MA4E0 Exercise Sheet 2

1. Find out the Lie algebra $\mathfrak{u}(n)$ associated to the unitary group $\mathrm{U}(n)$.

2. Find out the Lie algebra $\mathfrak{sp}(2n, \mathbb{R})$ of $\mathrm{Sp}(2n, \mathbb{R})$.

3. (a) Prove the exponention map $\exp : \mathfrak{so}(2) \to \mathrm{SO}(2)$ is surjective.
   (b) Prove the exponential map $\exp : \mathfrak{gl}(2, \mathbb{R}) \to \mathrm{GL}_+(2, \mathbb{R})$ is not surjective.
   (c) Prove the exponential map $\exp : \mathfrak{sl}(2, \mathbb{R}) \to \mathrm{SL}(2, \mathbb{R})$ is not surjective.

4. Let $H \subset \mathrm{GL}(3, \mathbb{R})$ be the Heisenberg group, *i.e.* the group of all 3×3 upper triangular real matrices whose diagonal entries are 1.
   (a) Find the Lie algebra $\mathfrak{h}$ of H.
   (b) Find the center $Z(H)$ of H, find its Lie algebra.
   (c) Prove the exponential map $\exp : \mathfrak{h} \to H$ is surjective.

5. Let $\mathfrak{g}$ be a two dimensional Lie algebra. Prove either $\mathfrak{g}$ is abelian, or there exists a basis $\{X, Y\}$ of $\mathfrak{g}$ so that $[X, Y] = Y$. Can you find the simply connected Lie group with the latter Lie algebra?

6. Suppose H is a Lie subgroup of G. Show that the Lie algebra $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$.

7. Show that $g(\exp X)g^{-1} = \exp(Ad_g X)$.