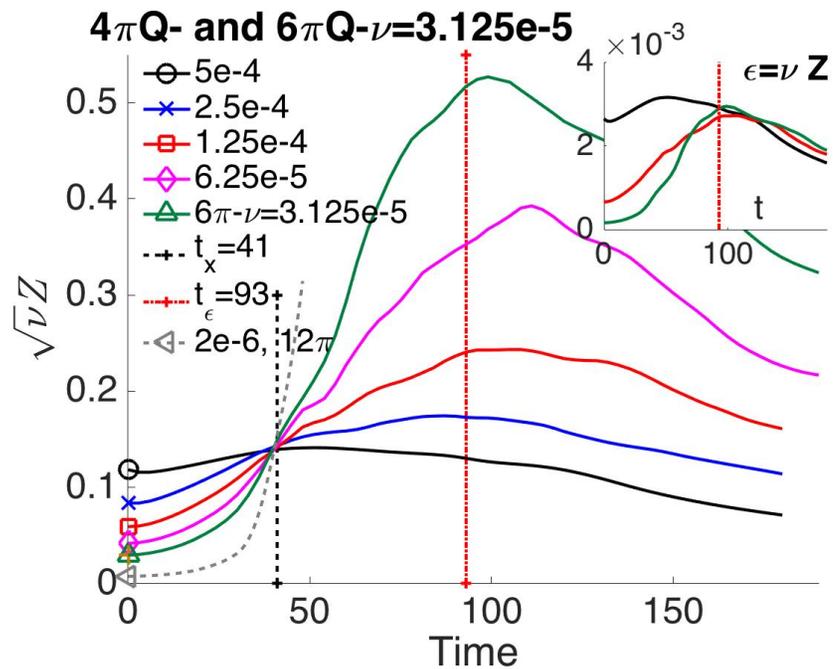
- Width of 150mm, $\tilde{r} = 75.5\text{mm}$ and circulation of $\Gamma = 20,000 \text{ mm}^2/\text{s}$.
- If the last time they have clear helicity results is $t'_f = 2.8$, then $t_f = 2.8 \times 75.5^2/2e4=800\text{ms}$.



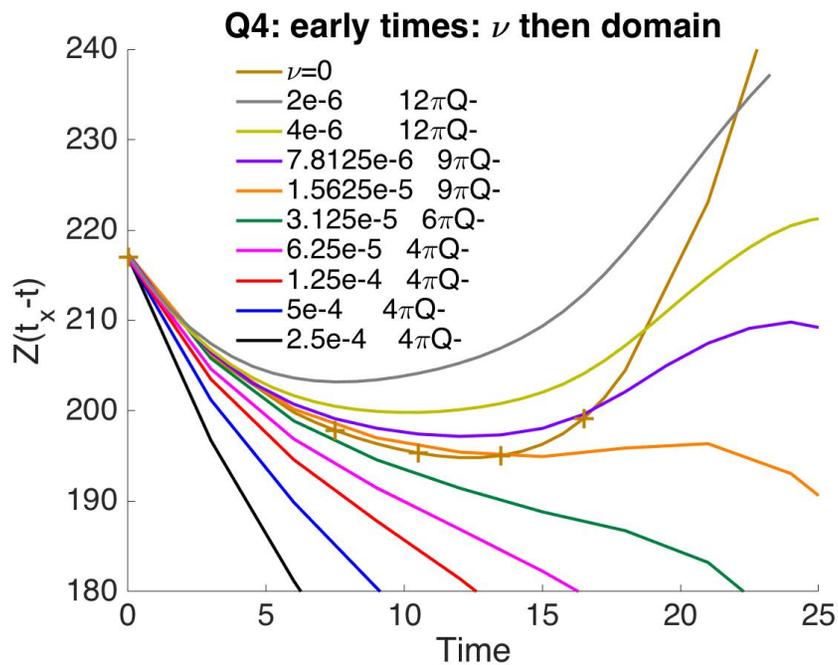
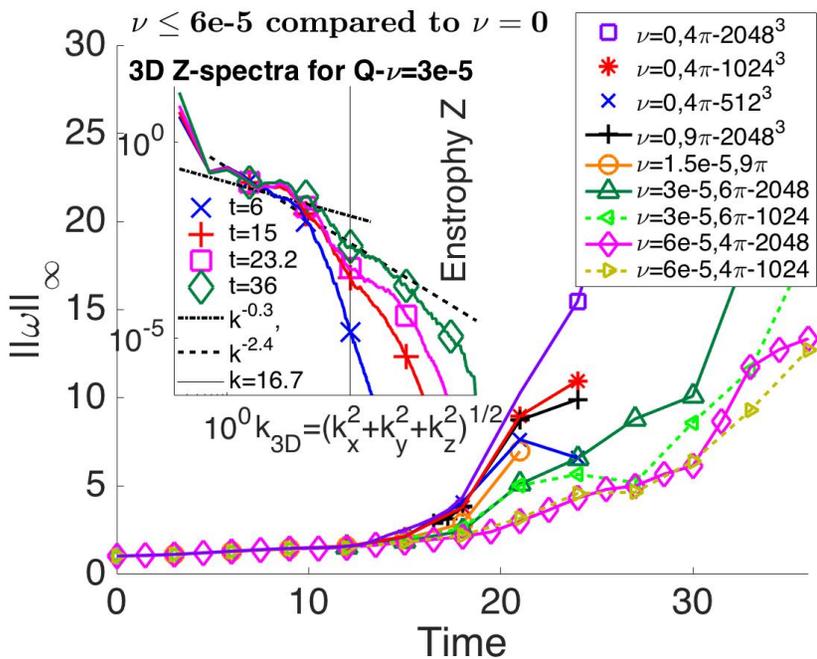
- Based upon the circulation, initial size of the trefoil and the separation between its two loops, the characteristic time for reconnection to start in the experiment would be at $t_r \approx 560\text{ms}$ and for the simulations at $t = 31$.
- This would indicate that the simulation time of $t \approx 53$ would correspond to when the experiment ended at $t \approx 800\text{ms}$.



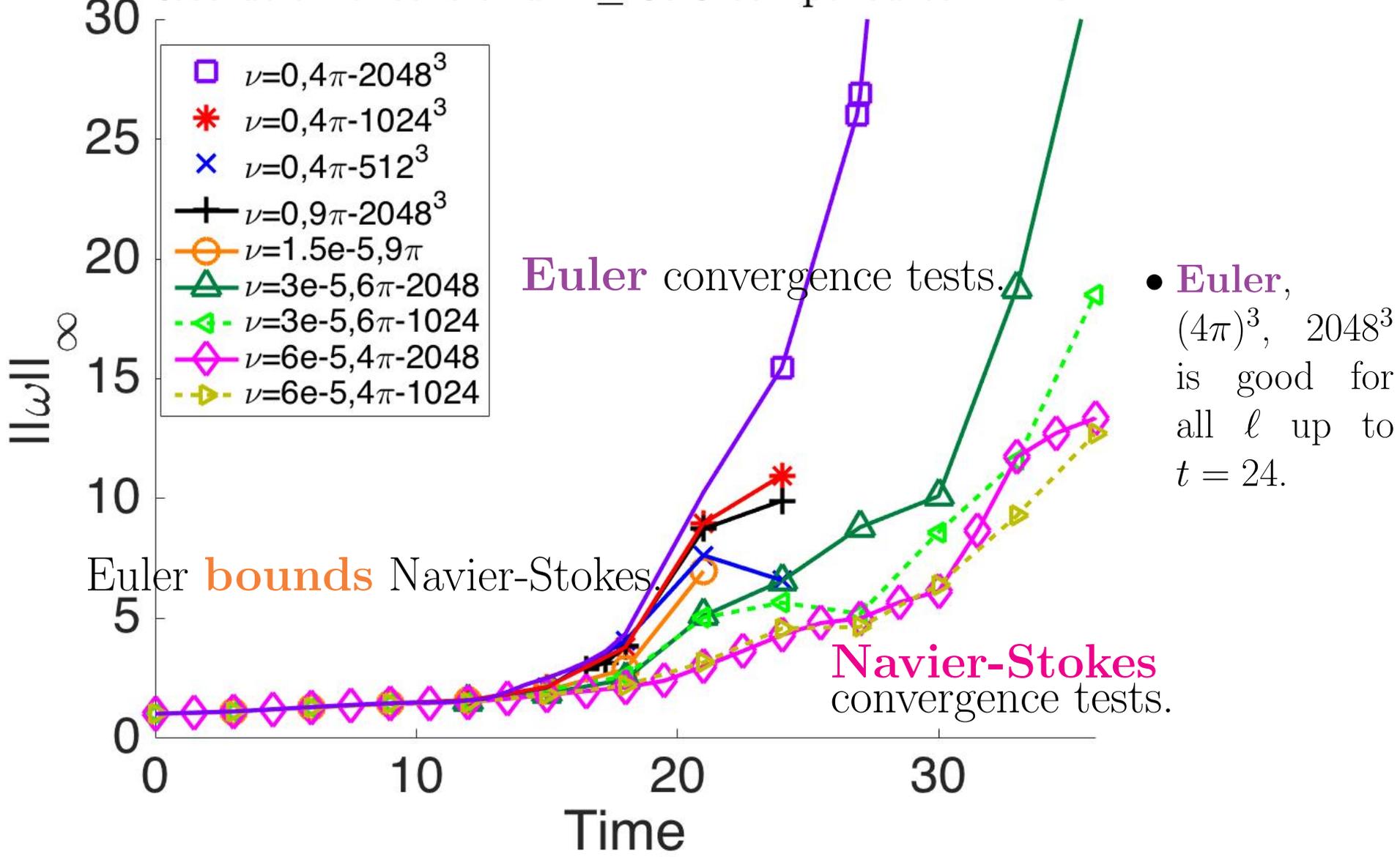




Address



Resolution checks and $\nu \leq 6e-5$ compared to $\nu = 0$



• **Euler**, $(4\pi)^3, 2048^3$ is good for all ℓ up to $t = 24$.

- **Navier-Stokes low ν** : $\nu = 6.25e-5, \ell = 4\pi, 2048^3$ is only calculation that is resolved in terms of $\|\omega\|_\infty$ and an exponential Z -spectrum tail for all times.
- $\nu = 1.5625e-5$, lowest ν , $\|\omega\|_\infty$ **bounded by Euler** $\|\omega\|_\infty$ at all times.

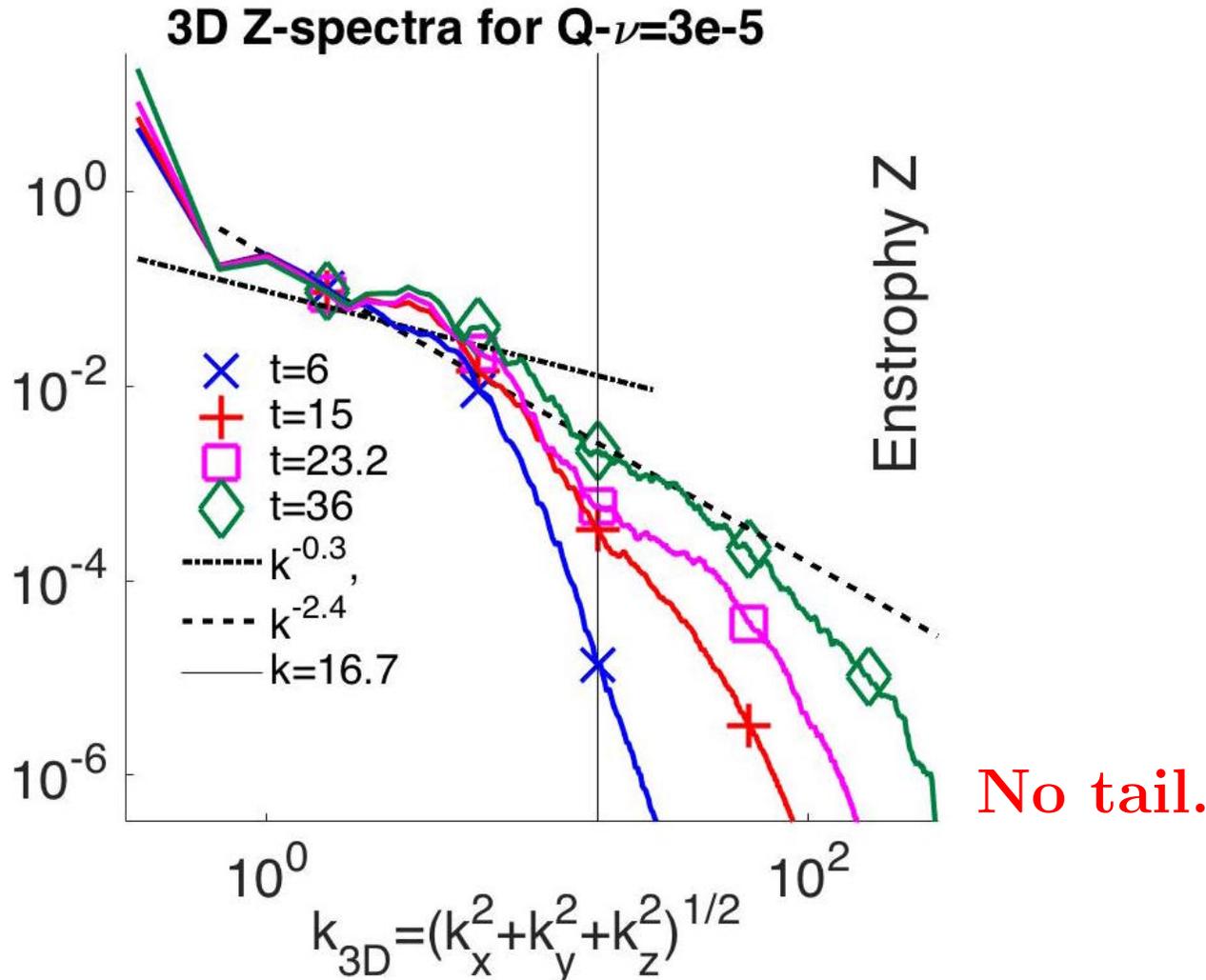
- Spectra at low wavenumbers ($k^{-0.3}$ regime) gives global enstrophy Z .

- Spectra at highest wavenumbers: No exponential tail, $\omega_m = \|\omega\|_\infty$ is under-resolved.

- Spectra at intermediate wavenumbers ($k^{-2.4}$ regime) connects.

If too short, the global enstrophy Z is unreliable.

Which calculations are resolved?



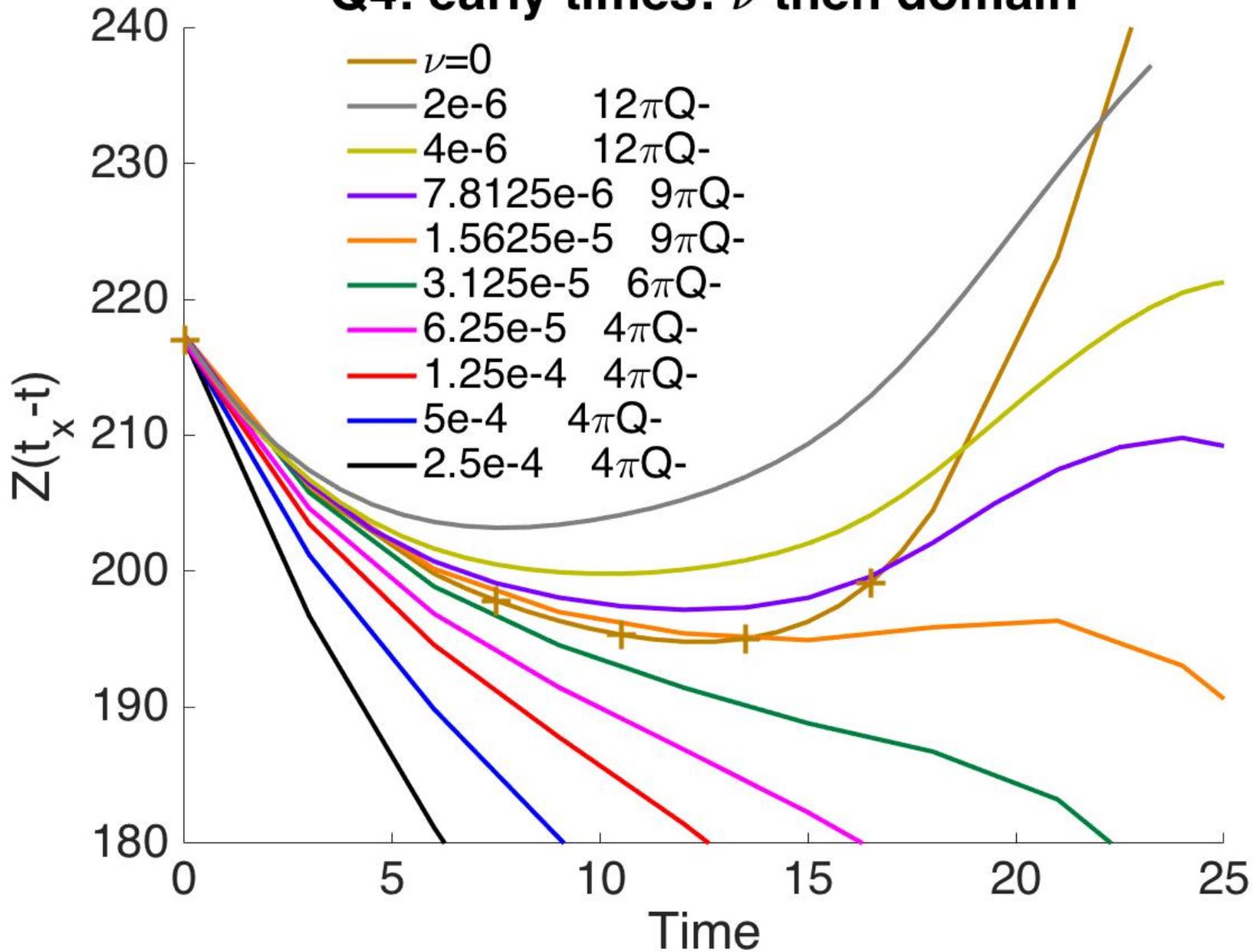
- $\nu = 6.25e-5$, $\ell = 4\pi$, 2048^3 is **only low ν calculation with an exponential Z -spectral tail for all times.**

- **Enstrophy Z** for $\nu = 6.25e-5$, $\ell = 4\pi$, 1024^3 and $\nu = 3.125e-5$, $\ell = 6\pi$, 2048^3 are **reliable for all times.**

- $\nu = 1.5625e-5$ and $\nu = 7.8e-6$ with $\ell = 9\pi$ should have **reliable Z for all times.**

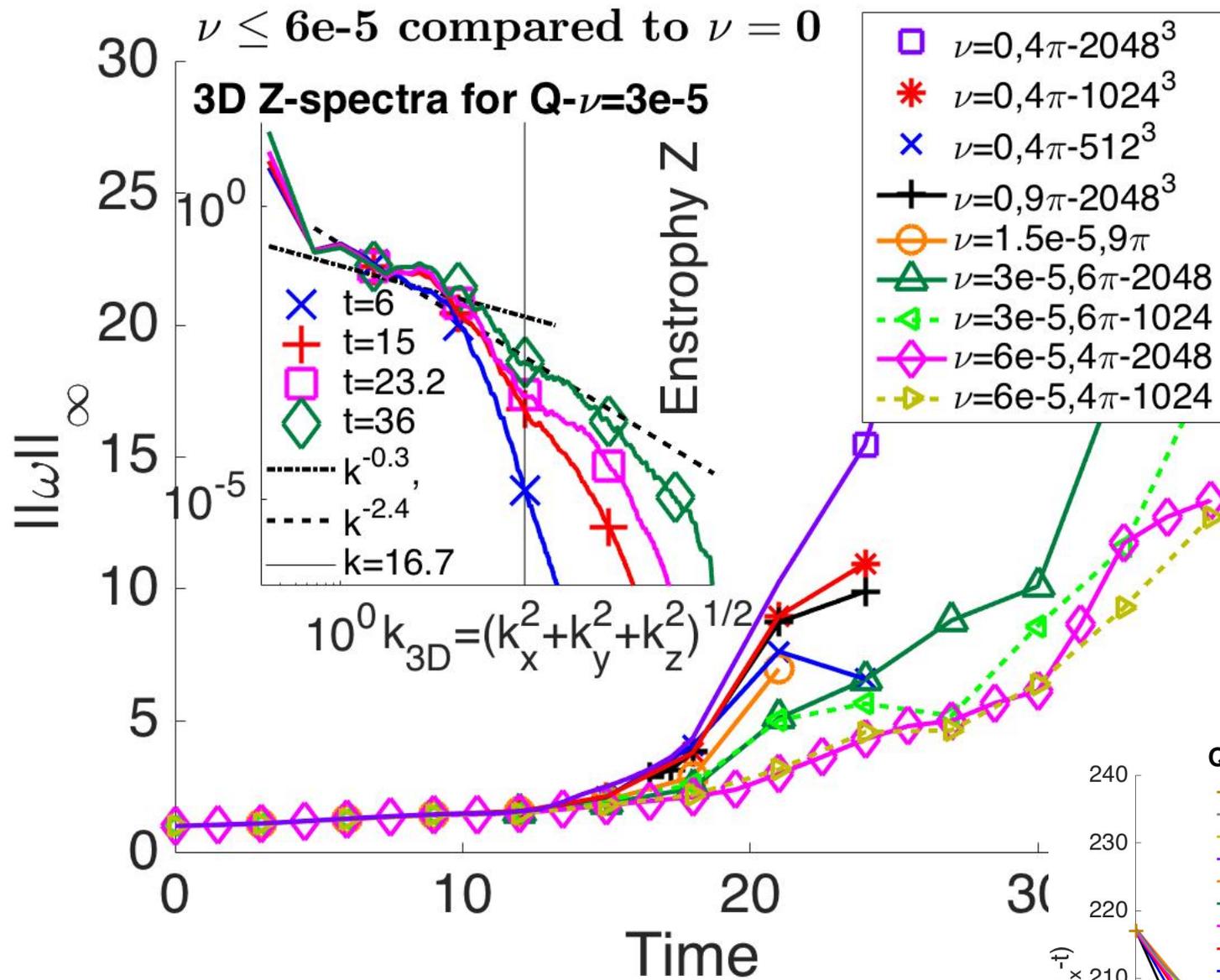
- Even $\nu = 4e-6$ and $\nu = 2e-6$ with $\ell = 9\pi$ are OK for short times.

Q4: early times: ν then domain

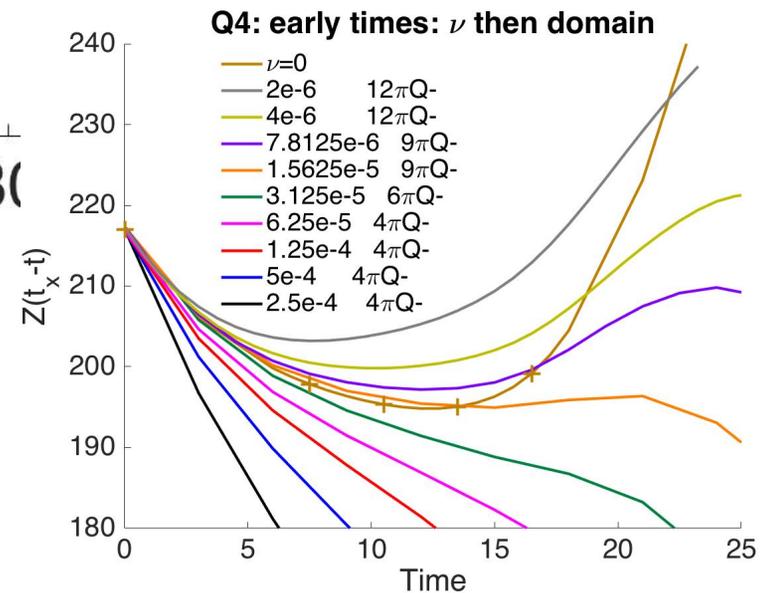


● Navier-Stokes $\nu < 3e-5$ exceed Euler Z for short period.

● Masmoudi (2007) seems to be irrelevant for enstrophy Z and dissipation $\epsilon = \nu Z$.

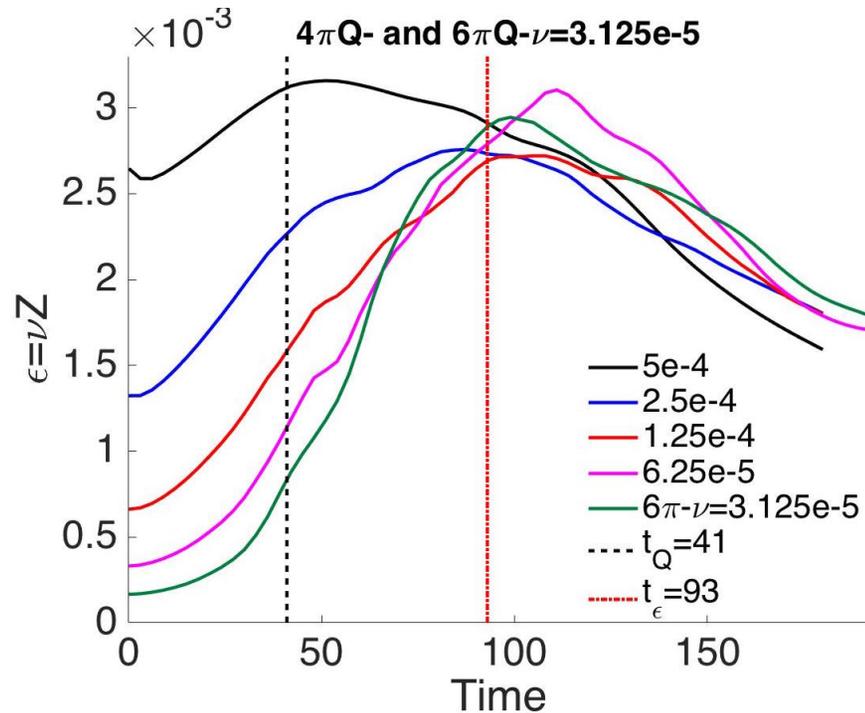
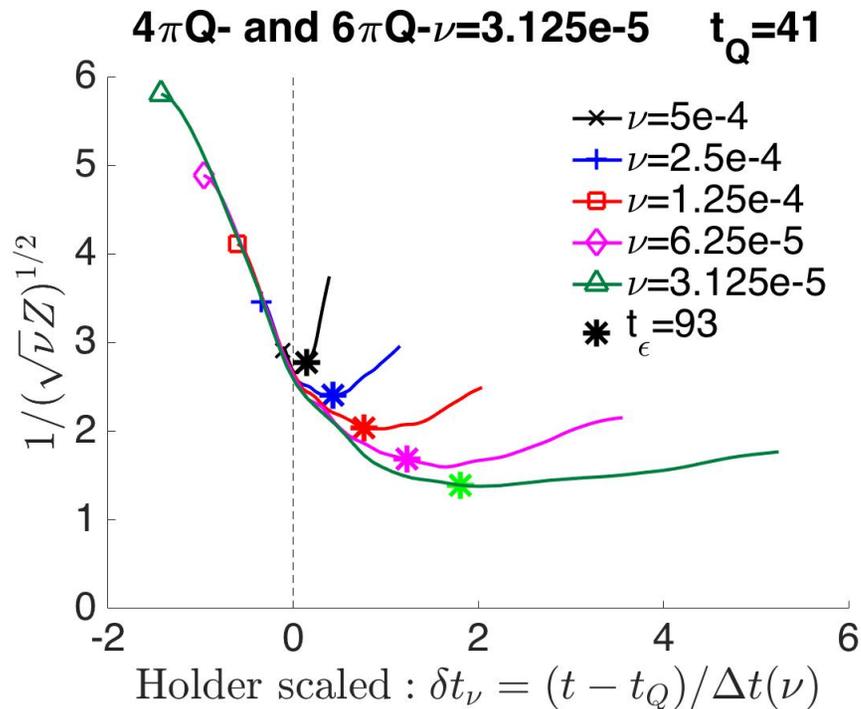


- Euler, $\ell = 4\pi, 2048^3$ is good for all ℓ up to $t = 24$.
- Euler ω_m exceed Navier-Stokes ω_m for all times and all ν .
- Enstrophy Z are reliable for all ν up to $t = 24$.
- $\nu < 3e-5$ exceed Euler Z for short period.



What is new? Dissipation Anomaly?

- Finite energy dissipation in a finite time: Finite ΔE .
- This is preceded by a scaling regime covering a change of 16 in ν .
- I get these regimes at roughly the same times for case Q, R, S, my thinner cases.



and

But do these results contradict the mathematics?

In particular Constantin (1986) and Masmoudi (2007). **No.**

What new physics makes this possible?

(3D graphics. In particular how negative helicity is ejected from the trefoil.)

Where next?

- These calculations while perhaps provocative, leave many questions unanswered.
For example: What is the mechanism that is responsible for the influence of the distant boundaries upon the inner dynamics of the trefoil? And is there over-looked, or unproven, mathematics that could still suppress the experimentally observed growth of the enstrophy at very, very small viscosities?
- **Backwards cascade of negative helicity?** Numerically, the idea that wavenumber modes of opposite sign can move to opposite extremes of the wavenumber spectra was introduced by Biferale & Kerr (1995) in the context of GOY-like shell models. Furthermore, the growth large-scale negative helicity ($\mathcal{H} < 0$) might be what compensates for the formation of additional small-scale positive helicity ($\mathcal{H} > 0$) that is needed to maintain the self-similar growth of $\sqrt{\nu}Z(t)$. If ℓ is too small, $\mathcal{H} < 0$ has no place to go and the growth of Z would be suppressed. But if ℓ increased, both the outer $\mathcal{H} < 0$ and the inner $\mathcal{H} > 0$ can grow together. How viscous terms can generate both positive and negative helicity in physical space during reconnection is demonstrated in the $t = 36$ figure.
- **New mathematics.** If this eventually rules out the $\sqrt{\nu}Z(t)$ behaviour reported here in the $\nu \rightarrow 0$ limit, how can the observed *dissipation anomaly* be explained? It could be that the suppression of enstrophy growth in the $\nu \rightarrow 0$ limit is logarithmically weak. Or in shell model language, there could be bursts in the cascade that cross several wavenumber shells, such that it would take only a few of these bursts to reach infinite wavenumber and give the appearance of finite dissipation in a finite time.

References

- Beale, JT, Kato, T, & Majda, A 1984 Remarks on the breakdown of smooth solutions of the 3-D Euler equations. *Commun. Math. Phys.* **94**, 61.
- Berger MA, Field GB (1984) The topological properties of magnetic helicity. *J Fluid Mech* 147:133148.
- Biferale, L. 2003 Shell models of energy cascade in turbulence. *Annu. Rev. Fluid Mech.* **35**, 441-468.
- Biferale, L., & Kerr, R.M. 1995 On the role of inviscid invariants in shell models of turbulence. *Phys. Rev. E* **52**, 6113–6122.
- Bustamante, MD, & Kerr, RM 2008 3D Euler about a 2D symmetry plane. *Physica D* **237**, 1912–1920.
- Calugareanu, G. 1959 L'intégral de Gauss et l'analyse des noeuds tridimensionnels. *Res. Math. Pures Appl.* **4**, 5–20.
- Constantin, P. 1986 Note on Loss of Regularity for Solutions of the 3D Incompressible Euler and Related Equations. *Commun. Math. Phys.* **104**, 311–326.
- Donzis, D., Gibbon, J.D., Gupta, A., Kerr, R. M., Pandit, R., & Vincenzi., D. 2013 Vorticity moments in four numerical simulations of the 3D Navier-Stokes equations. *J. Fluid Mech.* **732**, 316331.
- Escauriaza, L., Seregin, G., & Sverák, V. 2003 $L_{3,\infty}$ -solutions to the Navier-Stokes equations and backward uniqueness. *Russian Math. Surveys* **58**, 211–250. translation from *Uspekhi Mat. Nauk*, 58 (2003), 344 (Russian);

Gibbon, JD 2012 Conditional regularity of solutions of the 3D Navier-Stokes equations & implications for intermittency. *J. Math. Phys.* **53**, 115608.

Gibbon, JD Dynamics of scaled norms of vorticity for the three-dimensional Navier-Stokes and Euler equations. *Topological Fluid Dynamics II* 2012; This volume. arxiv:1212.0684.

Holm, D., & Kerr, R.M. 2007 Helicity in the formation of turbulence. *Phys Fluids* **19**, 025101.

Hou, TY 2008 Blow-up or no blow-up? The interplay between theory and numerics.. *Physica D* **237**, 1937–1944.

Kerr, RM 1993 Evidence for a singularity of the three-dimensional, incompressible Euler equations. *Phys. Fluids A* **5**, 1725–1746.

Kerr, RM 1996 Cover illustration: vortex structure of Euler collapse.. *Nonlinearity* **9**, 271–272.

Kerr, RM 2011 Vortex stretching as a mechanism for quantum kinetic energy decay. *Phys. Rev. Lett.* **106**, 224501.

Kerr, RM 2012 Dissipation and enstrophy statistics in turbulence: Are the simulations and mathematics converging?. *J. Fluid Mech.* **700**, 1–4.

Kerr, RM 2013 Swirling, turbulent vortex rings formed from a chain reaction of reconnection events. *Phys. Fluids* **25**, 065101.

- Kerr, R.M. 2013 Bounds for Euler from vorticity moments and line divergence.. *J. Fluid Mech.* **729**, R2.
- Kleckner, D., & Irvine, W.T.M. 2013 Creation and dynamics of knotted vortices. *Nature Phys.* **9**, 253–258.
- Masmoudi, N. 2007 Remarks about the inviscid limit of the Navier-Stokes system.. *Commun. Math. Phys.* **270**, 777–788.
- Moffatt, H.K. 1969 The degree of knottedness of tangled vortex lines. *J. Fluid Mech.* **35**, 117–129. doi:10.1017/S0022112069000991
- Moffatt, H.K. 2014 and singular structures in fluid dynamics. *Proc. Nat. Acad. Sci.* **111**, 3663–3670.
- Moffatt H, Ricca R (1992) Helicity and the Calugareanu invariant. *Proc R Soc Math Phys Eng Sci* 439(1906):411-429.
- Necas, J., Ruzicka, M., & Sverák, V. 1996 On Leray's self-similar solutions of the Navier-Stokes equations. *Acta Math.* **176**, 283–294.
- Pohl, W.F. 1968 The self-linking number of closed space curve. *J. Math. Mech* **17**, 975–985.
- Proment, D., Onorato, M. & Barenghi, C. Vortex knots in a Bose-Einstein condensate. *Physical Review E* 85, 1–8 (2012).
- Ricca, R. L. & Berger, M. A. Topological ideas and fluid mechanics. *Physics Today* 49(12), 28–34 (1996).

Renzo L. Ricca, David C. Samuels and Carlo F. Barenghi, Evolution of vortex knots. *J. of Fluid Mech.* 391, 29–44 (1999). DOI: <http://dx.doi.org/10.1017/S0022112099005224>

Scheeler, Martin W., Kleckner, Dustin, Proment, Davide, Kindlmann, Gordon L., & Irvine, William T. M. 2014 Helicity conservation in topology-changing reconnections: the flow of linking and coiling across scales. *arXiv* **1404.6513v2**, [physics.flu-dyn].

Scheeler, Martin W., Kleckner, Dustin, Proment, Davide, Kindlmann, Gordon L., & Irvine, William T. M. 2014 Helicity conservation by flow across scales in reconnecting vortex links and knots.. *Proc. Nat. Acac. Sci.* **111**, 15350–15355.

Scheeler, Martin W., Kleckner, Dustin, Proment, Davide, Kindlmann, Gordon L., & Irvine, William T. M. Supporting information for Scheeler *et al.* (2014). www.pnas.org/cgi/content/short/1407232111.

Seregin, G.A. 2011 Necessary conditions of a potential blow-up for Navier-Stokes equations. *J. Math. Sci.* **178**, 345–352.

Waleffe, F. 1992 The nature of triad interactions in homogeneous turbulence. *Phys. Fluids* **5**, 677–685.

Yeung, PK, Donzis, DA, & Sreenivasan, KR 2012 Dissipation, enstrophy and pressure statistics in turbulence simulations at high Reynolds numbers. *J. Fluid Mech.* **700**, 5–15.