Geometry of curve graphs and mapping class groups

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Young Researchers in Mathematics, 1–4 August 2016
Curves on surfaces

A *simple closed curve* in a surface $S$ is an embedding of the circle into $S$, in particular, it has no self-intersections. A curve is *essential* if it does not bound a disc in $S$ and *non-peripheral* if it does not cobound an annulus with a component of the boundary of $S$.

**Examples (Good curves)**

**Examples (Bad curves)**

From now on, a *curve* will mean an essential, non-peripheral simple closed curve.
The curve graph $C(S)$ has:

- vertices - isotopy classes of curves in $S$;
- an edge between vertices if they have representative curves which are disjoint.

**Example**

We make $C(S)$ into a metric space by giving each edge length 1.
A metric space is $k$-hyperbolic if every geodesic triangle in the space has a $k$-centre, that is, a point at distance at most $k$ from each of the three sides.

Theorem (Masur-Minsky)
The curve graph of each surface is $k$-hyperbolic, for some $k$. 
Mapping class groups

The mapping class group $\text{MCG}(S)$ of a surface $S$ is the group of isotopy classes of homeomorphisms from $S$ to itself.

Example (Torus $S_1$)

$$\text{MCG}(S_1) \cong \text{GL}_2(\mathbb{Z})$$

\[
\begin{pmatrix}
1 & 1 \\
0 & 1 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & 0 \\
1 & 1 \\
\end{pmatrix}
\]
The mapping class group $MCG(S)$ acts on the curve graph $\mathcal{C}(S)$ by isometries since:

- homeomorphisms take curves to curves (vertices $\rightarrow$ vertices)
- homeomorphisms take disjoint curves to disjoint curves (edges $\rightarrow$ edges)
- all the isotopy relations work out.

We can use this action to study the *geometry* of the mapping class group.
Geometry of groups

Let $G = \langle X \rangle$ be a group, $X = \{x_1, \ldots, x_n\}$ a finite generating set. The word metric on $G$ is given by defining $d(g, h)$ to be the length of the shortest word in the generators and their inverses representing $g^{-1}h$.

**Generating sets**

The word metric depends on the generating set $X$. However, the metrics given by two different generating sets are “approximately the same” - the distance between two elements doesn't change too much. This is made precise by the notion of quasi-isometry, a very useful equivalence relation when we are thinking about the geometry of groups.
Example (→)

\( F_2 = \langle x, y \rangle \), the free group on \( \{x, y\} \).

\[
d(1, x^2 y y^{-1} x y x^{-1}) = d(1, x^3 y x^{-1}) = 5
\]

Example (←)

\( \mathbb{Z}^2 = \langle a, b \rangle \), where \( a = (1, 0) \), \( b = (0, 1) \).

\[
d((0, 0), 3a + b - a) = d((0, 0), 2a + b) = 3
\]
To build a CAT(0) cube complex, start with cubes $[0, 1]^n$.

Glue them together in a nice way to get a simply-connected space. In particular, the link of every vertex should be a flag complex.
Medians

Let $X$ be a CAT(0) cube complex, and let $I(a, b)$ be the set of all shortest paths between $a$ and $b$ in the 1-skeleton of $X$.

**Proposition**

If $a$, $b$, $c$ are three vertices in $X$ then $I(a, b)$, $I(b, c)$ and $I(c, a)$ intersect in a unique vertex $m$, called the *median* of $a$, $b$ and $c$.
Recall: a metric space is $k$-hyperbolic if every geodesic triangle in the space has a $k$-centre.

The $k$-centre of a triangle need not be unique, but there is a bound on the diameter of the set of $k$-centres of a triangle in a $k$-hyperbolic space which depends only on $k$.

A $k$-centre is close to geodesics between all three pairs of points, so is “almost” a median of these three points.
Gromov hyperbolic spaces are “tree-like”

Any tree is a 0-hyperbolic space as in any geodesic triangle the median of three points is contained in all three of the geodesics joining pairs of points.

On the other hand, in an arbitrary $k$-hyperbolic space the distances between a finite number of points can be approximated by the distances in a tree joining those points.
Coarse median spaces

One way of generalising the concept of Gromov hyperbolicity is to think of spaces which are “like CAT(0) cube complexes” in the same way that Gromov hyperbolic spaces are “like trees”.

*Coarse median spaces* have the property that every finite set of points in the space can be approximated as vertices of a CAT(0) cube complex.

For any three points in the space we can find a bounded set of points which are “close” to being a median.

**Theorem (Bowditch)**

For each surface, the mapping class group $MCG(S)$ is coarse median.
Where next?

- The separating curve graph - the subgraph of $C(S)$ where we use only curves which cut the surface in two. This isn’t Gromov hyperbolic in general, but is it coarse median? (Work in progress.)

- Maybe this is more general. Is any sensible graph of curves we could associate to a surface coarse median?

- What else can we tell about the geometry of these graphs? A “distance formula” in terms of distances in curve graphs of subsurfaces exists for the mapping class group. Is this more general too?
Thank you!