# Differential Geometry

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Chapter 1 - Review of basic notions on smooth manifolds

1.1 First Principles

Hausdorff
A topological space $M$ is called Hausdorff if for all $x, y \in M$, there exist open sets $U, V$ such that $x \in U$ and $y \in V$ and $U \cap V = \emptyset$.

Second Countable
A topological space $M$ is second countable if there exists a countable basis for the topology of $M$.

Locally Euclidean of dimension $n$
A topological space $M$ is locally Euclidean of dimension $n$ if for every point $x \in M$, there exists an open set $U \subset M$ and an open set $W \subset \mathbb{R}^n$ so that $U$ and $W$ are homeomorphic.

Definition 1.1
A topological manifold of dimension $n$ is a topological space that is Hausdorff, second countable and locally Euclidean of dimension $n$.

Remark
1. Unless otherwise specified, all manifolds are assumed to be connected.
2. Some people assume “paracompactness” which is used to prove the existence of a partition of unity.

Definition 1.2
A smooth atlas $\mathcal{A}$ of a topological space $M$ is given by

a. An open covering $\{U_i\}_{i \in I}$ where $U_i \subset M$ open and $M = \bigcup_{i \in I} U_i$

b. A family $\{\phi_i : U_i \to W_i\}_{i \in I}$ of homeomorphisms $\phi_i$ onto open subsets $W_i \subset \mathbb{R}^n$

So that if $U_i \cap U_j \neq \emptyset$ then the map $\phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$ is a diffeomorphism.

Definition 1.3
If $U_i \cap U_j \neq \emptyset$ then the diffeomorphism $\phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$ is known as the transition map.

Definition 1.4
A smooth structure on a Hausdorff topological space is an equivalence class of atlases, with two atlases $\mathcal{A}$ and $\mathcal{B}$ being equivalent if, for $(U_i, \phi_i) \in \mathcal{A}$ and $(V_j, \psi_j) \in \mathcal{B}$ with $U_i \cap V_j \neq \emptyset$ then the transition map $\phi_i(U_i \cap V_j) \to \psi_j(U_i \cap V_j)$ is a diffeomorphism (as a map between open sets of $\mathbb{R}^n$).

It is easy to check that this is an equivalence relation, since the inverse of a diffeomorphism and composition of two diffeomorphisms are diffeomorphisms.
Definition 1.5
A smooth manifold $M$ of dimension $n$ is a topological manifold of dimension $n$ together with a smooth structure.

Definition 1.6
Let $M, N$ be two manifolds of dimension $m, n$ respectively. A map $F: M \to N$ is called smooth at $p \in M$ if there exist charts $(U, \phi), (V, \psi)$ with $p \in U \subset M$ and $F(p) \in V \subset N$ with $F(U) \subset V$ and the composition $\psi \circ F \circ \phi^{-1}: \phi(U) \to \psi(V)$ is smooth (as a map between open sets of $\mathbb{R}^n$). $F$ is called smooth if it is smooth at every $p \in M$.

In particular:
- A smooth map $f: M \to \mathbb{R}$ is a smooth function.
- A smooth map $c: I \subset \mathbb{R} \to M$ from an open interval $I \subset \mathbb{R}$ is called a smooth curve in $M$.

Definition 1.7
A map $F: M \to N$ is called a diffeomorphism if it is smooth, bijective and its inverse $F^{-1}: N \to M$ is also smooth.

Definition 1.8
A map $F$ is called an immersion (respectively submersion) if $F$ is injective (respectively surjective) and its differential is injective (respectively surjective).

Definition 1.9
A map $F$ is called an embedding if $F$ is an immersion and homeomorphic onto its image.

Definition 1.10
If $F: M \to N$ is an immersion then $F(M)$ is an immersed submanifold of $N$.

Definition 1.11
If $F: M \to N$ is an embedding, then $F(M)$ is an embedded submanifold (or a regular submanifold) of $N$; equivalently $F(M)$ is a submanifold of $N$ using the subspace topology.

1.2 Tangent Space and Vector Fields

Notation
- Let $C^\infty(M, N) := \{\text{smooth maps from } M \text{ to } N\}$ and let $C^\infty(M) := \{\text{smooth functions on } M\}$
- Given a point $p \in M$, denote $C^\infty(p) := \{\text{functions defined on some open neighbourhood of } p \text{ and smooth at } p\}$

Definition 1.12
a. The tangent vector $X$ to the curve $c: (-\epsilon, \epsilon) \to M$ at $t = 0$ is the map $c'(0): C^\infty(c(0)) \to \mathbb{R}$ given by the formula:

$$X(f) = c'(0)(f) := \left. \frac{d(f \circ c)}{dt} \right|_{t=0} \quad \forall f \in C^\infty(c(0))$$
b. A tangent vector $X$ at $p \in M$ is the tangent vector at $t = 0$ of some curve $\alpha: (-\epsilon, \epsilon) \to M$ with $\alpha(0) = p$. That is, $X = \alpha'(0): C^\infty(p) \to \mathbb{R}$

Remark
A tangent vector at $p$ is known as a linear functional defined on $C^\infty(p)$ which satisfies the Leibniz property:

$$X(fg) = X(f)g + fX(g), \quad \forall f, g \in C^\infty(p)$$

**Definition 1.13**
The tangent space at $p \in M$, denoted by $T_p M$ is the set of all tangent vectors at $p$.

Remark
1. $T_p M$ admits a structure of a real vector space.
2. Given a chart $(U, \phi)$ with $p \in U$ and coordinates $(x^1, ..., x^n)$ the coordinate vectors $\frac{d}{dx^1}\big|_p, ..., \frac{d}{dx^n}\big|_p$ form a basis of $T_p M$. Note that there is no standard choice of basis as the coordinates are dependent upon which chart you are using.

In other words, any tangent vector $X \in T_p M$ can be written as $X = \sum_{i=1}^n a^i \frac{d}{dx^i}\big|_p$

**Proposition (Change of variables formula)**
Given two coordinate systems $(x^1, ..., x^n)$ and $(y^1, ..., y^n)$ and a vector field $X$ then if

$$X = \sum_{i=1}^n a^i \frac{d}{dx^i}\big|_p = \sum_{i=1}^n b^i \frac{d}{dy^i}\big|_p$$

Then $b^i = \sum_{j=1}^n a^j \frac{dy^i}{dx} |_p$ for all $i \in \{1, ..., n\}$.

**Proof**
Pick $k \in \{1, ..., n\}$ and apply $X$ to the function $y^k \in C^\infty(p)$. Then

$$X(y^k) = \sum_{i=1}^n a^i \frac{d}{dx^i}\big|_p (y^k) = \sum_{i=1}^n a^i \frac{dy^k}{dx^i}$$

$$X(y^k) = \sum_{i=1}^n b^i \frac{d}{dy^i}\big|_p (y^k) = b^k$$

Hence $b^k = \sum_{i=1}^n a^i \frac{dy^k}{dx^i}$.

**Einstein Summation convention**
The vector $X = \sum_{i=1}^n a^i \frac{d}{dx^i}\big|_p$ in Einstein summation convention is written instead as

$$X = a^i \frac{d}{dx^i}\big|_p$$

**The Differential**
Given $F \in C^\infty(M, N)$ and $p \in M$ and $X \in T_p M$, choose a curve $\alpha: (-\epsilon, \epsilon) \to M$ with $\alpha(0) = p$ and $\alpha'(0) = X$ (this is possible due to the theorem about existence of solutions of linear first order ODEs). Then consider the map $F_p: T_p M \to T_{F(p)} N$ mapping $X \to F_p(X) := (F \circ \alpha)'(0)$. This is a linear map between two vector spaces and it is independent of the choice of $\alpha$. 
**Definition 1.14**
The linear map $F_p$ defined above is called the derivative or differential of $F$ at $p$, while the image $F_p(X)$ is called the push forward of $X$ at $p \in M$.

**Remark**
Sometimes $F_p$ is denoted as $dF_p$.

**Definition 1.15**
The tangent bundle of a smooth $n$ dimensional manifold $M$, denoted $TM$ is the disjoint union of the tangent spaces at $p \in M$. That is:

$$TM := \bigsqcup_{p \in M} T_p M$$

The elements of $TM$ will be represented by either $v \in T_p M$ or the pair $(p, v)$.

**Remark:**
$TM$ admits a smooth structure which makes $TM$ itself a smooth manifold of dimension $2n$.

**Definition 1.16**
Given a smooth manifold $M$, a vector field $V$ is a map $V: M \to TM$ mapping $p \to V_p \equiv V(p)$, and $V$ is called smooth if it is smooth as a map from $M$ to $TM$.

**Notation**

$\mathfrak{X}(M) := \{\text{smooth vector fields on } M\}$

$\mathfrak{X}(M)$ is an $\mathbb{R}$-vector space: for $Y, Z \in \mathfrak{X}(M)$, $p \in M$ and $a, b \in \mathbb{R}$, $(aY + bZ)_p = aV_p + bZ_p$ and for $f \in C^\infty(M)$, $Y \in \mathfrak{X}(M)$ define $fY: M \to TM$ mapping $p \to (fY)_p := f(p)V_p$

**Definition 1.17**
For $X, Y \in \mathfrak{X}(M)$, the Lie bracket $[X, Y] := XY - YX$.

Observe that for $X, Y, Z \in \mathfrak{X}(M)$, then the Lie bracket satisfies the Jacobi identity

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$$

**Proof**
Take $f \in C^\infty(M)$ then

$$[[X, Y], Z](f) + [[Y, Z], X](f) + [[Z, X], Y](f)$$

$$= [XY - YX, Z](f) + [YZ - ZY, X](f) + [ZX - XZ, Y](f)$$

$$= (XYZ - YXZ - ZXY + ZYX)(f) + (YZX - ZYX - XYZ + XZY)(f)$$

$$+ (ZXY - ZYX - YZX + YXZ)(f) = 0$$

**1.3 Cotangent Space, Vector Bundles and Tensor Fields**

**Definition 1.18**
Let $M$ be a smooth $n$-manifold, and $p \in M$. We define cotangent space at $p$, denoted by $T^*_p M$ to be the dual space of the tangent space at $p$: $T^*_p M \equiv (T_p M)^* = \{f: T_p M \to \mathbb{R}: f\text{ smooth}\}$.

Elements of $T^*_p M$ are called cotangent vectors or tangent covectors at $p$.

**Remark:**
1. For $f: M \to \mathbb{R}$ smooth, the composition $T_pM \xrightarrow{f_p} T_{f(p)}\mathbb{R} \cong \mathbb{R}$ is called $df_p$ and referred to as the differential of $f$. Note that $df_p \in T^*_pM$, so it is a cotangent vector at $p$.

2. For a chart $(U, \phi, x^i)$ of $M$ and $p \in U$, then $\{dx^i\}_{i=1}^n$ is a basis of $T^*_pM$. In fact $\{dx^i_p\}$ is the dual basis of $\left\{\frac{d}{dx^i}\right\}_{i=1}^n$.

**Definition 1.19**

The disjoint union $T^*M := \bigsqcup_{p \in M} T^*_pM$ is called the cotangent bundle of $M$.

**Remark**

$T^*M$ is a smooth manifold of dimension $2n$.

**Review of Tensor Product of Vector Spaces**

**Definition 1.20**

The tensor product of finitely many vector spaces $V_1, ..., V_k$ is defined on the set $V_1 \otimes ... \otimes V_k := \{T: V_1 \times ... \times V_k \to \mathbb{R}: T$ is multi-linear$\}$.

**Definition 1.21**

The elements in the tensor product $V^r_s := V \otimes ... \otimes V \otimes V^* \otimes ... \otimes V^*$ are called $(r, s)$-tensors or $r$-contravariant, $s$-covariant tensors.

**Remark**

$V^r_s$ is a vector space.

**Definition 1.22**

Given $T \in V^r_s$, $\bar{T} \in V^r_s$ then the tensor product $T \otimes \bar{T}$ is an element of $V^r_{s+\delta}$ given by $(T \otimes \bar{T})(L_1, ..., L_r, v_1, ..., v_{s+\delta}) := T(L_1, ..., L_r, v_1, ..., v_s) \otimes \bar{T}(L_{r+1}, ..., L_{r+\delta}, v_{s+1}, ..., v_{s+\delta})$

**Remark**

The Tensor Product is bilinear and associative however it is in general not commutative; that is $T_1 \otimes T_2 \neq T_2 \otimes T_1$ in general.

**Definition 1.23**

$T \in V^r_s$ is called reducible if it can be written in the form $T = V_1 \otimes ... \otimes V_r \otimes L^1 \otimes ... \otimes L^s$ for $V_i \in V, L^j \in V^*$ for $1 \leq i \leq r, 1 \leq j \leq s$.

**Definition 1.24**

Choose two indices $(i, j)$ where $1 \leq i \leq r, 1 \leq j \leq s$. For any reducible tensor $T = V_1 \otimes ... \otimes V_r \otimes L^1 \otimes ... \otimes L^s$. Let $C_i^j(T) := L^j(V_i) \otimes V_1 \otimes ... \otimes V_r \otimes L^1 \otimes ... \otimes D \otimes ... \otimes L^s$. Then $C_i^j(T) \in V^r_{s-1}$. We extend this linearly to get a linear map $C_i^j: V^r \to V^r_{s-1}$ which is called tensor-contraction.

**Example**

Let $V$ be a vector space with basis $\{e_i\}$ and dual basis $\{\theta^j\}$. Let $T = \sum_{i,j} T^j_i e_i \otimes \theta^j \in V \otimes V^*$ then $C_i^j(T) = \sum_{i,j} T^j_i C_i^j(e_i \otimes \theta^j) = \sum_{i,j} T^j_i \theta^j(e_i) = \sum_{i=1}^n T^j_i$. 

Remark
From the example above we deduce that $C_1^1: V_1^1 = (V \otimes V^*) \to V_0^0(= \mathbb{R})$ is the trace operation $T_r: \text{End}(V) \to \mathbb{R}$

Definition 1.25
An antisymmetric (or alternating $(0,k)$-tensor) $T \in V_k^0$ is called a $k$-form on $V$, and the space of all $k$-forms on $V$ is denoted $\Lambda^k V^* = \{ T \in V_k^0 : T \text{ alternating} \}$

Definition 1.26
A (smooth real) vector bundle of rank $k$, denoted $(E, M, \pi)$ is a smooth manifold $E$ of dimension $n + k$ (the total space) a smooth manifold $M$ of dimension $n$ (the base manifold) and a smooth surjective map $\pi: E \to M$ (projection map) with the following properties:

1. There exists an open cover $\{V_{\alpha}\}_{\alpha \in I}$ of $M$ and diffeomorphisms $\psi_\alpha: \pi^{-1}(V_{\alpha}) \to V_{\alpha} \times \mathbb{R}^k$

2. For any point $p \in M$, $\psi_\alpha(\pi^{-1}(p)) = \{p\} \times \mathbb{R}^k \cong \mathbb{R}^k$ and we get a commutative diagram (in this case $\pi_1: V_{\alpha} \times \mathbb{R}^k \to V_{\alpha}$ is projection onto the first component).

3. Whenever $V_\alpha \cap V_\beta \neq \emptyset$ the diffeomorphism $\psi_\alpha \circ \psi_\beta^{-1}: (V_\alpha \cap V_\beta) \times \mathbb{R}^k \to (V_\alpha \cap V_\beta) \times \mathbb{R}^k$ takes the form $\psi_\alpha \circ \psi_\beta^{-1}(p, a) = (p, A_{\alpha\beta}(p)(a))$, $a \in \mathbb{R}^k$ where $A_{\alpha\beta}: V_\alpha \cap V_\beta \to GL(k, \mathbb{R})$ is smooth. For each $p \in M$, the set $E_p := \pi^{-1}(p)$ is called the fibre over $p$. Observe that $\psi_\alpha(E_p) = \{p\} \times \mathbb{R}^k \cong \mathbb{R}^k$

The pairs $(V_\alpha, \psi_\alpha)$ are called local trivialisations. The maps $A_{\alpha\beta}: V_\alpha \cap V_\beta \to GL(k, \mathbb{R})$ are called transition maps.

Examples:

1. The trivial bundle $(M \times \mathbb{R}^k, M, \pi)$ where $\pi$ is projection onto the first component.
2. The tangent bundle $(TM, M, \pi)$
3. The infinite Möbius band:
   Take an infinite strip, and identify the two edges with each other, giving one a twist.
   The infinite Möbius band is a non-orientable manifold and a non-trivial vector bundle over $S^1$ with fibre $\mathbb{R}$.

Definition 1.27
A smooth section of a vector bundle $(E, M, \pi)$ is a map $s: M \to E$ so that $\pi \circ s = id_M$ that is $s(p) \in E_p$ for all $p \in M$. $s$ is called a smooth section if it is smooth as a map from $M$ to $E$.
Denote $\Gamma(E) = \{ \text{smooth sections of } (E, M, \pi) \}$

Example
Take $E = TM$ and then $\Gamma(TM) = \mathfrak{X}(M)$
Suppose a rank $k$ vector bundle admits $k$ sections which are everywhere linearly independent, then we call such a collection of sections a frame (or global frame or frame field). When these $k$ sections are only locally defined, we call the collection a local frame.

**Remark**

A Global frame does not always exist, but a local frame always exists (as a manifold is locally Euclidean).

**Construction of a local frame using a local trivialisation**

Let $\{e_1, ..., e_k\}$ be the standard basis of $\mathbb{R}^k$ when within an open set $V_\alpha \subset M$, a local frame is given by $\psi_\alpha^{-1}(p, e_i)$ for $p \in V_\alpha$ and $i = 1, ..., k$.

**Examples**

Let $E = TM$ and let $(U, \phi, x^i)$ be a chart. A local frame is given by $\{ \frac{d}{d x^i}\}_{i=1}^n$.

Let $E = T^*M$ and let $(U, \phi, x^i)$ be a chart. A local frame is given by $\{dx^i\}_{i=1}^n$.

**Vector Bundle Construction Lemma (Proof non-examinable)**

**Lemma 1.28**

Let $M$ be an $n$-dimensional smooth manifold and $\{V_\alpha\}_{\alpha \in I}$ an open cover of $M$. Suppose for each $\alpha, \beta \in I$ that there exists a smooth map $A_{\alpha \beta}: V_\alpha \cap V_\beta \to GL(k, \mathbb{R})$ satisfying the following condition:

$$\forall \alpha, \beta, \gamma \in I \quad \forall p \in V_\alpha \cap V_\beta, \quad A_{\alpha \beta} A_{\beta \gamma}(p) = A_{\alpha \gamma}(p)$$

Then there exists a smooth real vector bundle $(E, M, \pi)$ of rank $k$ with local trivialisations $\psi_\alpha^{-1}(V_\alpha) \to V_\alpha \times \mathbb{R}^k$. The transition maps are given by $A_{\alpha \beta}$; the change of basis matrix between the coordinate system induced by $V_\alpha$ and the coordinate system induced from $V_\beta$.

**Idea of Proof**

Consider $\left( \bigsqcup_{\alpha \in I} V_\alpha \times \mathbb{R}^k \right)/\sim$, where $\sim$ is defined as follows: For $p \in V_\alpha \cap V_\beta$ and $a \in \mathbb{R}^k$ then $(p, a) \sim (p, A_{\alpha \beta}(p)a) \in V_\alpha \times \mathbb{R}^k$ for all transition matrices $A_{\alpha \beta}(p)$. Then we prove that $\left( \bigsqcup_{\alpha \in I} V_\alpha \times \mathbb{R}^k \right)/\sim$ is a vector bundle. The Details are in [G-H-L].

**Example**

$TM$ is a vector bundle. We use the Lemma.

Define $E_p = T_pM$ and let $E := \bigsqcup_{p \in M} T_pM$. Let $(V_\alpha, \phi_\alpha)$ and $(V_\beta, \phi_\beta)$ be two charts for $M$. For $p \in V_\alpha \cap V_\beta$ define $A_{\alpha \beta}(p) = \phi_\alpha \circ \phi^{-1}_\beta(p)$ which is the transition map in $\mathbb{R}^n$ between the coordinate system of $V_\beta$ and $V_\alpha$. Being a linear map this is smooth and clearly for all $\alpha, \beta, \gamma \in I$, $A_{\alpha \gamma}(p) = \phi_\alpha \circ \phi^{-1}_\beta(p) = \phi_\alpha \circ (\phi^{-1}_\beta \circ \phi_\beta) \circ \phi^{-1}_\gamma(p) = A_{\alpha \beta} A_{\beta \gamma}(p)$. Hence by the lemma there exists a smooth real vector bundle $(TM, M, \pi)$ of rank $n$ with local trivialisation $\psi_\alpha^{-1}(V_\alpha) \to V_\alpha \times \mathbb{R}^n$ which is simply $\psi_\alpha = id_{V_\alpha \times \mathbb{R}^n}$.

**1.4 Bundle Maps and isomorphisms**

**Definition 1.29**

Suppose $(E, M, \pi)$ and $(\tilde{E}, \tilde{M}, \tilde{\pi})$ are two vector bundles. A smooth map $F: E \to \tilde{E}$ is called a smooth bundle map from $(E, M, \pi)$ to $(\tilde{E}, \tilde{M}, \tilde{\pi})$ if
1. there exists a smooth map \( f: M \to \tilde{M} \) such that the following diagram commutes

\[
\begin{array}{ccc}
E & \xrightarrow{F} & \tilde{E} \\
\downarrow \pi & & \downarrow \tilde{\pi} \\
M & \xrightarrow{f} & \tilde{M}
\end{array}
\]

That is, \( \tilde{\pi}(F(q)) = f(\pi(q)) \) for all \( q \in E \)

2. \( F \) induces a linear map from \( E_p \) to \( \tilde{E}_{f(p)} \) for any \( p \in M \)

Definition 1.30
Two vector bundles \((E, M, \pi)\) and \((\tilde{E}, \tilde{M}, \tilde{\pi})\) are called smoothly isomorphic if there exists a smooth bundle map between them which is a diffeomorphism.

Remarks
1. If two vector bundles are isomorphic they are viewed as “the same”
2. A vector bundle which is isomorphic to the trivial bundle \((M \times \mathbb{R}^k)\) is called trivial.

Claim
A vector bundle is trivial if and only if it admits a global frame.

Example
\((0,2\pi) \times \mathbb{R}\) is a trivial vector bundle. By joining the two ends there are two possibilities either the result is \( S^1 \times \mathbb{R} \) if there is no twisting, while if there is twisting we get the infinite Mobius band.

Construction of new vector bundles from old

Dual Bundle
Take a vector bundle \((E, M, \pi)\) where \( E = \bigsqcup_{p \in M} E_p \). Replace \( E_p \) with its dual \( E^*_p \) and consider \( E^* = \bigsqcup_{p \in M} E^*_p \).

Let \( V_\alpha, \psi_\alpha, A_{\alpha\beta} \) by as in Definition 1.26. Then the transition maps for the dual bundle \( E^* \) are denoted \( (A^\text{dual})_{\alpha\beta} = (A_{\alpha\beta}^{-1})^T \). Observe that \((A^\text{dual})_{\alpha\beta} (A^\text{dual})_{\beta\gamma} = (A^{-1}_{\alpha\beta})^T (A^{-1}_{\beta\gamma})^T = (A_{\alpha\beta} A_{\beta\gamma})^{-1})^T = (A^{-1}_{\alpha\gamma})^T = (A^\text{dual})_{\alpha\gamma} \). Then we apply Lemma 1.28.

Example
Take \( TM \) the tangent bundle over \( M \). The cotangent bundle \( T^*M \) is the dual bundle of \( TM \) and thus \( T^*M = (TM)^* \).

Tensor Product of Vector Bundles
Suppose \((E, M, \pi)\) is a vector bundle of rank \( k \) and \((\tilde{E}, \tilde{M}, \tilde{\pi})\) is a vector bundle of rank \( l \) over the same base manifold \( M \) then define \( E \otimes \tilde{E} = \bigsqcup_{p \in M} E_p \otimes \tilde{E}_p \).

This is well defined because \( E_p \) and \( \tilde{E}_p \) are vector spaces. Let \( V_\alpha \) be an open cover of \( M, \psi_\alpha, \tilde{\psi}_\alpha, A_{\alpha\beta}, \tilde{A}_{\alpha\beta} \) be the local trivialisations and transition maps to \( E \) and \( \tilde{E} \) respectively. Then the transition maps and local trivialisations for \( E \otimes \tilde{E} \) are given by
Subbundle
A vector bundle \((E, \bar{M}, \bar{\pi})\) is called a subbundle of a vector bundle \((E, M, \pi)\) if for each point \(p \in M\), the fibre \(E_p\) is a linear subspace of \(E\).

Example
The space of alternating \(k\)-tensors on \(E\), \(\wedge^k E = \bigsqcup_{p \in M} \wedge^k E_p\) is a sub bundle of \(\bigotimes^k E := E \otimes \ldots \otimes E\) (which is a subbundle of \(\bigotimes^k E\)).

Pullback Bundle
Let \(f: M \rightarrow N\) be a smooth map and \((E, N, \pi)\) be a vector bundle over \(N\). Then the pullback bundle \(f^* E := \bigsqcup_{p \in M} E_{f(p)}\) is a vector bundle over \(M\).

In conclusion, if \((E, M, \pi)\) is a vector bundle then the following are all vector bundles:

- \(E^*\)
- \(E \otimes \ldots \otimes E \otimes E^* \otimes \ldots \otimes E^*\) (\(r\) terms \(s\) terms)
- \(\wedge^k E\) (which is a subbundle of \(\bigotimes^k E\))
- \(f^* E\) for a smooth map \(f: N \rightarrow M\)

Tensor Fields and Differential Forms

Definition 1.31
Define the \((r, s)\)-tensor bundle (or tensor bundle of \((r, s)\)-type) of a manifold \(M\) to be the vector bundle

\[ T^r_s M := TM \otimes \ldots \otimes TM \otimes T^* M \otimes \ldots \otimes T^* M \]

An \((r, s)\)-tensor field (or an \(r\)-contravariant, \(s\) covariant tensor field) is a section of \(T^r_s M\).

Denote \(\Gamma(T^r_s M):=\{\text{smooth } (r, s)\text{-tensor fields}\}\) we adopt the convention that \(\Gamma(T^0_0 M) = C^\infty(M)\), the space of all smooth functions over \(M\).

Using local coordinates \((x^i)\) we can write an \((r, s)\)-tensor field \(T\) as

\[ T = T_{i_1 \ldots i_r}^{j_1 \ldots j_s} \frac{d}{dx^{i_1}} \otimes \ldots \otimes \frac{d}{dx^{i_r}} \otimes dx^{j_1} \otimes \ldots \otimes dx^{j_s} \]

Here \(\frac{d}{dx^{i_1}} \otimes \ldots \otimes \frac{d}{dx^{i_r}} \otimes dx^{j_1} \otimes \ldots \otimes dx^{j_s}\) is a local frame for \(T^r_s M\).

Note that Tensor Fields are \(C^\infty(M)\)-linear

Definition 1.32
Let \(F: M \rightarrow N\) be a smooth map between two smooth manifolds and \(\omega \in \Gamma(T^0_0 N)\) be a \(k\) covariant tensor field. We define a \(k\)-covariant tensor field \(F^* \omega\) over \(M\) by

\[ (F^* \omega)_p(v_1, \ldots, v_k) := \omega_{F(p)} \left( \begin{array}{c} F_{*p}(v_1) \\ \vdots \\ F_{*p}(v_k) \end{array} \right) \quad \forall v_1, \ldots, v_k \in T_p M \]

In this case, \(F^* \omega\) is called the pullback of \(\omega\) by \(F\).
Proposition 1.33 (Properties of Pullback)
Suppose $F:M \to N$ is a smooth map and $G:N \to Q$ a smooth map for $M, N, Q$ smooth
manifolds and $\omega \in T(T^0_0 N), \eta \in T(T^0_1 N)$ and $f \in C^\infty (N)$. Then

\begin{itemize}
  \item[(a)] $(g \circ F)^* = F^* \circ G^*$
  \item[(b)] $F^*(\omega \otimes \eta) = F^*\omega \otimes F^*\eta$. In particular, $F^*(f \circ \omega) = (f \circ F)F^*\omega$
  \item[(c)] $F^*(df) = d(f \circ F)$
  \item[(d)] If $p \in M$ and $(y^i)$ are local coordinates in a chart containing the point $F(p) \in N$ then
    \[ F^*(\omega_{j_1,...,j_k} dy^{i_1} \otimes ... \otimes dy^{i_k}) = \left( \omega_{j_1,...,j_k} \circ F \right) d(y^{j_1} \circ F) \otimes ... \otimes d(y^{j_k} \circ F) \]
\end{itemize}

Proof
The proofs are all exercises. □

Definition 1.34
A differential $k$-form on a smooth manifold $M$ is a smooth section of the vector bundle
$\Lambda^k M := \Lambda^k T^* M$ (which is a subbundle of $\bigotimes^k T^* M = T^k_0 M$)

Notation
We denote the set of all differential $k$-forms on $M$ by $\Omega^k (M) = \Gamma(\Lambda^k M)$. We adopt the
convention that $\Omega^0 (M) = C^\infty (M)$. Note that a $(0,1)$ tensor field is a $1$-form.
Locally we write a differential $k$-form $\omega$ as $\omega = \omega_{i_1,...,i_k} dx^{i_1} \wedge ... \wedge dx^{i_k}$. In this case $dx^{i_1} \wedge ... \wedge dx^{i_k}$ is a local frame for $\Lambda^k M$

Exterior derivative
The exterior derivative is a map $d:\Omega^k (M) \to \Omega^{k+1} (M)$ which is $\mathbb{R}$-linear such that $d \circ d = 0$
and if $f$ is a $C^\infty$ function and $X$ a $C^\infty$ vector field on $M$ then $(df)(X) = Xf$.

Integration of differential forms
$\int_M \omega$ is well defined only if $M$ is orientable, $\dim(M) = n$ and has a partition of unity and $\omega$
has compact support and is a differential n-form on $M$.

Theorem (Partition of unity) (Proof non-examinable)
Let $M$ be a smooth manifold which is countable at infinity (that is a countable union of
compact sets). Then there exists a partition of unity for $M$. The partition of unity statement is

Remarks
1. All the “natural” manifolds we are considering will be countable at infinity. In
   particular, all compact manifolds are countable at infinity and $\mathbb{R}^n = \bigcup_{j=1}^{\infty} \{ x \in 
   \mathbb{R}^n : |x|^2 \leq j \}$
2. In some textbooks, they assume manifolds are paracompact in order to get a partition
   of unity.

Remarks about tensor fields
1. Tensor contractions $C^j_i$ for tensors extend to tensor fields.
2. One can consider symmetric and antisymmetric tensor fields (smooth anti-symmetric \((0,k)\)-tensor fields = differential \(k\)-forms)

3. If \(T \in T^r_s \mathcal{M}\) the set of \((r,s)\)-tensor fields then write

\[
T = T_{j_1...j_r}^{i_1...i_r} \frac{d}{dx^{i_1}} \otimes ... \otimes \frac{d}{dx^{i_r}} \otimes dx^{j_1} \otimes ... \otimes dx^{j_s}
\]

Let \(X \in T_p \mathcal{M}\) where \(X = a^i \frac{d}{dx^i} = \tilde{a}^j \frac{d}{dx^j}\) for coordinate systems \(\{x^i\}, \{\tilde{x}^j\}\)

Then \(\frac{d}{dx^j} = \frac{dx^i}{d\tilde{x}^j} \frac{d}{dx^i}, \quad d\tilde{x}^j = \frac{dx^i}{d\tilde{x}^j} dx^i\)

this is the covariant transformation

this is the contravariant transformation
Chapter 2 – Riemannian Manifolds

2.1 Definition of Riemannian Manifolds

Definition 2.1
An inner product (or scalar product) on a vector space $V$ is a function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ that is:

1. Symmetric: $\langle u, v \rangle = \langle v, u \rangle$ for all $u, v \in V$
2. Bilinear: $\langle au + bv, w \rangle = a \langle u, w \rangle + b \langle v, w \rangle$ and $\langle u, av + bw \rangle = a \langle u, v \rangle + b \langle u, w \rangle$ for all $a, b \in \mathbb{R}$ and $u, v, w \in V$
3. Positive-definite: $\langle u, u \rangle > 0$ for all $u \neq 0$.

Definition 2.2
A Riemannian metric on a smooth manifold $M$ is a map that assigns every point $p \in M$ an inner product $g_p = \langle \cdot, \cdot \rangle_p$ on $T_p M$, which depends smoothly on $p$; that is, for all $X, Y \in \mathfrak{X}(M)$ the function $M \to \mathbb{R}$ mapping $p \to \langle X_p, Y_p \rangle_p$ is smooth.

Equivalently, a Riemannian metric is a symmetric, positive definite $(0,2)$-tensor field on $M$.

Definition 2.3
A pair $(M, g)$ of a manifold $M$ equipped with a Riemannian metric $g$ is called a Riemannian manifold.

Remark
If the Positive Definite condition in Definition 2.1 is replaced with

4. Non-degenerate: $\langle u, v \rangle = 0$ for all $v \in V$ implies that $u = 0$

Then $g$ is called a semi or pseudo-Riemannian metric.

2.2 Expression of the metric in local coordinates

Given a Riemannian manifold $(M, g)$ and a chart $(U, x^i)$ consider the functions $g_{ij} : U \to \mathbb{R}$ mapping $p \to g_{ij}(p) := \langle \left( \frac{d}{dx^i} \right)_p, \left( \frac{d}{dx^j} \right)_p \rangle$

For each $p \in U$, $(g_{ij}(p))_{1 \leq i, j \leq n}$ is an $n \times n$ matrix that is:

- Symmetric: $g_{ij}(p) = g_{ji}(p)$
- Positive Definite: $g_{ij}(p)(v^i v^j) > 0$ for any $(v^1, \ldots, v^n) \neq 0$

$\Rightarrow \left( (g_{ij}(p)) \right)$ is an invertible matrix

Definition 2.4
These functions $g_{ij}$ are called the local representations of the Riemannian metric $g$ with respect to the coordinates $(U, x^i)$.

Recall that $g \in T(T^*_2 M)$ so using the local frame $\{dx^i\}$ for $T^* M$ we write $g = g_{ij} d x^i \otimes d x^j$. 
Remark
Using the symmetric part of two 1 forms,
\[ \omega \cdot \eta = \text{sym}(\omega \otimes \eta) := \frac{1}{2} (\omega \otimes \eta + \eta \otimes \omega) \quad \forall \omega, \eta \in \Omega^1(M) \]
we can also write \( g = g_{ij} dx^i dx^j \)

2.3 Length and Angle between tangent vectors

Definition 2.5
Suppose \((M, g)\) is a Riemannian manifold and \(p \in M\). We define the length (or norm) of a tangent vector \(v \in T_p M\) to be \(|v| := \sqrt{\langle v, v \rangle_p}\) (Recall \(g(\ldots) = \langle \ldots \rangle\)) and the angle \(\alpha\) between \(v, w \in T_p M\) \((v \neq 0 \neq w)\) by \(\cos \alpha = \frac{\langle v, w \rangle_p}{|v||w|}\).

Two vectors \(v\) and \(w\) are orthogonal if \(\langle v, w \rangle = \frac{\pi}{2}\) or equivalently if \(\langle v, w \rangle_p = 0\). Vectors \(e_1, \ldots, e_k\) are called orthonormal if \(|e_i| = 1\) for all \(i\) and \(\langle e_i, e_j \rangle_p = 0\) for all \(i \neq j\).

Definition 2.6
Given a smooth curve \(c: [a, b] \to (M, g)\), the length of \(c\) between \(a\) and \(b\) is defined by the formula
\[ L_g(c) := \int_a^b |c'(t)|_g \, dt \]
Where \(c'(t_0) = c_*, \left( \frac{d}{dt} \right|_{t=t_0} \) \( \in T_{c(t_0)}M\)

2.4 Examples of Riemannian metrics

1. Euclidean Metric (canonical metric) \(g_{\text{Eucl}}\) on \(\mathbb{R}^n\)
\[ g_{\text{Eucl}} := \delta_{ij} dx^i \otimes dx^j = dx^1 \otimes dx^1 + \cdots + dx^n \otimes dx^n = dx^1 dx^1 + \cdots + dx^n dx^n \]

2. Induced Metric
Let \((M, g)\) be a Riemannian manifold and \(f: N \to (M, g)\) an immersion where \(N\) is a smooth manifold (that is \(f\) is a smooth map and \(f\) is injective). Then the induced metric on \(N\) is defined
\[ (f^* g)_p(v, w) := g_{f(p)}(f_*(v), f_*(w)) \quad \forall v, w \in T_p N, p \in N \]
Note that we require \(f\) to be an immersion in order to get the positive definite condition on \(f^* g\).

3. Induced Metric \(i^* g_{\text{Eucl}}\) on \(S^n\)
The induced metric on \(S^n\) sometimes denoted \(g_{\text{Eucl}}|_{S^n}\) from the Euclidean space \((\mathbb{R}^{n+1}, g_{\text{Eucl}})\) by the inclusion \(i: S^n \hookrightarrow \mathbb{R}^{n+1}\) is called the standard (or round) metric on \(S^n\) (clearly \(i\) is an immersion).
Consider stereographic projection \(S^2 \to \mathbb{R}^3\) and denote the inverse map \(u: \mathbb{R}^2 \to (S^2, g_{\text{round}})\). Then \(u^* g_{\text{round}} = \)? Gives the Riemannian metric for \(\mathbb{R}^2\).

4. Product metric
If \((M_1, g_1), (M_2, g_2)\) are two Riemannian manifolds then the product \(M_1 \times M_2\) admits a Riemannian metric \(g = g_1 \oplus g_2\) called the product metric defined by
\[ g(u_1 \oplus u_2, v_1 \oplus v_2) = g_1(u_1, v_1) \oplus g_2(u_2, v_2) \]
Where \(u_i, v_i \in T_{p_i} M_i\) for \(i = 1, 2\). We use the fact that \(T_{p_1,p_2}(M_1 \times M_2) \cong T_{p_1}M_1 \oplus T_{p_2} M_2\)
5. Warped Product

Suppose \( (M_1, g_1), (M_2, g_2) \) are two Riemannian manifolds. Then \( (M_1 \times M_2, g_1 \oplus f^2 g_2) \) is the warped product of \( g_1, g_2 \) (or denoted \( (M_1, g_1) \times_f (M_2, g_2) \)) where \( f: M_1 \to \mathbb{R} \) is a smooth positive function.

\[
(g_1 \oplus f^2 g_2)_{p_1,p_2}(u_1 \oplus u_2, v_1 \oplus v_2) = g_{1,p_1}(u_1, v_1) \oplus f^2(p_1)g_{2,p_2}(v_2, w_2)
\]

Example
In polar coordinates, \((r, \theta)\) the Euclidean metric on \(\mathbb{R}^2 \setminus \{0\}\) can be written as \(g_{\text{Eucl}} = dr \otimes dr + r^2 d\theta \otimes d\theta\). It can be considered the warped product of \((\mathbb{R}^+, g_{\text{Eucl}}) \times_f (S^1, g_{\text{round}})\) where \(f(r) = r\).

6. Conformal Metric

Let \((M, g)\) be a Riemannian manifold and \(\lambda: M \to \mathbb{R}^+\) be a smooth positive function. Then \((M, \lambda g)\) is also a Riemannian manifold. This change of metric preserves the angle between two vectors \(v, w \in T_p M\) (Check) \((M, \lambda g)\) is said to be conformal to \((M, g)\). We can define an equivalence relation \(g \sim g'\) if \(g'\) is conformal to \(g\). Denote the equivalence classes by \([g]\). In particular when \(\lambda\) is constant the two Riemannian metrics are said to be homothetic.

Remarks:

1. Suppose \((M, g)\) is a Riemannian Manifold and \(\{e^i\}\) a local frame for \(TM\) and \(\{\theta^i\}\) is its dual frame then \(g = g_{ij} \theta^i \otimes \theta^j\) where \(g_{ij} := \langle e_i, e_j \rangle\). If \(\{e_i\}\) is an orthonormal basis then \(\{\theta^i\}\) is the dual frame implies that \(g = \sum_{i=1}^n \theta^i \otimes \theta^i\).

2. \(u_1 \oplus u_2 \in T_{p_1,p_2}(M_1 \times M_2) \cong T_{p_1}M_1 \oplus T_{p_2}M_2\)

2.5 Existence of a Riemannian Metric

Theorem 2.7

On any smooth manifold \(M\) there exists at least one Riemannian Metric.

Proof

Let \(\{U_i, \phi_i\}_{i \in I}\) be an atlas of \(M\) such that \(\{U_i\}_{i \in I}\) is locally finite. For each \(i \in I\) we define on \(U_i\) the Riemannian metric \(g_i := \phi_i^* g_{\text{Eucl}}\) (recall that \(\phi_i: U \to \phi(U) \subset \mathbb{R}^n\) by definition \(\phi_i\) is a diffeomorphism onto its image hence an immersion.)

\[g_i(u, v) \equiv g_{\text{Eucl}}\left(\phi_i(u), \phi_i(v)\right)\] for all \(u, v \in T_p M\), for all \(p \in U_i\).

Let \(\{\alpha_i\}_{i \in I}\) with \(\alpha_i: M \to \mathbb{R}\) be a partition of unity with respect to the open cover \(\{U_i\}_{i \in I}\). Then define \(g = \sum_{i \in I} \alpha_i g_i\) is a Riemannian metric on \(M\). Note that the sum is well defined because \(\{U_i\}_{i \in I}\) is locally finite.

The three requirements for a Riemannian metric are easy to check:

Note that \(g\) is positive definite: for all \(v \neq 0 \in T_p M, p \in M\) there exists \(i \in I\) so that \(p \in U_i\). Then \(\alpha_i(p) > 0\) and \(g(v, v) \geq \alpha_i(p)g_i(v, v) > 0\) because \(g_i\) is positive definite so \(g(v, v) > 0\). ■
2.6 Conformal Maps and Isometries

Definition
A smooth map \( f : (M, g) \to (N, h) \) between two Riemannian manifolds is called a conformal map with conformal factor \( \lambda : M \to \mathbb{R}^+ \) if \( f^* h = \lambda^2 g \).

Note
A conformal map preserves angles; that is \( \mathcal{A}(v, w) = \mathcal{A}(f(v), f(w)) \) for all \( v, w \in T_p M \) and \( p \in M \).

Example:
\( S^2 \subset \mathbb{R}^3 \). We consider stereographic projection. \( S^2 \setminus p_N \to \mathbb{R}^2 \). As stereographic projection is a diffeomorphism its inverse \( u : \mathbb{R}^2 \to S^2 \setminus p_N \) is a conformal map. It follows from an exercise sheet that \( u \) is a conformal map with conformal factor \( \rho(x, y) = \frac{2}{1 + x^2 + y^2} \).

Definition 2.9
A Riemannian Manifold \( (M, g) \) is locally flat if for every point \( p \in M \) there exists a conformal diffeomorphism \( f : U \to V \) between an open neighbourhoods \( U \) of \( p \) and \( V \subset \mathbb{R}^n \) of \( f(p) \).

Definition 2.10
Given two Riemannian Manifolds \( (M, g) \) and \( (N, h) \), they are called isometric if there is a diffeomorphism \( f : M \to N \) such that \( f^* h = g \). Such a diffeomorphism \( f \) is called an isometry.

Remark
In particular an isometry \( f : (M, g) \to (M, g) \) is called an isometry of \( (M, g) \). All Isometries on a Riemannian manifold form a group.

Example:
\( (\mathbb{R}^2 \setminus \{0\}, g_{\text{Eucl}}) \) is isometric to the warped product \( (\mathbb{R}, g_{\text{Eucl}}) \times_f (S^1, g_{\text{round}}) \) where \( f(r) = r^2 \).

Definition 2.11
\( (M, g), (N, h) \) are called locally isometric if for every point \( p \in M \) there is an isometry \( f : U \to V \) from an open neighbourhood \( U \) of \( p \) in \( M \) and an open neighbourhood \( V \) of \( f(p) \) in \( N \).

Definition 2.12
A Riemannian metric \( g \) on \( M \) is called a flat metric if \( (M, g) \) is locally isometric to \( (\mathbb{R}^n, g_{\text{Eucl}}) \).

Definition 2.13
Suppose \( f : (M, g) \to (N, h) \) is an immersion. Then \( f \) is isometric if \( f^* h = g \).

2.7 Extra advantages of having a Riemannian Metric

Riemannian Volume Forms
Lemma 2.14
On any Oriented Riemannian \( n \)-manifold \( (M, g) \) there is a unique \( n \)-form \( dV_g \) satisfying \( dV_g(e_1, \ldots, e_n) = 1 \) where \( e_1, \ldots, e_n \) is an orthonormal basis for some \( T_p M, p \in M \).
This unique $n$-form $dV_g$ is called the Riemannian Volume Form. In local coordinates $(x^i)$ then

$$dV_g = \sqrt{\det(g_{ij})} \, dx^1 \wedge dx^2 \wedge \ldots \wedge dx^n$$

Recall that $g = g_{ij}dx^i \otimes dx^j$. See Sheet 3 for details.

**Integral of Functions**

**Definition 2.15**

Let $(M, g)$ be an oriented Riemannian $n$-manifold with its Riemannian volume form $dV_g$. If $f$ is a compactly supported smooth function on $M$ then $f \, dV_g$ is a new $n$-form which is compactly supported. We can define the integral of $f$ over $M$ as

$$\int_M f := \int_M f \, dV_g$$

(recall the integration of $n$-forms over $n$-manifolds)

**Musical Isomorphisms and gradient of a function**

Let $(M, g)$ be a Riemannian manifold. Then $T_pM \cong T_p^*M$. The isomorphisms between $T_pM$ and $T_p^*M$ are called the musical isomorphisms.

For $X \in T_pM$, consider the map $Y \mapsto \langle X, Y \rangle \in T_p^*M$. The map $X \mapsto (Y \mapsto \langle X, Y \rangle)$ is an isomorphism for Riemannian manifolds $b$ called “flat” $b: T_pM \to T_p^*M$ mapping $X \mapsto \langle X, \cdot \rangle = X^b (:= b(X))$ and its inverse $\# = b^{-1}$ called “sharp” $\#: T_p^*M \to T_pM$.

Together $b, \#$ are known as the musical isomorphisms.

They extend to vector fields and 1-forms to give a linear map $b: \mathfrak{X}(M) \to \Omega^1(M)$ and $\#: \Omega^1(M) \to \mathfrak{X}(M)$ for the local expressions of $b$ and $\#$ see below:

In local coordinates $(x^i)$ write $X = a^i \frac{d}{dx^i}$ then $X^b = g_{ij} a^i dx^j$.

Write $g^{ij}$ to denote the $i,j$th component of the inverse matrix $(g_{ij})^{-1}$. Then write $\omega = \omega^i dx^i$ so $\omega^\# = g^{ij} \omega_i \frac{d}{dx^j}$.

Let $f \in C^\infty(M)$. The gradient of $f$ is defined as $\text{grad} f := (df)^\# \in \mathfrak{X}(M)$

For the Properties of the gradient of a function, see exercise sheet 3.

Take $f \in C^\infty(\mathbb{R}^n)$. Then $\text{grad} f = \left( \frac{df}{dx^1}, \ldots, \frac{df}{dx^n} \right) \in \mathfrak{X}(\mathbb{R}^n)$.

**Remark**

The musical isomorphisms can be defined for a vector space $V$ equipped with an inner product $\langle \cdot, \cdot \rangle$. Then (flat) $b: V \to V^*$ and (sharp) $\#: V^* \to V$

**Extending the inner product to tensors and tensor fields of $(r, s)$-type**

Suppose $V$ is a vector space with an inner product $\langle \cdot, \cdot \rangle$.

1. Define the inner product of two $(0, 1)$ tensors $\alpha, \beta \in V^1_1 (= V^*)$ as $\langle \alpha, \beta \rangle := \langle \alpha^\#, \beta^\# \rangle_{\mathbb{R}^V}$
2. We extend the inner product to reducible \((r,s)\)-tensors

\[
\langle x_1 \otimes x_2 \otimes \ldots \otimes x_r \otimes \alpha^1 \otimes \ldots \otimes \alpha^s, y_1 \otimes y_2 \otimes \ldots \otimes y_r \otimes \beta^1 \otimes \ldots \otimes \beta^s \rangle
= \langle x_1, y_1 \rangle \times \ldots \times \langle x_r, y_r \rangle \times \langle \alpha^1, \beta^1 \rangle \times \ldots \times \langle \alpha^s, \beta^s \rangle
\]

3. We extend the inner product to general \((r,s)\) tensors by linearity.

Let \((M, g)\) be a Riemannian Manifold and \(g = \langle \cdot, \cdot \rangle\). Given \(\omega, \eta \in T(V_p^*)\) then \(\langle \omega, \eta \rangle(p) := \langle \omega_p, \eta_p \rangle_p\)

Let \((E, M, \pi)\) be a vector bundle and \(E = \bigsqcup_{p \in M} E_p\)

### 2.8 Bundle Metrics

**Recall from linear algebra:**

On a vector space \(V\), a bilinear form \(B : V \times V \to \mathbb{R}\) can be considered as an element \(B \in E^* \otimes E^*\). Given a vector bundle \((E, M, \pi)\) a bundle metric is a map that assigns each fibre \(E_p\) an inner product \(\langle \cdot, \cdot \rangle_p\) which depends smoothly on \(p \in M\).

**Definition 2.16**

A bundle metric \(h\) on the vector bundle \((E, M, \pi)\) is an element of \(\Gamma(E^* \otimes E^*)\) which is symmetric and positive definite.

**Notation**

Given \(v, w \in E_p\) we write their inner product as \(h(v, w)\) or \(\langle v, w \rangle_h\)

**Example:**

Any vector bundle admits a bundle metric (use partition of unity).

**Remark**

Given a Vector bundle \((E, M, \pi)\) with a bundle metric \(h\) we can define an isomorphism \(E \to E^*\). \(W\) can extend \(h\) to any \((r,s)\) tensor products of \(E\) and \(E^*\).

### 2.9 Hodge Star Operator and Laplace Operator

**Recall Linear Algebra:**

Let \(V\) be a vector space with an inner product \(\langle \cdot, \cdot \rangle\) and \(\dim V = n\). Consider the \(k\)-fold exterior product \(\bigwedge^k V := \{T \in V^k_0 : T\text{ is antisymmetric}\}\) these are the \(k\)-vectors while \(\bigwedge^k V^* := \{T \in V^k_0 : T\text{ is antisymmetric}\}\) is the set of \(k\)-forms. We have an inner product on \(\bigwedge^k V = \langle v_1 \wedge \ldots \wedge v_k, w_1 \wedge \ldots \wedge w_k \rangle = \det(\langle v_i, w_j \rangle)\) and extend it linearly to \(T \in \bigwedge^k V\). If \(e_1, \ldots, e_n\) are an orthonormal basis for \(V\), then \(\{e_{i_1} \wedge \ldots \wedge e_{i_k} : 1 \leq i_1 \leq i_2 \leq \ldots \leq i_k \leq n\}\) is an orthonormal basis of \(\bigwedge^k V\).

**Definition**

Two ordered bases \(\{e_1, \ldots, e_n\}\) and \(\{\tilde{e}_1, \ldots, \tilde{e}_n\}\) are said to have the same orientation if \(e_1 \wedge \ldots \wedge e_n\) is a positive multiple of \(\tilde{e}_1 \wedge \ldots \wedge \tilde{e}_n\) and have negative orientation if the multiple is negative.

\(V\) is said to be oriented if an ordered basis is distinguished as positive.

Suppose \(V\) is oriented. We define the Hodge Star Operator \(* : \bigwedge^k V \to \bigwedge^{n-k} V\) for \(0 \leq k \leq n\) by \(* (e_1 \wedge \ldots \wedge e_k) = e_{k+1} \wedge \ldots \wedge e_n\) where \(\{e_1, \ldots, e_n\}\) is a positively oriented basis of \(V\).
Exercise
This operation is independent of basis but dependent on the orientation.

Let \((M, g)\) be a Riemannian Manifold which is orientable and has dimension \(n\). Since \(M\) is oriented, select an orientation for all tangent spaces \(T_pM\). Also on all cotangent spaces \(T^*_pM\). \(g\) induces an inner product on each \(T^*_pM\), thus we obtain the Hodge Star Operator

\[
*: \bigwedge^k(T^*_pM) \to \bigwedge^{n-k}(T^*_pM)
\]

Which extends to a bundle map \(*: \Gamma\left(\bigwedge^k(T^*M)\right) \to \Gamma\left(\bigwedge^{n-k}(T^*M)\right)\), i.e. \(*: \Omega^k(M) \to \Omega^{n-k}(M)\).

Remarks

- \(dV_g = * (1)\) where \(1 \in \Omega^0(M) = \mathcal{C}^\infty(M)\)

- For \(d: \Omega^k(M) \to \Omega^{k+1}(M)\) define the codifferential \(d^* : \Omega^k(M) \to \Omega^{k-1}(M)\) by mapping \(d^* := (-1)^{n(k+1)+1} * d *\). Sometimes books write \(\delta\) instead of \(d^*\). It is convention so set \(d^* f = 0\) for \(f \in \Omega^0(M) = \mathcal{C}^\infty(M)\).

- Laplace Operator
  Define \(\Delta : \Omega^k(M) \to \Omega^k(M)\) by \(\Delta := dd^* + d^* d\). Let \(\omega \in \Omega^k(M)\). Then \(\omega\) is called harmonic if \(\Delta(\omega) = 0\).
Chapter 3: Affine connection and the Levi-Civita Connection

3.1 Affine Connections

Motivation
An affine connection is a kind of directional derivative of vector fields on a manifold. Imagine a vector field $V$ on $\mathbb{R}^n$ (that is a map $\mathbb{R}^n \to \mathbb{R}^n$). Take a point $p \in \mathbb{R}^n$ and choose a tangent vector $X \in T_p \mathbb{R}^n \cong \mathbb{R}^n$.

Denote by $\nabla_X V$ the directional derivative of $V$ at $p$ in the direction $X$. Write $X = a^t \frac{d}{dx^t}$. Then $\nabla_X V = a^t \frac{dV}{dx^t} \in T_p \mathbb{R}^n$.

We shall generalise to the case that $V \in \mathfrak{X}(M)$ $p \in M$ and $X \in T_p M$.

We could define $\nabla_X V$ in local coordinates, however in order to obtain a definition of $\nabla_X V$ that is independent of the local coordinates we need some additional structure on $M$; an affine connection.

Definition 3.1
An affine/linear connection on a manifold $M$ is a map $\nabla: \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ mapping $(X, V) \to \nabla_X V$ satisfying:

C1. For each fixed $V \in \mathfrak{X}(M)$ then the map $X \to \nabla_X V$ is $C^\infty(M)$-linear. That is $\nabla_{fX + gY} V = f\nabla_X V + g\nabla_Y V$ for all $f, g \in C^\infty(M)$ and $X, Y \in \mathfrak{X}(M)$.

C2. For each $X \in \mathfrak{X}(M)$ then the map $V \to \nabla_X V$ is $\mathbb{R}$-linear. That is $\nabla_X (aV + bW) = a\nabla_X V + b\nabla_X W$ for all $a, b \in \mathbb{R}$ and $V, W \in \mathfrak{X}(M)$.

C3. $\nabla_X (fV) = X(f) \cdot V + f\nabla_X V$ for all $f \in C^\infty(M)$, $X, V \in \mathfrak{X}(M)$.

Example
The canonical connection on the Euclidean space $\mathbb{R}^n$ is $\left(\nabla_X V\right)_p = \lim_{\varepsilon \to 0} \frac{V(p + \varepsilon X_p) - V(p)}{\varepsilon}$.

Remarks
1. More generally we can define a connection on a vector bundle $(E, M, \pi)$ as the map $\nabla: \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$ satisfying the conditions similar to C1-C3 in Definition 3.1.

An affine connection on $M$ is then a connection on $TM$.

2. $\nabla_X V$ is called the directional derivative on $V$ in the direction of $X$.

3. An affine connection is not a tensor field because $\nabla_X V$ is not $C^\infty(M)$-linear in $V$.

4. $\nabla$ is also denoted by $D$.

Lemma 3.2 (The local Expression)
Let $\nabla$ be an affine connection on a manifold $M$ and let $X, Y \in \mathfrak{X}(M)$ be expressed in terms of a local frame $\{e_i\}$ on $U \subset M$ for $TM$ by $X = x^i e_i$ and $Y = y^i e_i$. Then

$\nabla_X Y = \nabla_{x^i e_i} (y^j e_j) = x^i \nabla_{e_i} (y^j e_j) = x^i (e_i(y^j) e_j + y^j \nabla_{e_i} e_j) = X(y^j) e_k + x^i y^j \Gamma^k_i e_k$

Where the functions $\Gamma^k_i$ on $U$ are called the Christoffel Symbols of $\nabla$ with respect to the local frame $\{e_j\}$ are defined by $\nabla_{e_i} e_j = \sum_k \Gamma^k_{ij} e_k$. 


Remark
($\nabla_X Y)_p$ depends only on the value of $X$ at $p$ and the values of $Y$ in a neighbourhood of $p$. See Exercise Sheet 4 for further discussions.

Take $e_i = \frac{d}{dx^i}$ for local coordinates $(x^i)$.

Example
By the Motivation Section, the canonical connection on $\mathbb{R}^n$ locally looks like $\nabla_X Y = X(Y^j) \frac{d}{dx^i} = X \left( \frac{dY^j}{dx^i} \frac{d}{dx^i} \right)$. The Christoffel Symbols of the canonical connection with respect to the coordinate frame $\frac{d}{dx^i}$ are all identically 0.

Proposition 3.3 (Existence of an affine connection)
Every smooth manifold admits at least one affine connection.

Proof
Non-examinable. The proof uses a partition of unity.

3.2 Extension of connections to tensor fields
We extend $\nabla$ to $T_s^r M := T M \otimes ... \otimes T M \otimes T^* M \otimes ... \otimes T^* M$

Lemma 3.4
Given an affine connection on $M$, then there is a unique map $\nabla: \mathfrak{X}(M) \times \Gamma(T^r_s M) \rightarrow \Gamma(T^r_s M)$ mapping $(X, T)$ to $\nabla_X T$ which coincides with the given connection on $\Gamma(T^r_s M) = \mathfrak{X}(M)$ satisfying conditions C1-C3 and additionally:

C4. On $\Gamma(T^0_0 M) = C^\infty(M)$ then $\nabla$ is given by $\nabla_X f = X(f)$

C5. $\nabla$ satisfies the following product rule with respect to the tensor products:

$\nabla_X (T_1 \otimes T_2) = \nabla_X T_1 \otimes T_2 + T_1 \otimes \nabla_X T_2$, $\forall T_1 \in \Gamma(T^r_1 M)$, $\forall T_2 \in \Gamma(T^r_2 M)$

C6. $\nabla$ commutes with all tensor contractions: $\nabla_X (c(T)) = c(\nabla_X T)$ where $c$ represents any tensor contraction

$c_{ij}(v_1 \otimes ... \otimes v_\eta \otimes a^1 \otimes ... \otimes a^n)
= a^j(v_i)(v_1 \otimes \otimes \hat{v}_i \otimes \otimes v_\eta \otimes a^1 \otimes ... \otimes \hat{a}^j \otimes ... \otimes \hat{a}^n)$

Remarks:

1. For any $\omega \in \Gamma(T^0_0 M) = \Omega^1(M)$, $X, Y \in \mathfrak{X}(M)$ we have $X(\omega(Y)) = \nabla_X (\omega(Y)) = \nabla_X (c(Y \otimes \omega)) = c(\nabla_X Y \otimes \omega) = c(\nabla_X Y \otimes \omega) + Y \otimes \nabla_X \omega = \omega(\nabla_X Y) + (\nabla_X \omega) Y$

Therefore we have proved that $(\nabla_X \omega)(Y) = X(\omega(Y)) - \omega(\nabla_X Y)$

2. Similarly, for $T \in \Gamma(T^r_s M)$ we have

$(\nabla_X T)(\omega^1, ..., \omega^r, Y_1, ..., Y_s) = X(\nabla_Y T(\omega^1, ..., \omega^r, Y_1, ..., Y_s)) - \sum_{j=1}^r T(\omega^1, ..., \nabla_X \omega^j, ..., \omega^r, Y_1, ..., Y_s) - \sum_{j=1}^r T(\omega^1, ..., \omega^r, Y_1, ..., \nabla_X Y_j, ..., Y_s)$

Definition 3.5
Let $\nabla$ be an affine connection on $M$. For $T \in \Gamma(T^r_s M)$ we define $\nabla T \in \Gamma(T^r_{s+1} M)$ by
The $(r, s + 1)$ tensor field $\nabla T$ is called the total covariant derivative.

**Definition 3.6**
A tensor field $T$ is called parallel if $\nabla T = 0$

**Remark**
More generally for $k \geq 2$ we can define $\nabla^k : \Gamma(T^k \!_x \!_y M) \to \Gamma(T^{k+1} \!_x \!_y M)$ inductively by $\nabla^k = \nabla \circ \nabla^{k-1}$

### 3.3 Connection along curves and parallel transport

**Lemma 3.7**
For $p \in M \ (\nabla_X Y)_p$ depends only on the value of $X$ at $p$ and the values of $Y$ along any curve $\gamma : (-\epsilon, \epsilon) \to M$ such that $\gamma(0) = p$ and $\gamma'(0) = Y_p$

**Proof**
Exercise Sheet 4. ■

**Definition 3.8**
Let $\gamma : I \to M$ be a curve in $M$. A map $V : I \to TM$ is said to be a vector field along $\gamma$ if $V(t) \in T_{\gamma(t)}M$ for all $t \in I$.

Denote by $\mathfrak{X}(\gamma) := \{\text{smooth vector fields along } \gamma\}$

**Definition 3.9**
Let $\gamma : I \to M$ be a curve in $M$. For any $t_0 \in I$ choose a neighbourhood $U$ of $\gamma(t_0)$ and a local frame $\{E_i\}_{i=1}^n$ for $TM$. Then we can write any $V \in \mathfrak{X}(\gamma)$ as $V(t) = V^i(t)E_i(\gamma(t))$ for $t$ near $t_0$.

We define the covariant derivative of $V$ along $\gamma$ as

$$D_t V(t_0) := (V^i)'(t_0)E_i(\gamma(t_0)) + V^i(t_0)\nabla_{\gamma'(t_0)}E_i$$

$$D_t V := \frac{dV^i}{dt}E_i + V^i \nabla_{\gamma'}E_i$$

(Recall that $\nabla_{\gamma'}E_i = \gamma'(t_0)\ i \ \frac{dE_i}{dx} \ i \ \frac{d}{dx}$)

**Remark:**
1. The above definition is well defined and independent of the choice of local frame $E_i$
   To check take $\tilde{E}_i$
2. $D_t V : \mathfrak{X}(\gamma) \to \mathfrak{X}(\gamma)$ satisfies the following properties ($\mathbb{R}$-linear but not $C^\infty(M)$-linear)
   $$D_t(aV + bW) = aD_tV + bD_tW$$
   for all $a, b \in \mathbb{R}, V, W \in \mathfrak{X}(\gamma)$. Moreover for all $f \in C^\infty(M)$
   we have $D_t(fV) = fV + f \cdot D_t V$
3. Suppose there exists a smooth vector field $\mathbf{V}$ on $U$ such that $V(t) = V_{y(t)}$ for all $t \in I$.
   Then $D_t V(t_0) = \nabla_{\gamma'(t_0)} \mathbf{V}$

**Definition 3.10**
$V \in \mathfrak{X}(\gamma)$ is called parallel along $\gamma$ if $D_t V = 0$. If the velocity vector field $\gamma'(t) := \gamma_{y(t)} \left( \frac{d}{dt} \right)$ of a curve $\gamma$ is parallel along $\gamma$ that is $D_t \gamma'(t) = 0$ then the $\gamma$ is said to be a self-parallel curve or geodesic.
Proposition 3.11
Given an affine connection $\nabla$ on $M$ a curve $\gamma : I \to M$ and a tangent vector $V_0$ at a point $\gamma(t)$ on the curve there exists a unique parallel vector field $V(t)$ along $\gamma$ such that $V(t_0) = V_0$.

Proof
Take a local frame $\{E_i\}$ for $TM$ around $\gamma(t_0)$. Let $\Gamma^k_{ij}$ be the Christoffel Symbols with respect to $\{E_i\}$. We write $\gamma'(t) = a^i(t)E_i(\gamma(t))$ for $t$ near $t_0$.
Suppose $V(t) = V^i(t)E_i(\gamma(t))$ is a solution then
\[
0 = D_t V(t) = \sum_{k=1}^{n} \left[ (V^k)'(t) + a^i(t)V^j(t)\Gamma_{ij}^k(\gamma(t)) \right] E_k(\gamma(t))
\]
\[
V^i(t_0)E_i(\gamma(t_0)) = V_0 = V^i_0E_i(\gamma(t_0))
\]
So
\[
\begin{cases}
(V^k)'(t) + a^i(t)V^j(t)\Gamma_{ij}^k(\gamma(t)) = 0 & \forall k \\
V^i(t_0) = V^i_0 & \forall i
\end{cases}
\]
We find the terms of the solution $V^k(t)$ by solving this system of linear ODEs. ■

Definition 3.12
Let $\gamma : I \to M$ be a curve in $M$ and $\gamma(t_0)$ be a point on the curve. Then the map
\[
P_{t_0,\gamma}(v) : T_{\gamma(t_0)}M \to T_{\gamma(t)}M \text{ mapping } P_{t_0,\gamma}(v) = V(t)
\]
Where $V \in T_{\gamma(t_0)}M$ and $V(t)$ is the unique extension of $v$ to a parallel vector field along the curve $\gamma$. The map $P$ is called the parallel transport from $\gamma(t_0)$ to $\gamma(t)$.

3.4 The Levi-Civita Connection

Definition 3.13
Let $\nabla$ be an affine connection on a Riemannian Manifold $(M, g)$. $\nabla$ is called compatible with $g$ if for every curve $\gamma : I \to M$ and every pair of parallel vector fields $X, Y$ along $\gamma$ then $g(X, Y)$ is constant.

Remark
1. $\nabla$ is compatible with $g$ if and only if $\nabla g = 0$. For more equivalent definitions of compatibility see Exercise Sheet 4. One such definition is
\[
X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z) \quad \forall X, Y, Z \in \mathfrak{X}(M)
\]
2. For a vector bundle $(E, M, \pi)$ with a bundle metric $h$. (that is $h$ restricted to a fibre $E_p$ is just $g$ on $p$ and $h$ depends smoothly on $p$.) Then for a connection $\nabla$ on $(E, M, \pi)$ we have
$\nabla$ is compatible with $h$ if and only if $X(h(V, W)) = h(\nabla_X V, W) + h(V, \nabla_X W)$ for all $X \in \mathfrak{X}(M)$ and $V, W \in \Gamma(E)$.
Since $\nabla : \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$ mapping $(X, V) \to \nabla_X V$ is $C^\infty(M)$-linear in $X$, $\nabla$ can be thought of as a map
\[
\nabla : \Gamma(E) \to \Gamma(T^*M \otimes E)
\]
More generally we can extend the definition so that for $h \in \Gamma(E^* \otimes E^*)$ and
\[
\nabla : \Gamma \left( \bigotimes_{r \text{ terms}} E \bigotimes \bigotimes_{s \text{ terms}} E^* \right) \to \Gamma \left( T^*M \otimes E \bigotimes \bigotimes_{r \text{ terms}} E \bigotimes E^* \bigotimes \bigotimes_{s \text{ terms}} E^* \right)
\]
\( \nabla \) is compatible with \( h \) if and only if \( \nabla h = 0 \). Observe that when \( E = TM \) we return to the case in part 1.

**Definition 3.14**
An affine connection \( \nabla \) on a manifold \( M \) is called symmetric or torsion free if \( \nabla_X Y - \nabla_Y X = [X, Y] \) for all \( X, Y \in \mathfrak{X}(M) \).

Define \( T: \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M) \) mapping \((X, Y) \to T(X, Y) := \nabla_X Y - \nabla_Y X - [X, Y] \). Then \( T \) is \( C^\infty(M) \)-linear and antisymmetric. \( T \) is called the torsion tensor of \( \nabla \). See Exercise Sheet 4 for details.

**Theorem 3.15**
Given a Riemannian Manifold \((M, g)\) there exists a unique affine connection \( \nabla \) on \( M \) that is symmetric and compatible with \( g \). Moreover it is determined by the formula(\(*\)). This desired connection is known as the Levi-Civita Connection: 

\[
\langle \nabla_X Y, Z \rangle = \frac{1}{2} \left( \langle X(Y, Z) + Y(Z, X) - Z(X, Y) - \langle X, [Y, Z] \rangle + \langle Y, [Z, X] \rangle + \langle Z, [X, Y] \rangle \right) \quad \forall X, Y, Z \in \mathfrak{X}(M)
\]

**Idea of Proof:**
Assume \( \nabla \) exists with the desired properties. Using compatibility of \( \nabla \) with \( g \) we get

\[
\langle \nabla_X Y, Z \rangle = X(Y, Z) - \langle Y, \nabla_X Z \rangle
\]

Since \( \nabla \) is symmetric we have \( \nabla_X Z + \nabla_Z X = [Z, X] \) so replacing this in the expression above gives

\[
\langle \nabla_X Y, Z \rangle = X(Y, Z) - \langle Y, [Z, X] \rangle - \nabla_Z X = X(Y, Z) - \langle Y, [Z, X] \rangle + \langle Y, \nabla_Z X \rangle \quad (1)
\]

Cycling \( X, Y, Z \) gives two similar formulae:

\[
\langle \nabla_Y Z, X \rangle = Y(Z, X) - \langle Z, [X, Y] \rangle + \langle Z, \nabla_Y X \rangle \quad (2)
\]

\[
\langle \nabla_Z X, Y \rangle = Z(X, Y) - \langle X, [Y, Z] \rangle + \langle X, \nabla_Y Z \rangle \quad (3)
\]

By computing (1) + (2) - (3) we obtain

\[
\langle \nabla_X Y, Z \rangle - \langle \nabla_Y Z, X \rangle + \langle \nabla_Z X, Y \rangle = X(Y, Z) - \langle Y, [Z, X] \rangle + \langle Y, \nabla_Z X \rangle - Y(Z, X) - \langle Z, [X, Y] \rangle + \langle Z, \nabla_Y X \rangle + Z(X, Y) - \langle X, [Y, Z] \rangle + \langle X, \nabla_Y Z \rangle
\]

Now using symmetry of the metric, we get the desired formula (\( * \)) this implies uniqueness of the Levi-Civita Connection.

Moreover (\( * \)) determines an affine connection. It is simple to see that this connection satisfies conditions C1-C3 in Definition 3.1. ■

**Remarks:**

1. We can find the local expressions of the Christoffel Symbols of the Levi-Civita connection with respect to the local coordinates \((x^i)\). See Sheet 4 for details. The result is that

\[
\Gamma_{ij}^k = \frac{1}{2} g^{kl} \left( \frac{dg_{ji}}{dx^l} + \frac{dg_{il}}{dx^j} - \frac{dg_{ij}}{dx^l} \right)
\]

2. \( P_{t_0}: T_{Y(t)}M \to T_{Y(t)}M \) is an isometry.
Chapter 4: Geodesics and Riemannian Distance

Assume throughout this chapter that we have a Riemannian Manifold \((M,g)\) with a Levi-Civita connection \(\nabla\).

### 4.1 Geodesics

A Geodesic is a curve \(\gamma: I \to M\) which has zero acceleration; that is the derivative of \(\gamma'(t)\) is identically zero.

**Definition 4.1**

A curve \(\gamma: I \to M\) from an open interval \(I \subset \mathbb{R}\) is called a geodesic if \(D_t \gamma'(t) = 0\) for all \(t \in I\). That is \(\gamma'(t)\) is parallel along \(\gamma\). Moreover, the restriction of \(\gamma\) to some closed interval \([a,b] \subset I\) is called a geodesic segment or arc.

**Theorem 4.2 (Local Existence and Uniqueness of Geodesics)**

Given \(p \in M\) and \(V \in T_p M\) there exists \(\epsilon > 0\) and a geodesic \(\gamma: (-\epsilon, \epsilon) \to M\) such that \(\gamma(0) = p\) and \(\gamma'(0) = V\). Moreover, any two geodesics agree on their common domain.

**Proof**

Choose local coordinate chart \((U, x^i)\) of \(p\). Without loss of generality we may assume that \(p\) corresponds to the origin. Let \(\Gamma^k_{ij}\) be the Christoffel Symbols of the Levi-Civita Connection on \(M\) with respect to the local frame \(\frac{d}{dx^i}\); that is \(\nabla \frac{d}{dx^i} = \Gamma^k_{ij} \frac{d}{dx^k}\). Suppose \(\gamma: I \to M\) is a geodesic such that \(\gamma(0) = p\) and \(\gamma'(0) = V\). Therefore we can write \(\gamma(t) = x^i(t) \frac{d}{dx^i} \left(\gamma(t)\right)\). Write \(V = V^i \frac{d}{dx^i} \left(\gamma(0)\right) \in T_p M\) then

\[
V = V^i \frac{d}{dx^i} \left(\gamma(0)\right) = V^i \frac{d}{dx^i} \left(\gamma(t)\right).
\]

By the Definition of Geodesic, and the definition of Covariant Derivative:

\[
0 = D_t \gamma'(t) = D_t \left( x^i'(t) \frac{d}{dx^i} \left(\gamma(t)\right) \right) = (x^i')''(t) \frac{d}{dx^i} \left(\gamma(t)\right) + (x^i')'(t) \frac{d}{dx^i} \left(\gamma(t)\right) + (x^i)''(t) \frac{d}{dx^i} \left(\gamma(t)\right)
\]

Now we treat the two components of the sums separately: to emphasis this we change the index of the first component from \(i\) to \(k\) and expand the second component involving the connection:

\[
0 = (x^k)''(t) \frac{d}{dx^k} \left(\gamma(t)\right) + (x^i')'(t) \frac{d}{dx^i} \left(\gamma(t)\right) + (x^i')'(t) \frac{d}{dx^i} \left(\gamma(t)\right) + (x^i)''(t) \frac{d}{dx^i} \left(\gamma(t)\right)
\]

\[
= (x^k)''(t) \frac{d}{dx^k} \left(\gamma(t)\right) + (x^i')'(t) \frac{d}{dx^i} \left(\gamma(t)\right) + (x^i')'(t) \frac{d}{dx^i} \left(\gamma(t)\right) \Gamma^i_{jk} \frac{d}{dx^j} \left(\gamma(t)\right)
\]

Therefore we get the initial value problem:
From ODE Theory, there exists $\epsilon > 0$ such that the system of second order linear equations has a unique solution for $t \in (-\epsilon, \epsilon)$.

Now we can repeat the entire argument for any chart $(V, y^i)$ for $M$. By repeating as necessary we can extend the domain of the geodesic $\gamma$. By the local uniqueness, it follows that any two geodesics coincide on their common domains. ■

**Remark**
The solutions to the geodesic equation are smooth as functions depending on the initial conditions. It follows from the local existence and uniqueness (Theorem 4.2) that for any point $p \in M$ and $V \in T_pM$ there is a unique maximal geodesic. That is, a geodesic defined on some open interval $I$ containing 0 and cannot be extended to any larger domain as a geodesic.

**Examples:**

1. **Geodesics in $\mathbb{R}^n_+, g_{Euc}$.** In this case the Christoffel Symbols $\Gamma_{ij}^k = 0$ for all $i, j, k$. Given $p \in \mathbb{R}^n$ a vector $V \in \mathbb{R}^n$. In the statement of Theorem 4.2, we need to solve the equation $(x^k)''(t) = 0$ for all $k$ and $x^k(0) = 0$ and $(x^k)'(0) = V^k$ for all $k$. Therefore $x^k = p^k + tV^k$ and thus $\gamma(t) = p + tV$ is the Geodesic.

2. **Geodesics on $\mathbb{S}^2 \subset \mathbb{R}^3$.**

   We use polar coordinates $(\theta, \phi) = (x, y, z) = p$. We obtain the formulae:
   
   $$x = \sin \theta \cos \phi, \quad y = \sin \theta \sin \phi, \quad z = \cos \theta$$
   
   Project $p$ to $p'$ in the plane spanned by $x$ and $y$. Then $\mathcal{X}(x, p') = \phi$ and $\mathcal{X}(z, p) = \theta$.

   $g_{ij} = d\theta^2 + \sin^2 \theta \, d\phi^2$. By associating $x^1$ to $\theta$ and $x^2$ to $\phi$ then $(g_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2 \theta \end{pmatrix}$

   That is $g_{11} = 1$, $g_{12} = 0 = g_{21}$, $g_{22} = \sin^2 \theta$. Also $(g^{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2 \theta \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\sin^2 \theta} \end{pmatrix}$

   so $g^{11} = 1$, $g^{12} = g^{21} = 0$, $g^{22} = \frac{1}{\sin^2 \theta}$. Recall the formula for the Christoffel Symbols:

   $$\Gamma^k_{ij} = \frac{1}{2} g^{kl} \left( \frac{dg_{jl}}{dx^i} + \frac{dg_{il}}{dx^j} - \frac{dg_{ij}}{dx^l} \right)$$

   So

   $$\Gamma^1_{11} = 0 = \Gamma^1_{12} = \Gamma^1_{21} = \Gamma^2_{22} = \Gamma^2_{11}$$

   $$\Gamma^1_{22} = -\frac{1}{2} \left( \frac{d\sin^2 \theta}{d\theta} \right) = -\sin \theta \cos \theta$$

   $$\Gamma^2_{21} = \frac{1}{2} g^{21} \left( \frac{dg_{21}}{dx^1} + \frac{dg_{11}}{dx^2} - \frac{dg_{12}}{dx^1} \right) + \frac{1}{2} g^{22} \left( \frac{dg_{22}}{dx^1} + \frac{dg_{12}}{dx^2} - \frac{dg_{12}}{dx^2} \right)$$

   $$= \frac{1}{2} \frac{1}{\sin^2 \theta} \left( \frac{d\sin^2 \theta}{d\theta} \right) = \cos \theta$$
Recall the Geodesic equations:
Let \( \gamma(t) = (x^1(t), \ldots, x^n(t)) \) be a geodesic then
\[
(x^k)''(t) + (x^i)'(t)(x^j)'(t)\Gamma^k_{ij}(x(t)) = 0 \quad \forall k
\]

Then if \( \gamma(t) = (\theta(t), \phi(t)) \) is a geodesic on \((S^2, g_{S^2})\) then
\[
\theta''(t) - \sin \theta(t) \cos \theta(t) (\phi'(t))^2 = 0
\]
\[
\phi''(t) + \frac{2 \cos \theta(t)}{\sin \theta(t)} \theta'(t) \phi'(t) = 0
\]

We will find a special solution. When \( \theta(t) = t \) and \( \phi(t) = \phi_0 \) we see that it is a solution to the differential equations above, and this solution corresponds to a portion of a great circle passing through \( p \) and \( p_n \). By uniqueness of geodesics and using rotations, the geodesics on \((S^2, g_{S^2})\) are portions of great circles.

3. **Geodesics on the Upper Half Hyperbolic Plane**

Let \( H^2 = \{(x, y) \in \mathbb{R}^2 : y > 0\} \) and a metric \( g_{H^2} = \frac{1}{y^2} (dx^2 + dy^2) \). Then using the same method above by computing \( g_{ij} \) and \( g^{ij} \) and then \( \Gamma^k_{ij} \) we can determine all geodesics. They are of the forms of semicircles and vertical lines as shown below:

See Exercise Sheet 5 for details.

4. **Geodesics on the Poincaré Disc**

Let \( D = \{(x, y) \in \mathbb{R}^2: x^2 + y^2 < 1\} \) and \( g_D = \left( \frac{2}{1-x^2-y^2} \right)^2 (dx^2 + dy^2) \),
then as before we can show that all geodesics are of the form of arcs normal to the tangent of the disc:
4.2 Exponential Map
This is map from $T_p M$ to $(M, g)$. First we need a Stronger Version of Theorem 4.2:

**Theorem 4.2’**
For $p_0 \in M$ there exists a neighbourhood $U$ of $p_0$ and $\epsilon > 0$ so that for all $p \in U$ and $v \in T_p M$ with $|v| < \epsilon$ then there exists a unique geodesic $\gamma_{p,v}: (-1,1) \to M$ with $\gamma_{p,v}(0) = p$ and $\gamma_{p,v}'(0) = v$. Moreover the map $\gamma: TU \times (-1,1) \to M$ mapping $(p, v), t) \to \gamma_{p,v}(t)$ is smooth.

**Definition 4.4**
Let $O_p \subset T_p M$ be the set of all vectors $v$ so that $\gamma_{p,v}$ is defined on an interval containing $[0,1]$. Then we define the exponential map at $p$ by $\exp_p : O_p \to M$ mapping $v \to \exp_p(v) := \gamma_{p,v}(1)$ where $\gamma_{p,v}$ is the unique geodesic as in Theorem 4.2’.

Take $O := \bigsqcup_{p \in M} O_p \subset TM$ and define the exponential map by $\exp: O \to M$ mapping $(p, v) \to \exp_p(v) := \gamma_{p,v}(1)$.

**Example:**
We take $(S^2, g_{\text{round}})$

\[ \exp_{PN} : T_{PN} S^2 \to S^2 \]. In fact $\exp_{PN} : D(0, \pi) \subset T_{PN} S^2 \to S^2 \setminus \{p_S\}$ is surjective and a diffeomorphism.

**Lemma 4.5 (Homogeneity of Geodesics)**
Given $(p, v) \in TM$ and $\lambda > 0$ we have $\gamma_{p,\lambda v}(t) = \gamma_{p,v}(\lambda t)$ whenever either side is defined.

**Proof**
Proof not examinable. ■

**Proposition 4.6 (Properties of the Exponential Map)**

1. $O$ is an open subset of $TM$ containing all zero-vectors and hence $\exp$ is a smooth map between smooth manifolds $O$ and $M$. (Any open subset of a manifold is itself a manifold).
2. For any $(p, v) \in TM$ the maximal geodesic $\gamma_{p,v}(t)$ is given by $\gamma_{p,v}(t) = \exp_p(t, v)$ for all $t$ such that both sides are well defined.
3. $\exp_p(0) = p$ and $\left(\exp_p \right)_{*,0}(v) = v$ for all $v \in T_p M$
Proof
Non-examinable.

Remarks:
2. Means that \( \exp_p \) maps the radial lines in \( T_pM \) to radial geodesics in \( M \) passing through \( p \).
3. As \( \left( \exp_p \right)_{*0} = 1d \) then \( \exp_p \) is locally a diffeomorphism.

4.3 Normal Coordinates

Lemma 4.7 (Normal Neighbourhood Theorem)
Given \( p \in (M, g) \) there exists a neighbourhood \( U \) of 0 in \( T_pM \) and another neighbourhood \( V \) of \( p \in M \) such that the exponential map \( \exp_p : U \to V \) is a diffeomorphism.

Such \( V \) is called a normal neighbourhood of \( p \). In particular, there exists \( \varepsilon > 0 \) such that \( \exp_p : B(0, \varepsilon) \subset T_pM \to \exp_p \left( B(0, \varepsilon) \right) \subset M \) is a diffeomorphism where \( B(0, \varepsilon) = \{ v \in T_pM : |v|_g < \varepsilon \} \).

Terminology: For \( 0 < r < \varepsilon \) with \( \varepsilon \) as in Lemma 4.7, we call \( \exp_p B(0, r) \) an open geodesic ball and \( \exp_p B(0, r) \) a closed geodesic ball. Also, \( \exp_p \left( \partial B(0, r) \right) \) is called a geodesic sphere.

Given \( p \in (M, g) \) choose an orthonormal basis \( \{ e_i \} \) of \( T_pM \). This gives an isomorphism \( E : \mathbb{R}^n \to T_pM \) mapping \( (x^1, ..., x^n) \to E(x^1, ..., x^n) := x^i e_i \).

Let \( V \) be a normal neighbourhood of \( p \), we define the chart map \( \phi := E^{-1} \circ \exp_p^{-1} : V \to \mathbb{R}^n \).

The coordinates associated to the chart \( (V, \phi) \) are called normal coordinates centred at \( p \).

They map \( \exp_p(v) \in V \to v = x^i e_i \in T_pM \to x = (x^1, ..., x^n) \in \mathbb{R}^n \).

Then \( \phi : V \to \mathbb{R}^n \) maps \( \exp_p(v) \to (x^1, ..., x^n) \).

It is extremely useful to take normal coordinates centred at \( p \in M \), since then \( g_{ij}(p) = \delta_{ij} \) which implies that \( T^k_j(p) = 0 \) for all \( i, j, k \).

4.4 Riemannian Distance

Given a Riemannian manifold \( (M, g) \) and \( p, q \in M \) we can find a metric \( d_g(p, q) \) so \( (M, d) \) is a metric space whose induced topology is identical to the given topology on \( M \).

Definition 4.8
A map \( c : [a, b] \to M \) is called a smooth curve if \( c : (a, b) \to M \) is a smooth map (in the usual sense) and \( c : [a, b] \to M \) is continuous, and all derivatives of \( c \) extend continuously to \( a \) (with respect to a chart containing \( c(a) \)). Similarly for \( b \).
For some $\epsilon > 0 \tilde{c} : [a, a + \epsilon) \to \mathbb{R}^n$ where $\tilde{c} = \phi \circ c$. $\tilde{c}^{(k)}(t)$ is defined for all $a < t < a + \epsilon$ and $c'(a) := \lim_{t \to a^+} c'(t)$, $c'(b) := \lim_{t \to b^-} c'(t)$

**Definition 4.9**
A continuous map $c : [a, b] \to M$ is called a piecewise smooth curve if there exists a finite subdivision $a = t_0 < t_1 < \ldots < t_k = b$ such that the restrictions $c |_{[t_{i-1}, t_i]}$ are smooth curves for each $i$.

**Remark**
1. The length of a piecewise smooth curve $c$ is defined as $L(c) = \sum_{i=1}^{k} L(c |_{[t_{i-1}, t_i]})$
2. We say $c$ joins the points $c(a)$ and $c(b)$.

Given any two points $p, q \in M$ consider the set of all piecewise smooth curves joining $p$ and $q$ namely $C_{pq}([a, b]) = \{\text{piecewise smooth curves } c : [a, b] \to M : c(a) = p, c(b) = q\}$

**Remark**
For a connected manifold $M$, $C_{pq} \neq \emptyset$ as every connected smooth manifold is path connected.

**Definition 4.10**
We define a map $d : M \times M \to \mathbb{R}$ by $d(p, q) := \inf_{c \in C_{pq}} L(c)$. $d(p, q)$ is called the Riemannian Distance between $p$ and $q$.

**Lemma 4.11**
Any Riemannian Manifold equipped with the Riemannian Distance $d$ is a metric space whose induced topology agrees with the topology of the underlying set $M$

**Proof**
See Exercise Sheet 6 Question 4. ■

**Remark**
For $\epsilon > 0$ sufficiently small, a geodesic ball $\exp_p(B(0, \epsilon))$ coincides with the metric ball $B_\epsilon(p) := \{q \in M : d(p, q) < \epsilon\}$

**4.5 Minimising Properties of Geodesics**
The statement “geodesics are shortest paths” is true under certain conditions.
Minimising Curves are Geodesics

**Definition 4.12**
A piecewise smooth curve \( \gamma: [a, b] \to M \) is called a minimising curve if \( L(\gamma) \leq L(c) \) for all \( c \in C_{pq}([a, b]) \); that is any other piecewise smooth curve \( c \) joining \( p = \gamma(a) \) and \( q = \gamma(b) \).

An equivalent condition on \( \gamma \) is \( L(\gamma) = d(\gamma(a), \gamma(b)) \)

**Definition 4.13**
Let \( c: [a, b] \to M \) be a smooth curve. A smooth variation of \( c \) is a smooth map \( \alpha: [a, b] \times (-\epsilon, \epsilon) \to M \) mapping \((t, s) \to \alpha(t, s)\) such that \( \alpha(t, 0) = c(t) \) for all \( t \in [a, b] \).

Consider, for \( s \in (-\epsilon, \epsilon) \) the curve \( \alpha_s: [a, b] \to M \) mapping \( t \to \alpha_s(t) := \alpha(t, s) \) and then their lengths. We will give a formula for \( \frac{d}{ds}_{s=0} L(\alpha_s) \) under the assumption \( \alpha_0(t) = c(t) \) has unit speed parameterisation; that is \(|c'(t)| = 1\) for all \( t \) (with respect to the metric).

**Definition 4.14**
The velocity vector of the curve \( \alpha_t: (-\epsilon, \epsilon) \to M \) mapping \( s \to \alpha_t(s) := \alpha(t, s) \) at \( s = 0 \) denoted by \( V(t) := \alpha_t'(0) = \frac{da}{ds}(t, 0) \) gives (as \( t \) varies) a vector field along the given curve \( c(t) \) called the variational vector field of \( \alpha \).

**Theorem 4.15 (First Variation Formula of Length)**
Suppose \( c: [a, b] \to M \) is a smooth curve with unit speed parametrisation; that is \(|c'(t)| = 1\). Take \( \alpha: [a, b] \times (-\epsilon, \epsilon) \to M \) is a smooth variation of \( c \) with variational vector field \( V(t) \) as above. Then

\[
\frac{d}{ds}_{s=0} L(\alpha_s) = \langle V(t), c'(t) \rangle \bigg|_a^b - \int_a^b \langle V(t), D_t c'(t) \rangle \, dt
\]

**Remark**
More generally, for \( c(t) \) with \(|c'(t)| > 0\) for all \( t \), we have a similar formula:

\[
\frac{d}{ds}_{s=0} L(\alpha_s) = \langle V(t), \frac{c'(t)}{|c'(t)|} \rangle \bigg|_a^b - \int_a^b \langle V(t), D_t \frac{c'(t)}{|c'(t)|} \rangle \, dt
\]

**Idea of Proof of Theorem 4.15**

\[
\frac{d}{ds}_{s=0} L(\alpha_s) = \frac{d}{ds}_{s=0} \int_a^b |\alpha'_s(t)| \, dt = \int_a^b \frac{d}{ds}_{s=0} |\alpha'_s(t)| \, dt
\]

Now,
By the Symmetry Lemma in Exercise Sheet 4, \( D_s \frac{da}{dt} = D_t \frac{da}{ds} \). Therefore

\[
\frac{d}{ds} \bigg|_{s=0} \frac{da}{dt} = \langle D_t \frac{da}{ds}, \frac{da}{dt} \rangle \bigg|_{s=0} \equiv \frac{1}{2} \left( D_t \frac{da}{ds} \right) \bigg|_{s=0} \frac{da}{dt} \bigg|_{s=0}
\]

Therefore

\[
\frac{d}{ds} \bigg|_{s=0} L(\alpha_s) = \int_a^b \frac{d}{dt} \left( \frac{da}{ds} \cdot \frac{da}{dt} \right) \bigg|_{s=0} dt - \int_a^b \langle D_s \frac{da}{dt}, D_t \frac{da}{dt} \rangle \bigg|_{s=0} dt
\]

\[
= \langle \frac{da}{ds} \bigg|_{s=0}, \frac{da}{dt} \bigg|_{s=0} \rangle \bigg|_a^b - \int_a^b \langle D_s \frac{da}{dt} \bigg|_{s=0}, D_t \frac{da}{dt} \bigg|_{s=0} \rangle dt
\]

\[
= \langle V(t), c'(t) \rangle \bigg|_a^b - \int_a^b \langle V(t), D_t c'(t) \rangle dt \quad \blacksquare
\]

**Theorem 4.16**

If a smooth curve \( c: [a, b] \to M \) is minimising and it has constant speed parameterisation then it is a geodesic.

**Idea of Proof of Theorem 4.16**

Assume \( |c'(t)| = |c'(a)| \) for all \( t \). If it is zero, then \( c(t) \) is constant so by definition a geodesic. If it is not zero, then we apply the first variation formula:

Take a smooth variation \( \alpha: [a, b] \times (-\epsilon, \epsilon) \to M \) such that \( \alpha(a, s) = \alpha(a, 0) = c(a) \) and \( \alpha(b, s) = \alpha(b, 0) = c(b) \) so \( V(a) = V(b) = 0 \). By the general first variation formula of length, we have

\[
\frac{d}{ds} \bigg|_{s=0} L(\alpha_s) = -\frac{1}{|c'(a)|} \int_a^b \langle V(t), D_t c'(t) \rangle dt
\]

Since \( c(t) \) is minimising, \( L(c(t)) = L(\alpha_s) \leq L(\alpha_0) \) for all \( s \in (-\epsilon, \epsilon) \) hence \( \frac{d}{ds} \bigg|_{s=0} L(\alpha_s) = 0 \) therefore for any \( V(t) \), we have

\[
0 = \frac{1}{|c'(a)|} \int_a^b \langle V(t), D_t c'(t) \rangle dt
\]

Such a \( V(t) \) was arbitrary (from the arbitrary choice of smooth variation) and the inner product \( \langle \cdot, \cdot \rangle \) is positive definite. Therefore \( D_t c'(t) = 0 \) for all \( t \in [a, b] \); that is \( c(t) \) is a geodesic. \( \blacksquare \)

Given any curve \( c: [a, b] \to M \) mapping \( t \to c(t) \) we can find a diffeomorphism to a unit speed curve.
**Geodesics are Locally Minimising**

Consider the Riemannian Manifold \((S^2, g_{s^2})\). Given two points \(p, q \in S^2\) we can find two geodesics joining \(p\) and \(q\), \(\gamma_1, \gamma_2\) where \(L(\gamma_1) \geq \pi\) and \(L(\gamma_2) \leq \pi\). \(\gamma_1\) and \(\gamma_2\) are geodesics because they are a portion of great circles. In fact \(\gamma_1\) is minimising, but \(\gamma_2\) is not minimising. Therefore a geodesic need not be globally minimising.

**Theorem 4.17 (Geodesics are Locally minimising)**

Let \(p \in M, U\) a normal neighbourhood of \(p\) and \(B \subset U\) be a geodesic ball centred at \(p\). Then every geodesic segment \(\gamma : [0,1] \rightarrow B\) with \(\gamma(0) = p\) is minimising. That is, \(L(\gamma) \leq L(c)\) for all piecewise smooth curves \(c : [0,1] \rightarrow B\) with \(\gamma(0) = c(0)\) and \(\gamma(1) = c(1)\). Moreover, this inequality is strict unless \(c\) is a monotone reparameterisation of \(\gamma\).

**Proof**

Non Examinable! ■

A useful technical Lemma (used in the Proof of 4.17) is the following:

**Lemma 4.18 (Gauss Lemma)**

Let \(p \in M\) and \(v \in T_p M\) such that \(\exp_p(v)\) is well-defined. Let \(w \in T_p M \cong T_v (T_p M)\). Then

\[
\langle (\exp_p)_\ast (v), (\exp_p)_\ast (w) \rangle = \langle v, w \rangle
\]

**Proof**

Non-Examinable. ■

**Remark**

In particular if \(\langle v, w \rangle = 0\) then \(\langle (\exp_p)_\ast (v), (\exp_p)_\ast (w) \rangle = 0\). This means that the radial geodesics are orthogonal to the geodesics spheres. (See Exercise Sheet 6).
Chapter 5: Curvature
In this chapter $(M, g)$ is always a Riemannian manifold with a Riemannian Metric $g$ and a Levi-Civita Connection $\nabla$.

5.1 The Riemannian Curvature Tensor
Given $Z \in \mathfrak{X}(M)$ then in general $\frac{\partial}{\partial x^i} \nabla_{\frac{\partial}{\partial x^j}} Z - \frac{\partial}{\partial x^j} \nabla_{\frac{\partial}{\partial x^i}} Z \neq 0$. So $\nabla$ is not commutative in general. The Riemannian curvature tensor measures the extent to which the covariant derivative $\nabla$ is not commutative.

Definition 5.1
Let $M$ be a manifold with an affine connection $\nabla$. The curvature tensor of $\nabla$ is the map

$$R(X, Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

Observe that this map is $C^\infty(M)$-linear in $X, Y, Z$. The curvature tensor of the Levi-Civita Connection $\nabla$ is called the Riemannian Curvature Tensor.

Remark
1. $R$ is a tensor field of type $(1,3)$ that is $R \in \Gamma(TM \otimes T^*M \otimes T^*M \otimes T^*M)$.
2. The value of $R(X, Y)Z$ at a point $p \in M$ depends only on the values of $X_p, Y_p, Z_p$.
3. $R(X, Y)Z$ is antisymmetric in $X$ and $Y$.
4. Some people would define $R(X, Y)Z := \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z$.

Given a Riemannian manifold then the curvature tensor of its Levi-Civita Connection is called the Riemannian Curvature Tensor.

Example
Consider $(\mathbb{R}^n, g_{\text{eucl}})$ then $R(X, Y)Z = 0$ for any $X, Y, Z \in \mathfrak{X}(\mathbb{R}^n)$; recall that $\nabla_{\frac{\partial}{\partial x^i}} \frac{d}{dx^j} = 0$ for all $i, j$ and $[\frac{d}{dx^i}, \frac{d}{dx^j}] = 0$ for all $i, j$ so $R \left( \frac{d}{dx^i}, \frac{d}{dx^j} \right) V = \nabla_{\frac{\partial}{\partial x^i}} \nabla_{\frac{\partial}{\partial x^j}} V - \nabla_{\frac{\partial}{\partial x^j}} \nabla_{\frac{\partial}{\partial x^i}} V$ for $V = V^k \frac{d}{dx^k}$

Therefore $\nabla_{\frac{\partial}{\partial x^i}} V = V^k \nabla_{\frac{\partial}{\partial x^i}} \frac{d}{dx^k} = 0$ hence $R \left( \frac{d}{dx^i}, \frac{d}{dx^j} \right) = 0$.

Using the metric $g = \langle \cdot, \cdot \rangle$ we define the Riemannian Curvature Tensor of type $(0,4)$ which is denoted again by $R$ (or $R_M$) as:

$$R(C^\infty(M)\text{-linear})$$

$$\langle R(X, Y)Z, W \rangle$$

for all $X, Y, Z, W \in \mathfrak{X}(M)$

Local Expressions
Let $(x^i)$ be local coordinates. Then

$$R = R_{ijkl} dx^i \otimes dx^j \otimes dx^k \otimes dx^l, \quad d_i := \frac{d}{dx^i}$$

$$R = R_{ijkl} dx^i \otimes dx^j \otimes dx^k \otimes dx^l$$
And it is possible to find formulas for $R_{ijl}^i$ and $R_{ijkl}$.  

By definition, $R(X,Y,Z,W)$ is antisymmetric in $X,Y$.  In fact there are more symmetric properties of $R$:

**Proposition 5.2 (Algebraic Properties of the Riemannian Curvature Tensor)**

1. $R(X,Y,Z,W) = -R(Y,X,Z,W)$
   $R(X,Y,Z,W) = -R(X,Y,W,Z)$
3. First Bianchi Identity:
4. Second Bianchi Identity:
   $$\nabla_Z R(X,Y,V,W) + \nabla_X R(Y,Z,V,W) + \nabla_Y R(Z,X,V,W) = 0$$
   [Fix the last two entries and cycle the first three]

**Proof**

1. By Compatibility of the Metric with $\nabla$,
   
   $$Y(Z,W) = (\nabla_Y Z, W) + (Z, \nabla_Y W)$$
   
   $$\Rightarrow X(Y(Z,W)) = (\nabla_X \nabla_Y Z, W') + (\nabla_Y Z, \nabla_X W) + (\nabla_X Z, \nabla_Y W) + (Z, \nabla_X \nabla_Y W) \quad (1)$$

   Similarly:
   
   $$Y(X(Z,W)) = (\nabla_Y \nabla_X Z, W) + (\nabla_X Z, \nabla_Y W) + (\nabla_X Z, \nabla_Y W) + (Z, \nabla_Y \nabla_X W) \quad (2)$$

   By computing (1) – (2) we have:
   
   $$[X,Y](Z,W) = ((\nabla_X \nabla_Y - \nabla_Y \nabla_X)Z, W) + (Z, (\nabla_Y \nabla_Y - \nabla_Y \nabla_X)W) \quad (3)$$

   On the other hand
   
   $$[X,Y](Z,W) = (\nabla_{[X,Y]} Z, W) + (Z, \nabla_{[X,Y]} W) \quad (4)$$

   Then
   
   $$(3) - (4) \Rightarrow 0 = ((\nabla_X \nabla_Y - \nabla_Y \nabla_X)Z - \nabla_{[X,Y]} Z, W) + (Z, (\nabla_Y \nabla_Y - \nabla_Y \nabla_X)W - \nabla_{[X,Y]} W)
   
   = (R(X,Y)Z, W) + (Z, R(X,Y)W) \Leftrightarrow R(X,Y,Z,W) = -R(X,Y,W,Z)$$

2. We Apply the First Bianchi Identity Four Times:
   
   $$R(Y,Z,W,X) + R(Z,W,Y,X) + R(W,Y,Z,X) = 0$$
   $$R(W,X,Y,Z) + R(X,Y,W,Z) + R(Y,W,X,Z) = 0$$

   Then by using antisymmetric properties in part 1, on the sum of these identities we achieve the desired result
   
   $$R(Z,X,Y,W) = R(Y,W,Z,X)$$

3. See Exercise Sheet 7 for a proof of the First and Second Bianchi Identities.

**Extending $R$ to general tensor fields**

Recall the affine connection $\nabla$ on $M$ can be extended to general tensor fields:

$$\nabla: \mathcal{X}(M) \times \Gamma(T^*_g M) \rightarrow \Gamma(T^*_g M)$$ mapping $(X,T) \rightarrow \nabla_X T$

So we can extend the Levi-Civita Connection and the Riemannian Curvature Tensor. We can consider $R: \mathcal{X}(M) \times \mathcal{X}(M) \times \Gamma(T^*_g M) \rightarrow \Gamma(T^*_g M)$ mapping $(X,Y,T) \rightarrow \nabla_X \nabla_Y T - \nabla_Y \nabla_X T - \nabla_{[X,Y]} T$

**Properties (Proofs are on the Example Sheet)**
a. $R(X,Y)f = 0 \ \forall f \in \Gamma(T^0_0M) = C^\infty(M)$
b. $R(X,Y)(T_1 \otimes T_2) = (R(X,Y)T_1) \otimes T_2 + T_1 \otimes R(X,Y)T_2$
c. For any tensor contraction $c$, $R(X,Y)c(T) = c(R(X,Y)T)$

5.2 Sectional Curvature

Definition 5.3
Given a point $p \in M$ and a two-dimensional subspace $\sigma_p$ of $T_pM$ we take an orthonormal basis $\{u, v\}$ of $\sigma_p$ and define the sectional curvature of $M$ at $p$ associated to $\sigma_p$ as $k(\sigma_p) := R(u,v,v,u) = \langle R(u,v)v,u \rangle$

Remarks

a. $k(\sigma_p)$ does not depend on choice of basis of the orthonormal basis of $\sigma_p$. That is $R(u,v,v,u) = R(\tilde{u},\tilde{v},\tilde{v},\tilde{u})$ for orthonormal bases $\{u,v\}, \{\tilde{u},\tilde{v}\}$ of $\sigma_p$.

b. If $\{u,v\}$ is any basis of $\sigma_p$ (not necessarily orthonormal) then

$$k(\sigma_p) = \frac{\langle R(u,v)v,u \rangle}{\langle u,u \rangle \langle v,v \rangle - \langle u,v \rangle^2}$$

If dim $M = 2$ then $k(T_pM)$ is the Gauss Curvature of $(M,g)$ at $p$. See exercise Sheet 8 for details.

The Riemannian Curvature tensor determines all sectional curvature. The converse is also true; the sectional curvature completely determines the Riemannian Curvature Tensor.

Proposition 5.4
Assume that $R_1, R_2 : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to C^\infty(M)$ are two $C^\infty(M)$-linear maps satisfying the properties a, b, c in proposition 5.2. Then if $R_1(X,Y,Y,X) = R_2(X,Y,Y,X)$ for all $X,Y \in \mathfrak{X}(M)$ then $R_1 = R_2$; that is $R_1(X,Y,Z,W) = R_2(X,Y,Z,W)$ for all $X,Y,Z,W \in \mathfrak{X}(M)$

Proof
See GHL.

Proposition 5.5
Given a point $p \in M$ then the following properties are equivalent:

1. $k(\sigma_p) = c \in \mathbb{R}$ for all 2-planes $\sigma_p \subset T_pM$
2. $R(u,v)w = c(\langle v,w \rangle u - \langle u,w \rangle v)$ for all $u,v,w \in T_pM$.

Definition 5.6
A Riemannian Manifold is called a constant curvature Riemannian Manifold if $k(\sigma_p) = c$ for all two-planes $\sigma_p \subset T_pM$ for all $p \in M$.

A Riemannian manifold whose Riemannian curvature tensor vanishes (equivalently, if all sectional curvature vanishes) is called flat.

Remark
$(M,g)$ is flat $\iff (M,g)$ is locally isometric to some Euclidean Space $(\mathbb{R}^n, g_{Eucl})$ $\iff$ The Riemannian metric $g$ is flat.
Example
\( \mathbb{R}^n \) is flat and so \( R \equiv 0 \). \( T^n \) is flat.

**Model Spaces with Constant Curvature:**
1. The Euclidean Space \((\mathbb{R}^n, g_{\text{Eucl}})\) has constant curvature 0
2. The unit sphere \((S^n, g_{\text{round}})\) has constant curvature 1
3. The Hyperbolic Space has constant curvature -1

   a. Poincare half space model:
   \[ H^n = \{(x^1, \ldots, x^n) \in \mathbb{R}^n : x^n > 0\}, \quad g = \frac{(dx^1)^2 + \cdots + (dx^n)^2}{(x^n)^2} \]
   In the case \( n = 2 \) see Exercise Sheet 2.

   b. Poincare ball model
   \[ B^n = \{(x^1, \ldots, x^n) \in \mathbb{R}^n : (x^1)^2 + \cdots + (x^n)^2 < 1\} \]
   \[ g = \frac{4}{1 - [(x^1)^2 + (x^2)^2 + \cdots + (x^n)^2]} \]
   In the case \( n = 2 \) see Exercise Sheet 2.

**Additional Exercise:**
Compute the Sectional Curvature of these model spaces.

### 5.3 Ricci and Scalar Curvature

**Definition 5.7**
The Ricci Curvature Tensor is the map \( \text{Ric} : \mathcal{X}(M) \times \mathcal{X}(M) \to C^\infty(M) \) mapping \((X,Y) \to \text{Ric}(X,Y)\) where for each point \( p \in M \),
\[
\text{Ric}(X,Y)(p) = \text{Ric}_p(X,Y) := \text{tr} \left\{ T_p M \to T_p M \atop w \to R_p(w,X_p)Y_p \right\}
\]
Where we extend \( w \) to a vector field \( \bar{w} \) then use the Riemann Curvature Tensor:
\[
R_p(w,X_p)Y_p = R(\bar{w},X)Y(p)
\]

**Remarks**

1. \( w \to R_p(w,X_p)Y_p \) is an endomorphism of \( T_pM \)
2. \( \text{Ric}(X,Y) \) is \( C^\infty(M) \)-linear in \( X \) and \( Y \)
3. If \( e_1, \ldots, e_n \) is an orthonormal basis for \( T_pM \) then \( \text{Ric}_p(U,V) = \sum_{i=1}^n R_p(e_i,U,V,e_i) \)
   where \( U, V \in T_pM \). Therefore \( \text{Ric}(U,V) \) is symmetric in \( U \) and \( V \).
4. The above definition of \( \text{Ric} \) is independent of the choice of orthonormal basis:
   \[
   \sum_{i=1}^n R_p(e_i,U,V,e_i) = \sum_{i=1}^n R_p(\bar{e}_i,U,V,\bar{e}_i)
   \]
5. Let \( \{e_i\} \) be a local frame with dual \( \{\theta^i\} \). Locally we can write \( X = x^i e_i \) and \( Y = y^i e_i \)
   then \( \text{Ric}(X,Y) = x^k y^l R_{k l} \) where \( R_{k l} = \text{Ric}(e_k,e_l) = g^{ij} R_{ij} = R(e_i,e_k,e_l,e_j) \)
   (In Exercise Sheet 7, take \( e_i = \frac{d}{dx^i} \).

**Definition 5.8**
The Ricci Curvature of a Riemannian Manifold \((M, g)\) is the quadratic form associated to the
Ricci Curvature Tensor; that is the map
\( \rho: \mathfrak{X}(M) \to C^\infty(M) \) mapping \( X \to \rho(X) = \text{Ric}(X, X) \)

Remarks

1. For a unit tangent vector \( u \in T_pM \) that is \( |u| = 1 \), we choose an orthonormal basis
\( \{e_1 = U, e_2, ..., e_n\} \) and compute \( \rho_p(U) := \text{Ric}_p(U, U) = \sum_{i=2}^n k(U, e_i) \)

2. As the Ricci Curvature Tensor Ric is symmetric it is completely determined by the Ricci Curvature.

Given a point \( p \) in \( (M, g) \) we define an endomorphism \( \overline{\text{Ric}}_p: T_pM \to T_pM \) mapping \( U \to \overline{\text{Ric}}_p U \) given by \( (\overline{\text{Ric}}_p U, V) = \text{Ric}_p(U, V) \)

Definition 5.9
The Scalar Curvature of a Riemannian Manifold \( (M, g) \) is the function \( s: M \to \mathbb{R} \) mapping \( p \to s(p) := \text{tr}\{\overline{\text{Ric}}_p: T_pM \to T_pM\} \)

Remark
If \( \{e_i\} \) is a basis of \( T_pM \) then \( s(p) = \text{tr}\overline{\text{Ric}}_p = g^{ij} \text{Ric}(e_i, e_j) = g^{ij} R_{ij} \)

If \( \dim M = 2 \) then \( s(p) = 2 \kappa(T_pM) = 2 \times \text{Gauss Curvature} \)

Definition 5.10
A Riemannian Manifold \( (M, g) \) is called an Einstein Manifold if there exists a constant \( \lambda \in \mathbb{R} \) such that \( \overline{\text{Ric}}_g \) is Einstein \( \iff \lambda g \)

Namely, \( \text{Ric}_p(U, V) = \lambda g(U, V) \) for all \( U, V \in T_pM \) and for all \( p \in M \).

Remark

1. If \( (M, g) \) has constant curvature \( c \) and \( \dim M = n \) then \( (M, g) \) is an Einstein Manifold with \( \lambda = (n - 1)c \).
2. For \( n = 2 \) the Riemannian Curvature Tensor is entirely given by the scalar curvature.
3. For \( n = 3 \) the Riemannian Curvature Tensor is entirely given by the Ricci Curvature.

In particular, if \( \dim M = 3 \) then \( (M, g) \) is Einstein \( \iff \lambda = (n - 1)c \).
**Chapter 6: Jacobi Fields**

Jacobi Fields lead to a geometric interpretation of curvature.

### 6.1 Basic Facts about Jacobi Fields

**Definition 6.1**

Let \( \gamma: [a, b] \to M \) be a geodesic on \( M \) and \( \epsilon > 0 \). A variation of \( \gamma \) on \( M \) through geodesics is a smooth map \( \alpha: [a, b] \times (-\epsilon, \epsilon) \to \alpha(t, s) \) such that

1. \( \alpha(t, 0) = \gamma(t) \) for all \( t \in [a, b] \)
2. For fixed \( s \), the curve \( \alpha(t, s) \) is a geodesic.

Consider the variational vector field of \( \alpha \), that is \( J(t) := \frac{d\alpha(t, s)}{ds} \bigg|_{s=0} \). Since \( \alpha \) is smooth, \( J(t) \in \mathfrak{X}(\gamma) \) is smooth.

**Theorem 6.2**

Let \( \gamma \) be a geodesic and \( \alpha \) a variation of \( \gamma \) through geodesics. If \( J(t) \in \mathfrak{X}(\gamma) \) is the variational vector field of \( \alpha \) then it satisfies the following equation known as the Jacobi Equation:

\[
D^2_{\gamma} J(t) + R(J(t), \gamma'(t))\gamma'(t) = 0
\]

**Definition 6.3**

Let \( \gamma: [a, b] \to M \) be a geodesic. A vector field \( J \) along \( \gamma \) is said to be a Jacobi Field if it satisfies the Jacobi equation for all \( t \in [a, b] \).

**Proposition 6.4 (Existence and Uniqueness of Jacobi Fields)**

Let \( \gamma: [a, b] \to M \) be a geodesic and \( p = \gamma(a) \). For any pair \( U, V \in T_pM \) there exists a unique Jacobi Field \( J \) along \( \gamma \) satisfying the initial condition \( J(a) = U \), \( D_{\gamma} J(a) = V \).

**Remarks**

1. Let \( \gamma \) be a geodesic on a Riemannian Manifold \((M, g)\), \( \dim M = n \). Denote by \( J(\gamma) := \{ \text{Jacobi Fields along the geodesic } \gamma \} \). Then \( J(\gamma) \) is a \( 2n \)-dimensional linear subspace of \( \mathfrak{X}(\gamma) \). Therefore there exist \( 2n \) linearly independent Jacobi Fields along \( \gamma \).

   Note that by using a parallel basis \( \{ E_i(t) \} \) along \( \gamma(t) \) then the Jacobi Equation is just a system of linear ODEs.

2. Notice that \( \gamma'(t) \) and \( t \gamma'(t) \) are linearly independent Jacobi Fields along \( \gamma \). In fact any \( J \in J(\gamma) \) can be written uniquely as \( J = J_0 + (at + b)\gamma'(t) \) where \( \langle J_0, \gamma' \rangle = 0 \).

**Examples**

1. \((\mathbb{R}^2, g_{\text{Eucl}})\). \( R \equiv 0 \) so \( J(t) = U + tV \) where \( U \) and \( V \) are constant vector fields along \( \mathbb{R}^2 \).
2. Riemannian Manifold of Constant Curvature:

Let $\gamma$ be a unit speed geodesic, $f(t)$ a Jacobi Field normal to $\gamma$ (that is $\langle f(t), \gamma'(t) \rangle = 0$ and $J(0) = 0$. Then $J(t) = h(t)E(t)$ where $E(t)$ is a parallel vector field along $\gamma$, and $h(t) \in C^\infty(M)$.

**Proposition 6.5**

A vector field $J$ along a geodesic $\gamma$ is a Jacobi Field if and only if $J$ comes from a variation of $\gamma$ through geodesics.

**Corollary 6.6**

Let $p \in M$, $v \in T_p M$ and $\gamma(t) = \exp_p(tv)$ be a geodesic and $w \in T_p(T_p M) \equiv T_p M$. Then the Jacobi Field $J(t)$ along $\gamma$ such that $J(0) = 0$ and $D_t J(0) = w$ is given by the formula:

$$J(t) = (\exp_p)_* tv(tw)$$

### 6.2 Jacobi Fields and Curvature

We shall relate the rate of spreading of geodesics that start from the same point $p \in M$ with the curvature at $p$.

**Proposition 6.7**

Let $\gamma(t) = \exp_p(tv)$ be a unit speed geodesic on $M$ and take $w \in T_p(T_p M)$ with $|w| = 1$ and $\langle v, w \rangle = 0$. Consider $J \in J(\gamma)$ with $J(0) = 0$ and $D_t J(0) = w$. Then the Taylor Expansion of $|J(t)|^2$ about $t = 0$ is given by $|J(t)|^2 = t^2 - \frac{1}{3} k(v, w)t^4 + E(t)$ where $\lim_{t \to 0} \frac{E(t)}{t^4} = 0$

**Corollary 6.8**

With the same conditions as above,

$$|J(t)| = t - \frac{1}{6} k(v, w)t^3 + \overline{E(t)} \text{ where } \lim_{t \to 0} \frac{\overline{E(t)}}{t^3} = 0$$

**Remark**

The geodesics $\gamma(t) = \exp_p(tv)$ spread apart less than the rays $tv$ in $T_p M$ if $k_p(v, w) < 0$.

Given $v \in \mathbb{R}^n$ we can consider it as either a point, a vector or a constant vector field on $\mathbb{R}^n$. $tv$ is a geodesic in $T_p M$ $tw$ is a Jacobi Field along $tv$. Using the exponential map gives $\exp_p(tv) = \gamma(t)$ and $J(t) = \exp_p(tw)$.  

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Now we give a geometrical interpretation of sectional curvature, which measures a deviation of a Riemannian Manifold from Euclidean Space.

**Corollary 6.9**

In a 2-plane \( \sigma_p \) of \( T_p M \), consider a circle of radius \( r \) centred at the origin \( 0_p \) and denote by \( l(r) \) the length of \( c_r \), which is the image of the above circle under the exponential map \( \exp_p \).

Then \( \lim_{r \to 0} \frac{2\pi - l(r)}{r^3} = \frac{3}{2} k(\sigma_p) \).

![Diagram](image)

**Remark**

Suppose \( g_{ij}(x^1, \ldots, x^n) = \delta_{ij} - \frac{1}{3} R_{klji} x^k x^l + o(|x^3|) \) see Exercise Sheet 8 to use the Jacobi Field to compute the Taylor Expansion.

Therefore

\[
\frac{dV_j}{\sqrt{|\det(g_{ij})| dx^1 \wedge \ldots \wedge dx^n}} = \left( 1 - \frac{1}{6} \frac{R_{ij}}{\text{Ricci curvature}} x^i x^j + o(|x^3|) \right) dx^1 \wedge \ldots \wedge dx^n
\]

6.3 Second Variation Formula

We now compute \( \frac{d^2}{ds^2} \bigg|_{s=0} L(\alpha_s) \).

**Theorem 6.10 (Second Variation Formula)**

Let \( \gamma: [a, b] \to M \) be a unit speed geodesic, \( \alpha: [a, b] \times (-\epsilon, \epsilon) \to M \) a smooth variation of \( \gamma \) and \( V \) its variational vector field. Then

\[
\frac{d^2}{ds^2} \bigg|_{s=0} L(\alpha_s) = \int_a^b \left( |D_t V^\perp|^2 - R(V^\perp, \gamma', V^\perp, V^\perp) \right) dt + \left. \left( D_s |_{s=0} \frac{d\alpha}{ds} \cdot \gamma' \right) \right|_a^b
\]

Where \( V^\perp := V - \langle V, \gamma' \rangle \gamma' \) tangent part

**Definition 6.11**

The index form of the geodesic \( \gamma \) is a symmetric bilinear map defined on all smooth vector fields along \( \gamma \) which vanish at the end points, given by \( V(a) = W(b) = 0 \).

\[
I(V, W) := \int_a^b \langle D_t V^\perp, D_t W^\perp \rangle - R(V^\perp, \gamma', V^\perp, W^\perp) \ dt
\]

**Proposition 6.12**

For \( V \in \mathfrak{X}(\gamma) \) with \( V(a) = V(b) = 0 \) and \( \langle V, \gamma' \rangle = 0 \), it is a Jacobi Field if and only if \( I(V, W) = 0 \) for all \( W \in \mathfrak{X}(\gamma) \) with \( W(a) = W(b) = 0 \).
Remark

**Second Variation formula for the energy**

For a smooth curve \( c: [a, b] \to M \) we have \( l(c) = \int_a^b |c'(t)| \, dt \) for the length. We define the energy of the curve as \( E(c) := \frac{1}{2} \int_a^b |c'(t)|^2 \, dt \). Then we can find

\[
\frac{d}{ds} \bigg|_{s=0} E(\alpha_s) =? \quad \frac{d^2}{ds^2} \bigg|_{s=0} E(\alpha_s) = \int_a^b |D_t V|^2 - R(V, \gamma', \gamma', V) \, dt + \langle D_s \gamma|_{s=0}, \gamma' \rangle^b_a
\]

We can also define \( I(V, W) := \int_a^b \langle D_t V, D_t W \rangle - R(V, \gamma', \gamma', W) \, dt \)
Chapter 7: Classical Theorems in Riemannian Geometry

7.1 Hopf-Rinow Theorem

Definition 7.1
A Riemannian Manifold \((M, g)\) is called geodesically complete at \(p \in M\) if every geodesic passing through \(p\) is defined on the whole real line; that is the exponential map \(\exp_p\) is defined over the whole tangent space \(T_pM\). \((M, g)\) is called geodesically complete if it is geodesically complete at any point \(p \in M\), that is \(\exp\) is defined on all of \(TM\).

Examples
- \(S^2\) is geodesically complete
- Take \((\mathbb{R}^2, g_{Eucl})\) then \((D, g_{Eucl})\) is not geodesically complete.
- \((D, g_{Hyp})\) the Poincare disc is geodesically complete
- \((\mathbb{R}^2 \setminus \{0\}, g_{Eucl})\) is not geodesically complete.

Lemma 7.2
If \((M, g)\) is connected, let \(p \in M\). Suppose \(M\) is geodesically complete at \(p\). Then for any \(q \in M\) there exists a minimising geodesic joining \(p\) and \(q\).

Theorem 7.3 (Hopf-Rinow)
Let \((M, g)\) be a connected Riemannian Manifold, and let \(d\) be the Riemannian distance function on \(M\) with respect to \(g\). Then the four following conditions are equivalent:

1. \((M, d)\) is complete as a metric space. (Any Cauchy Sequence converges)
2. \((M, g)\) is geodesically complete.
3. \((M, g)\) is geodesically complete at a point \(p \in M\).
4. The closed and bounded subsets of \(M\) are compact.

Moreover, any of the above conditions implies that for any two points \(p, q \in M\) there exists a minimising geodesic joining \(p\) and \(q\).

Proof
Non-examinable. ■

Remark
A Riemannian Manifold if called complete if any of the four conditions are satisfied.

7.2 Catan-Hadward Theorem

Theorem 7.4 (Catan-Hadward)
Let \(M\) be a complete Riemannian manifold of constant sectional curvature \(k\). Suppose \(\dim M = n\) then \(M\) is diffeomorphic to \(\mathbb{R}^n\). More precisely, for any point \(p \in M\) \(\exp_p: T_pM \to M\) is a diffeomorphism.

Theorem 7.5 (Classification of Riemannian Manifolds of constant sectional curvature)
Let \(M\) be a complete Riemannian Manifold of constant sectional curvature \(k\). Suppose \(\dim M = n\). The \((M, g)\) is isometric to the space form \(S_k\) where:
\[
S_k := \begin{cases} 
(\mathbb{R}^n, g_{\text{Euc}}) & \text{if } k = 0 \\
(S^n, g_{\text{Euc}}) \text{ of radius } r = \frac{1}{\sqrt{k}} & \text{if } k > 0 \\
\{(x^1, \ldots, x^n) \in \mathbb{R}^n: x^n > 0\}, \quad \frac{g_{\text{Euc}}}{\frac{|k|(x^n)^2}{(dx^1)^2 + \ldots + (dx^n)^2}} & \text{if } k < 0 
\end{cases}
\]

### 7.3 Bonnet-Myers Theorem

**Definition 7.6**
The Diameter of a Riemannian Manifold \((M, g)\) is \(\text{diam}(M) := \sup \{d_g(p, q): p, q \in M\}\).

**Theorem 7.7 (Bonnet-Myers)**
Suppose \(R > 0\) and \((M, g)\) is a complete and connected Riemannian Manifold of dimension \(n\) for which \(\text{Ric} \geq \frac{n-1}{R^2}g\).

Then \(M\) is compact and has diameter at most \(\pi R\).

**Remarks**

1. \(\text{Ric}(X, X) \geq \frac{n-1}{R^2}g(X, X) \quad \forall X \in \mathfrak{X}(M) \iff \text{Ric} \geq \frac{n-1}{R^2}g\)
2. This theorem is sharp for a round sphere \(S^n(R)\) of radius \(R\). That is \(\text{Ric} = \frac{n-1}{R^2}g\) and \(\text{diam}(S^n(R)) = \pi R\).
3. The hypothesis cannot be relaxed to \(\text{Ric} > 0\). Indeed, the paraboloid \(\{(x, y, z) \in \mathbb{R}^3: x^2 + y^2 = z\}\) equipped with the induced metric satisfies \(\text{Ric} > 0\) but it is not compact, though it is complete.