Without otherwise mentioned, all curves are smooth and regular.

1. Complete all the exercises mentioned in the class and in the lecture notes.

2. Show that the mean curvature $H$ at $p \in S$ is given by

$$H = \frac{1}{\pi} \int_0^{\pi} \kappa_n(\theta) d\theta,$$

where $\kappa_n(\theta)$ is the normal curvature at $p$ along a direction making an angle $\theta$ with a fixed direction.

3. If the surface $S_1$ and $S_2$ intersect along a regular curve $C$, then the curvature $k$ of $C$ at $p$ is given by

$$k^2 \sin^2 \theta = \lambda_1^2 + \lambda_2^2 - 2\lambda_1 \lambda_2 \cos \theta,$$

where $\lambda_1$ and $\lambda_2$ are the normal curvatures at $p$, along the tangent line to $C$, of $S_1$ and $S_2$, respectively, and $\theta$ is the angle made up by the normal vectors of $S_1$ and $S_2$ at $p$.

4. Show that the Gaussian curvature of the surface $z = f(x, y)$, where $f$ is a smooth function, is

$$K = \frac{f_{xx} f_{yy} - f_{xy}^2}{(1 + f_x^2 + f_y^2)^2}.$$

5. a. Show that if $\sigma$ is an isothermal parametrization, that is, $E = G = \lambda(u, v)$ and $F = 0$, then the Gaussian curvature

$$K = -\frac{1}{2\lambda} \Delta(\ln \lambda),$$

where $\Delta \phi$ denotes the Laplacian $\frac{\partial^2 \phi}{\partial u^2} + \frac{\partial^2 \phi}{\partial v^2}$ of the function $\phi$.

b. Calculate the Gaussian curvature of the surface (upper half-plane model) with first fundamental form

$$\frac{dv^2 + du^2}{u^2}.$$
6. Show that there exists no surface $\sigma(u, v)$ such that $E = G = 1$, $F = 0$ and $L = 1$, $M = 0$ and $N = -1$.

7. Find the Gaussian curvature of each surface:
   a. Paraboloid $x^2 + y^2 = 2pz$.
   b. Torus $\sigma(u, v) = ((a + b \cos u) \cos v, (a + b \cos u) \sin v, b \sin u); 0 < b < a, 0 \leq u, v \leq 2\pi$.

8. When $E = G = 1$ and $F = \cos \theta$, show that
   \[ K = -\frac{\theta_{uv}}{\sin \theta}. \]

9. Suppose that the first and second fundamental forms of a surface patch are $Edu^2 + Gdv^2$ and $Ldu^2 + Ndv^2$. Show that the principal curvatures $\kappa_1 = \frac{L}{E}$ and $\kappa_2 = \frac{N}{G}$ satisfy the equations
   \[ (\kappa_1)_v = \frac{E_v}{2E}(\kappa_2 - \kappa_1), (\kappa_2)_u = \frac{G_u}{2G}(\kappa_1 - \kappa_2). \]