MA3D9 Example Sheet 5

1. Let $v$ and $w$ be tangent vector fields along a curve $\gamma : I \to S$. Prove that

$$
\frac{d}{dt} \langle v(t), w(t) \rangle = \langle \nabla_\gamma v, w \rangle + \langle v, \nabla_\gamma w \rangle.
$$

Solution:

$$
\frac{d}{dt} \langle v(t), w(t) \rangle = \langle \frac{dv(t)}{dt}, w(t) \rangle + \langle v(t), \frac{dw(t)}{dt} \rangle = \langle \nabla_\gamma v + (\frac{dv}{dt} \cdot N)N, w \rangle + \langle v, \nabla_\gamma w + (\frac{dw}{dt} \cdot N)N \rangle
$$

since $N$ is orthogonal to the tangent plane $T_\gamma S$.

2. Let $\gamma(s)$ be a curve parametrized by arclength $s$, with nonzero curvature. Consider the parametrized surface $\sigma(s, v) = \gamma(s) + vb(s), s \in I, -\epsilon < v < \epsilon, \epsilon > 0$,

where $b$ is the binormal vector of $\gamma$. Prove that if $\epsilon$ is small, $\sigma(I \times (-\epsilon, \epsilon)) = S$ is a regular surface over which $\gamma(I) = \sigma(I \times 0)$ is a geodesic.

Solution:

At $\sigma(I \times 0)$, $\sigma_s = t(s) - vt(s)$, $\sigma_v = b(s)$. So $\sigma_v \times \sigma_s = n(s) + vt(s) \neq 0$. Since it is continuous, when $\epsilon$ is small, $\sigma_v \times \sigma_s$ is still nonzero at $\sigma(I \times (-\epsilon, \epsilon)) = S$. So $S$ is a regular surface.

When $v = 0$, the normal vector of surface $N = n$. So $\gamma'' = \kappa n$, which is perpendicular to the tangent plane. So $\gamma(I) = \sigma(I \times 0)$ is a geodesic.

3. Let $T$ be a torus of revolution which we shall assume to be parametrized by

$$
\sigma(u, v) = ((r \cos u + a) \cos v, (r \cos u + a) \sin v, r \sin u).
$$

Prove that

a. If a geodesic is tangent to the parallel $u = \frac{\pi}{2}$, then it is entirely contained in the region of $T$ given by $-\frac{\pi}{2} \leq u \leq \frac{\pi}{2}$.

b. A geodesic that intersects the parallel $u = 0$ under an angle $\theta$ ($0 < \theta < \frac{\pi}{2}$) also intersects the parallel $u = \pi$ if $\cos \theta < \frac{a-r}{a+r}$.
Hint: use Clairaut’s relation.

Solution:

a.) Clairaut’s relation says that for a geodesic, let $R$ be the radius of the parallel of intersections and $\theta$ be the angle with the parallel, $R \cos \theta$ is fixed, which is $a$ in this case (notice $\theta = 0$). So for $R \cos \theta = a$ to hold in general, $R \geq a$. So $\cos u \geq 0$, which is $-\frac{\pi}{2} \leq u \leq \frac{\pi}{2}$.

b.) In this case $R \cos \theta = (a + r) \cos \theta$. To intersect $u = \pi$, $a - r \geq (a + r) \cos \theta_x = (a + r) \cos \theta$. But if the equality holds, the geodesic must tangent to $u = \frac{\pi}{2}$. Notice $u = \frac{\pi}{2}$ is also a geodesic since it is a critical point of $f(u) = r \cos u + a$. It contradicts to the fact that there is unique geodesic passing through a given point and tangent to a given direction. Hence $\cos \theta < \frac{a - r}{a + r}$.

4. Surface of Liouville are those surfaces for which it is possible to obtain a system of local coordinates $\sigma(u,v)$ such that the coefficients of the first fundamental form are given as

$$E = G = U(u) + V(v), F = 0.$$ 

Prove that

a. The geodesic of a surface of Liouville may be obtained by

$$\int \frac{du}{\sqrt{U - c}} = \pm \int \frac{dv}{\sqrt{V + c}} + c_1,$$

where $c$ and $c_1$ are constants that depend on the initial conditions.

b. If $\theta$, $0 \leq \theta \leq \frac{\pi}{2}$, is the angle which a geodesic makes with the curve $v = \text{const}$, then

$$U \sin^2 \theta - V \cos^2 \theta = \text{const}.$$ 

(this is the analogue of Clairaut’s relation.)

Solution:

a.) We only need to check that the curves given are geodesics since $c_1$ and $c$ gives the initial conditions for point and tangent direction respectively. To check they are geodesics, we play the following trick: introduce new coordinates

$$du' = \sqrt{U - c} du + \sqrt{V + c} dv$$
$$dv' = \frac{du}{\sqrt{U - c}} \pm \frac{dv}{\sqrt{V + c}}$$

The first fundamental form becomes $du'^2 + (U - c)(V + c)dv'^2$. In this fundamental form, $v' = \text{const}$ (the so called $u'$ curves) are geodesics by geodesic equations. These are exactly the curves in our statement.

b.) Suppose our geodesic is unit-speed, so $(U + V)(u^2 + v'^2) = 1$. So

$$v'^2 = \frac{1}{(U + V)(1 + \left(\frac{du}{dv}\right)^2)} = \frac{V + c}{(U + V)^2}$$
\[ u'^2 = \frac{1}{(U + V)(1 + (\frac{dn}{\partial a})^2)} = \frac{U - c}{(U + V)^2} \]

We calculate that \( \cos^2 \theta = (U + V)v'^2 \) and \( \sin^2 \theta = (U + V)u'^2 \). So

\[
U \sin^2 \theta - V \cos^2 \theta = \frac{U(V + c)}{U + V} - \frac{V(U - c)}{U + V} = c.
\]