# On the Semi-Classical Brownian Bridge Measure

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**Abstract**

We prove an integration by parts formula for the probability measure induced by the semi-classical Riemmanian Brownian bridge over a manifold with a pole.

## 1 Introdcution

Let $M$ be a finite dimensional smooth connected complete and stochastically complete Riemannian manifold $M$ whose Riemannian distance is denoted by $r$. By stochastic completeness we mean that its minimal heat kernel satisfies that $\int p_t(x,y)dy = 1$. Denote by $C([0,1];M)$ the space of continuous curves: $\sigma : [0,1] \to M$, a Banach manifold modelled on the Wiener space. A chart containing a path $\sigma$ is given by a tubular neighbourhood of $\sigma$ and the coordinate map is induced from the exponential map given by the Levi-Civita connection on the underlying finite dimensional manifold. For $x_0, y_0 \in M$ we denote by $C_{x_0}M$ and $C_{x_0,y_0}M$, respectively, the based and the pinned space of continuous paths over $M$:

$$\begin{aligned} C_{x_0}M \quad &= \{\sigma \in C([0,1];M) : \quad \sigma(0) = x_0\}, \\ C_{x_0,y_0}M &= \{\sigma \in C([0,1];M) : \quad \sigma(0) = x_0, \quad \sigma(1) = y_0\}. \end{aligned}$$

The pullback tangent bundle of $C_{x_0}M$ consisting of continuous $v : [0,1] \to TM$ with $v(0) = 0$ and $v(t) \in T_{\sigma(t)}M$ where $\sigma \in C([0,1];M)$ which for each $\sigma$ can be identified by parallel translation with continuous paths on $T_{x_0}M$, the latter is identified with $\mathbb{R}^n$ with a frame $u_0$. To define gradient operators we make a choice of a family of $L^2$ sub-spaces together with an Hilbert space structure, and so we have a family of continuously embedded $L^2$ subspaces $\mathcal{H}_\sigma$ and the $L^2$ sub-bundle $\mathcal{H} := \cup_\sigma \mathcal{H}_\sigma$. Firstly denote by $H$ the Cameron-Martin space over $\mathbb{R}^n$,

$$H := \left\{ h \in C([0,1];\mathbb{R}^n) : h(0) = 0, |h|_{H^1} := \left( \int_0^1 |\dot{h}_s|^2 ds \right)^{\frac{1}{2}} < \infty \right\},$$

with $H^0$ its subset consisting of $h$ with $h(1) = 0$. If $/\!/.(\sigma)$ denotes stochastic parallel translation along a path $\sigma$ we denote by $\mathcal{H}_\sigma$ and $\mathcal{H}^0_\sigma$ the Bismut tangent spaces:

$$\mathcal{H}_\sigma = \{/\!/.(\sigma)h : h \in H\}, \qquad \mathcal{H}^0_\sigma = \{/\!/.(\sigma)h : h \in H, h(1) = 0\},$$

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specifying respectively the 'admissible' tangent vectors at $\sigma \in C_{x_0}M$ and vectors at $\sigma \in C_{x_0,y_0}M$. These vector spaces are given the inner product inherited from the Cameron-Martin space $H$.

For an $L^2$ analysis on $C_{x_0,y_0}M$ we need a probability measure on it which is usually taken to be the probability distribution of the conditioned Brownian motion. The heat kernel measure, the distribution of a Brownian sheet, offers also an alternative measure, see [25, 7, 27]. See also [23] for a study of the measure induced by a conditioned hypoelliptic stochastic process. If we suppose that $M$ has a pole $y_0$, by which we mean that the exponential map $\exp_{y_0} : T_{y_0}M \to M$ is a diffeomorphism, another probability measure, the probability distribution of the semi-classical Riemannian bridge, becomes available to us. For a simply connected Riemannian manifold with non-negative sectional curvature, every point is a pole. We denote this measure by $\nu = \nu_{x_0,y_0}$ and denote by $L^2(C_{x_0,y_0}M;\mathbb{R})$ the corresponding $L^2$ space.

A semi-classical Riemannian Brownian bridge $(\tilde{x}_s, s \leq 1)$ is a time dependent strong Markov process with generator $\frac{1}{2}\triangle + \nabla \log k_{1-s}(\cdot, y_0)$ where,

$$k_t(x_0,y_0) := (2\pi t)^{-\frac{n}{2}} e^{-\frac{r^2(x_0,y_0)}{2t}} J^{-\frac{1}{2}}(x_0),$$

and $J(y) = |\det D_{\exp_{y_0}^{-1}(y)} \exp_{y_0}|$ is the Jacobian determinant of the exponential map at $y_0$. Semi-classical Riemannian Brownian bridges (semi-classical bridge for short) were introduced by K. D. Elworthy and A. Truman [9]. For further explorations in this direction see [10] and [26]. If $p_t$ is the heat kernel, the Brownian bridge is a Markov process with generator $\frac{1}{2}\Delta + \nabla \log p_{1-t}(\cdot, y_0)$. Let us consider the two time dependent potential functions that drives the Brownian motion to the terminal value. They are close to each other as $t \to 1$, by Varadhan's asymptotic relations [29]: $(1-t)\log p_{1-t}(x,y_0) \sim -\frac{1}{2}r^2(x,y_0)$. There is also the relation $\lim_{t\to 1}(1-t)\nabla \log p_{1-t}(x,y) = -\dot{\gamma}(0)$ where $\gamma$ is normal geodesic from $y_0$ to $x$. The two drift vector fields $\nabla \log p_{1-t}(\cdot, y_0)$ and $\nabla \log k_{1-t}(\cdot, y_0)$ differ by $-\frac{1}{2}\nabla \log J$ near the terminal time.

Let us consider the unbounded linear differential operator $d$ on $L^2(C_{x_0,y_0}M;\mathbb{R})$ taking values in $L^2(\cup_\sigma \mathcal{H}_\sigma^*)$ where for $v \in \cup_\sigma \mathcal{H}_\sigma^*$,

$$\|v_\cdot(\cdot)\| := \left( \int_{C_{x_0,y_0}M} \left( |//_\cdot^{-1} v_\cdot(\sigma)|_H \right)^2 d\nu(\sigma) \right)^{\frac{1}{2}}.$$

Another norm can be given, taking into accounts of the damping effects of the Riccic curvature, which will be discussed later. As the distance function from the semi-classical bridge to the pole is precisely the $n$-dimensional Bessel bridge where $n = \dim(M)$, the advantage of the semi-classical Brownian bridge measure is that it is easier to handle, which we demonstrate by studying the elementary property of the divergence operator. Our main result is an integration by parts formula for $d$. Such a formula is believed to be equivalent to an integration by parts formula. A proof for the equivalence was given in [12] for compact manifold and for the Brownian motion measure by induction. The same method should work here. However since it is a bridge measure the current method has its advantages. First order Feyman-Kac type formulas together with estimates for the gradient of the Feyman-Kac kernel using semi-classical bridge process and the damped stochastic parallel translation was obtained in [24].

Denote by $OM$ the space of orthonormal frames over $M$ and $\{H_i\}$ the canonical horizontal vector fields on $OM$ associated to an orthonormal basis $\{e_i\}$ of $\mathbb{R}^n$ so that

$H_i$ is the horizontal lift of $ue_i$. For a tangent vector $v$ on $M$, we will denote by $\tilde{v}$ the horizontal lift of $v$ to $TOM$. Let $\{\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}\}$ be a filtered probability space on which is given a family of independent one-dimensional Brownian motions $\{B^i\}$. We define $B_t = (B_t^1, \dots, B_t^n)$. Let $u_0 \in \pi^{-1}(x)$ be a frame at $x$, $u_t$ and $\tilde{u}_t$ be the solution to the stochastic differential equations,

$$du_s = \sum_{i=1}^n H_i(u_s) \circ dB_s^i, \quad d\tilde{u}_s = \sum_{i=1}^n H_i(\tilde{u}_s) \circ dB_s^i + \tilde{A}_s(\tilde{u}_s)ds, \quad \tilde{u}_0 = u_0, \tag{1.1}$$

where $\circ$ denote Stratonovich integration and $A_s = \nabla \log k_{1-s}(\cdot, y_0)$. Then $\tilde{x}_s := \pi(\tilde{u}_s)$ is a semi-classical Brownian bridge from $x_0$ to $y_0$ in time 1. Let $\mathrm{Ric}_x$ denote the Ricci curvature at $x \in M$, by $\mathrm{Ric}_x^\sharp : T_xM \to T_xM$ we mean the linear map given by the relation $\langle \mathrm{Ric}_x^\sharp u, v\rangle = \mathrm{Ric}_x(u, v)$.

Denote $r = r(\cdot, y_0)$ for simplicity. We will need the following geometric conditions. Set

$$\Phi = \frac{1}{2} J^{\frac{1}{2}} \triangle J^{-\frac{1}{2}} = \frac{1}{4}|\nabla \mathrm{log} J|^2 - \frac{1}{4}\Delta(\mathrm{log} J). \tag{1.2}$$

**C1:** The Ricci curvature is bounded.
**C2:** $|\nabla\Phi| + |\nabla(\log J)| \le c(e^{ar^2} + 1)$ for some $c > 0$ and $a$ is an explicit constant to be given later.
**C3:** $\Phi$ is bounded from below.
**C4:** For each $t$, $k_t$ and $|\nabla k_t|$ are bounded, $|\nabla\Phi|$ is bounded.
The condition that the Ricci curvature is bounded ensures that the solution to the canonical SDE is differentiable in the sense of Malliavin calculus. It also implies that $|\tilde{W}_t|$ is bounded and that the integration by parts formula holds for the Brownian motion measure. Observe that $k_t$ and $|\nabla k_t|$ are bounded if $rJ^{-\frac{1}{2}}$ and $J^{-\frac{1}{2}}\nabla \log J^{-\frac{1}{2}}$ grow at most exponentially. Here we do not strive for the best possible conditions, as the optimal conditions will manifest themselves when Clark-Ocone formula and Poincaré inequalities are studied.

Our main results is the following integration by parts theorem.

**Theorem 1** *Assume* **C1- C4** *hold. Then for any $F, G \in \mathrm{Cyl}$ and $h \in H^0$ the following integration by parts formula hold.*

$$\begin{aligned}&\int_{C_{x_0,y_0}M} G(\tilde{x}_\cdot) dF\left(\tilde{u}_\cdot(\sigma)h_\cdot\right)\nu(d\sigma) + \int_{C_{x_0,y_0}M} F(\tilde{x}_\cdot) dG\left(\tilde{u}_\cdot(\sigma)h_\cdot\right)\nu(d\sigma)\\ =&\mathbf{E}\left[(FG)(\tilde{x}_\cdot)\int_0^1 \langle \dot{h}_s + \frac{1}{2}\tilde{u}_s^{-1}\mathrm{Ric}^\sharp(\tilde{u}_s h_s), d\tilde{B}_s\rangle\right] + \mathbf{E}\left[(FG)(\tilde{x}_\cdot)\int_0^1 d\Phi(\tilde{u}_s h_s)ds\right].\end{aligned}$$

*Here $d\tilde{B}_s = dB_s - \tilde{u}_s^{-1}\nabla \log k_{1-s}(\tilde{x}_s)\,ds$. In particular $d : \mathrm{Cyl} \subset L^2(C_{x_0,y_0}M) \to L^2(\cup_\sigma \mathcal{H}_\sigma)$ is closable, the domain of $d^*$ contains* Cyl *and*

$$d^*G = -dG + G\int_0^1 \langle \dot{h}_s + \frac{1}{2}\tilde{u}_s^{-1}\mathrm{Ric}^\sharp(\tilde{u}_s h_s), d\tilde{B}_s\rangle + G\int_0^1 d\Phi(\tilde{u}_s h_s)ds.$$

For based path space over a compact manifold, with Brownian motion measure (the Wiener measure), this was proved in [5], for non-compact manifolds see [12, 14], [16], [28], and [3]. For pinned manifolds with measure coming from the classical Brownian bridge measure, this was proved in [6] and [22].

Let us now clarify the definition of $d$. A common definition for $d$, which we use, is to take its initial domain to be Cyl, the set of cylindrical functions of the form $F(\sigma) =$

$f(\sigma_{t_1}, \ldots, \sigma_{t_m})$ where $m \in \mathcal{N}$, $0 < t_1 < t_2 < \cdots < t_m < 1$, and $f$ is a $BC^1$ function on the $m$-fold product space of $M$, or $\mathrm{Cyl}_0$ the subset containing $f(\sigma_{t_1}, \ldots, \sigma_{t_m})$ where $f$ is compactly supported. The $H$-derivative (Malliavin derivative) of $F$ in the direction of $u_\cdot(\sigma)h_\cdot \in T_\sigma C_{x_0}M$ is:

$$(dF)(/\!/_\cdot(\sigma)h_\cdot) = \sum_{k=1}^{m} \partial_k f(/\!/_{t_k}(\sigma)h_{t_k}),$$

where $\partial_k f$ denotes the derivative of $f$ in its $k$th component and $/\!/$ denotes parallel translation and identified with $u$ in the sequel. Denote by $G(s,t)$ and $G^0(s,t)$, respectively, the Green's functions of $\frac{d}{ds}$ on $(0,1)$ with suitable Dirichlet conditions: $G(s,t) = s \wedge t$ and $G^0(s,t) = s \wedge t - st$. Then

$$(\nabla F)(\sigma)(t) = \sum_{k=1}^{m} G(t_k, t)/\!/_{t_k,t}(\sigma)\nabla_k f(\sigma_{t_1}, \sigma_{t_2}, \ldots, \sigma_{t_m}),$$
$$(\nabla^0 F)(\sigma)(t) = \sum_{k=1}^{m} G^0(t_k, t)/\!/_{t_k,t}(\sigma)\nabla_k f(\sigma_{t_1}, \sigma_{t_2}, \ldots, \sigma_{t_m}),$$

where $\nabla_k f$ denotes the gradient of $f$ in the $k$th variable. We have

$$\|\nabla F\|^2 = \sum_{i,j=1}^{m} G(t_k, t_j)\langle /\!/_{t_k,t_j} \nabla_k f, \nabla_j f\rangle,$$
$$\|\nabla^0 F\|^2 = \sum_{i,j=1}^{m} G^0(t_k, t_j)\langle /\!/_{t_k,t_j} \nabla_k f, \nabla_j f\rangle.$$

It is an open problem whether the closure of $d$ with initial domain $BC^1$ agrees wth the closure of $d$ with initial value the cylindrical functions. This is the Markov uniqueness problem, this was studied In [13] where it was only proved that the latter including $BC^2$.

## 2 Proof of Theorem 1

To clarify the singularities at the terminal time we first prove a lemma concerning the divergence of a suitable vector field on the path space. Let $\tilde{u}_t$ be as defined by (1.1), $\tilde{x}_t = \pi(\tilde{u}_t)$. Recall that $k_t(x_0, y_0) = (2\pi t)^{-\frac{n}{2}} e^{-\frac{\rho^2(x_0,y_0)}{2t}} J^{-\frac{1}{2}}(x_0)$ and $\tilde{B}_s = B_s - \tilde{u}_s^{-1}\nabla \log k_{1-s}(\tilde{x}_s, y_0)\, ds$. The reference to $y_0$ will be dropped from time to time for simplicity. Define $\mathrm{ric}_u = u^{-1}\mathrm{Ric}^\sharp u$.

**Lemma 1** *Assume stochastic completeness,* **C2**, *and $h \in H^0$. Then the following integral exists,*

$$\int_0^1 \left\langle \dot{h}_s + \frac{1}{2}\mathrm{ric}_{\tilde{u}_s}(h_s), d\tilde{B}_s \right\rangle.$$

*Furthermore,*

$$\lim_{t\to 1} \mathbf{E}\left(\langle \nabla\, \mathrm{log} k_{1-t}(\cdot), \tilde{u}_t h_t\rangle\right)^2 = 0,$$

$\int_0^t \langle \dot h_s + \frac{1}{2}\mathrm{ric}_{\tilde u_s}(h_s), d\tilde B_s\rangle$ *converges, as* $t \to 1$*, in* $L^2(\Omega, \mathbb{P})$*; and*

$$\int_0^1 \left\langle \dot h_s + \frac{1}{2}\mathrm{ric}_{\tilde u_s}(h_s), d\tilde B_s \right\rangle = \int_0^1 \left\langle \dot h_s + \frac{1}{2}\mathrm{ric}_{\tilde u_s}(h_s), dB_s \right\rangle + \int_0^1 d\Phi(\tilde u_s h_s) ds$$
$$+ \int_0^1 \nabla d \left(\log k_{1-s}(\tilde x_s, y_0)\right)(\tilde u_s dB_s, \tilde u_s h_s).$$

**Proof** The singularities in the integral $\int_0^1 \left\langle \dot h_s + \frac{1}{2}\mathrm{ric}_{\tilde u_s}(h_s), d\tilde B_s \right\rangle$ come from the involvement of $\nabla \log k_{1-s}(\tilde x_s, y_0)$ and we only need to worry about

$$\alpha_t := \int_0^t \left\langle \dot h_s + \frac{1}{2}\mathrm{ric}_{\tilde u_s}(h_s), \tilde u_s^{-1} \nabla \log k_{1-s}(\tilde x_s, y_0) \right\rangle ds. \tag{2.1}$$

We integrate by parts to deal with $\int_0^t \left\langle \dot h_s, \tilde u_s^{-1} \nabla \log k_{1-s}(\tilde x_s, y_0) \right\rangle ds$, which involves the derivative of $h_s$. Since $\frac{D}{ds}(u_s h_s) = u_s \dot h_s$, by stochastic calculus applied to $d\left(\log k_{1-s}\right)(u_s h_s)$, where $d$ denotes spatial differentiation with respect to the $M$-valued variable, we see that

$$\begin{aligned}
&\langle \nabla \log k_{1-t}(\tilde x_t), \tilde u_t h_t\rangle \\
&= \int_0^t \left\langle \nabla \log k_{1-s}(\tilde x_s), \tilde u_s \dot h_s \right\rangle ds + \sum_{i=1}^n \int_0^t \nabla d\left(\log k_{1-s}\right)(\tilde u_s e_i, \tilde u_s h_s) dB_s^i \\
&\quad + \int_0^t \nabla d\left(\log k_{1-s}\right)\left(\nabla \log k_{1-s}(\tilde x_s), \tilde u_s h_s\right) ds \\
&\quad + \int_0^t \left(\frac{1}{2}\,\mathrm{trace}\,\nabla^2 + \partial_r\right)\left(D\left(\log k_{1-s}(\tilde x_s)\right)\right)(\tilde u_s h_s)\, ds,
\end{aligned}$$

the first term on the right hand side being $\alpha_t$. Since $\nabla \log k_{1-s}(x) = -\frac{r(x)\nabla r(x)}{1-s} + \nabla \log(J^{-\frac{1}{2}})$, $\Delta r = \frac{n-1}{r} + \langle \nabla r, \nabla \log J\rangle$, the following set of formulas are easy to verify.

$$\begin{aligned}
\Delta \log k_{1-s} &= -\frac{n}{1-s} - \frac{r\langle \nabla r, \nabla \log J\rangle}{1-s} - \frac{1}{2}\Delta(\log J), \\
\frac{\partial}{\partial s}\log k_{1-s} &= \frac{n}{2(1-s)} - \frac{r^2}{2(1-s)^2}, \\
|\nabla \log k_{1-s}|^2 &= \frac{r^2}{(1-s)^2} + \frac{1}{4}|\nabla \log J|^2 + \frac{r\langle \nabla r, \nabla \log J\rangle}{1-s}.
\end{aligned} \tag{2.2}$$

It follows that

$$\left(\frac{1}{2}\Delta + \frac{\partial}{\partial s}\right)(\log k_{1-s}) + \frac{1}{2}|\nabla \log k_{1-s}|^2 = \frac{1}{8}|\nabla \log J|^2 - \frac{1}{4}\Delta(\log J) = \Phi.$$

Let $\Delta^1 := (dd^* + d^* d)$ denote the Laplace-Beltrami Kodaira operator on differential 1-forms. By the Weitzenböck formula, $\left(\frac{1}{2}\,\mathrm{trace}\,\nabla^2 + \frac{\partial}{\partial s}\right) d = \left(\frac{1}{2}\Delta^1 d + \frac{1}{2}\mathrm{Ric}^\sharp d + \frac{\partial}{\partial s} d\right)$, and consequently,

$$\left(\frac{1}{2}\,\mathrm{trace}\,\nabla^2 + \frac{\partial}{\partial s}\right) d\left(\log k_{1-s}(\tilde x_s)\right)$$

$$
\begin{aligned}
&= d\left(\frac{1}{2}\Delta + \frac{\partial}{\partial s}\right)(\log k_{1-s}(\tilde{x}_s)) + \frac{1}{2}\mathrm{Ric}^\sharp\left(d\log k_{1-s}(\tilde{x}_s)\right) \\
&= -\frac{1}{2}d(|\nabla \log k_{1-s}(\cdot)|^2) + d\Phi + \frac{1}{2}\mathrm{Ric}^\sharp\left(d\log k_{1-s}(\tilde{x}_s)\right).
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\nabla d\left(\log k_{1-s}\right)\left(\nabla \log k_{1-s}(\tilde{x}_s), \tilde{u}_s h_s\right)\right) + \left(\frac{1}{2}\,\mathrm{trace}\,\nabla^2 + \frac{\partial}{\partial s}\right)\left(d\left(\log k_{1-s}\right)\right)(\tilde{u}_s h_s) \\
&= d\Phi(\tilde{u}_s h_s) + \frac{1}{2}\mathrm{Ric}\left(\nabla \log k_{1-s}(\tilde{x}_s), \tilde{u}_s h_s\right).
\end{aligned}
$$

Let us return to $\langle \nabla \log k_{1-t}(\tilde{x}_t), \tilde{u}_t h_t\rangle$:

$$
\begin{aligned}
&\langle \nabla \log k_{1-t}(\tilde{x}_t), \tilde{u}_t h_t\rangle \\
&= \int_0^t \left\langle \nabla \log k_{1-s}(\tilde{x}_s), \tilde{u}_s \dot{h}_s \right\rangle ds + \sum_{i=1}^n \int_0^t \nabla d\left(\log k_{1-s}\right)(\tilde{u}_s e_i, \tilde{u}_s h_s) dB_s^i \\
&\quad + \int_0^t d\Phi(\tilde{u}_s h_s)\, ds + \frac{1}{2}\int_0^t \mathrm{Ric}\left(\nabla \log k_{1-s}(\tilde{x}_s), \tilde{u}_s h_s\right)\, ds.
\end{aligned}
$$

We thus obtain the following relation:

$$
\begin{aligned}
\alpha_t &= \langle \nabla \log k_{1-t}(\tilde{x}_t), \tilde{u}_t h_t\rangle + \frac{1}{2}\int_0^t D\log k_{1-s}(\mathrm{Ric}^\sharp(\tilde{u}_s h_s))ds \\
&= \langle \nabla\ \log k_{1-t}(\cdot), \tilde{u}_t h_t\rangle - \int_0^t \langle \nabla \Phi, \tilde{u}_s h_s\rangle\, ds \\
&\quad - \sum_{i=1}^n \int_0^t \nabla d\left(\log k_{1-s}\right)(\tilde{u}_s e_i, \tilde{u}_s h_s)\, dB_s^i.
\end{aligned}
$$

We will prove that each of the terms on the right hand side converges as $t$ approaches 1. Furthermore $\langle \nabla\ \log k_{1-t}(\cdot), \tilde{u}_t h_t\rangle$ converges to zero. We first observe that there exists a constant $C$ such that $\mathbf{E}[r(\tilde{x}_t)^p] \le Ct^{\frac{p}{2}}$. Indeed $r_t := \rho(\tilde{x}_t, y_0)$ satisfies

$$
\begin{aligned}
r_t - r_0 &= \beta_t + \int_0^t \frac{1}{2}\Delta r(\tilde{x}_s)ds - \int_0^t \frac{r(\tilde{x}_s)}{1-s}ds - \frac{1}{2}\int_0^t \langle \nabla r, \nabla \log J\rangle_{\tilde{x}_s}\, ds \\
&= \beta_t + \int_0^t \frac{n-1}{2r_s}ds - \int_0^t \frac{r_s}{1-s}ds,
\end{aligned}
$$

where $\beta_t$ is a one dimensional Brownian motion and we have used the fact that $\Delta r = \frac{n-1}{r} + \langle \nabla r, \nabla \log J\rangle$. Thus $r_s$ is a Bessel bridge starting at $\rho(x_0, y_0)$ and ending at 0 at time 1. In particular $\lim_{t\uparrow 1} \tilde{x}_t = y_0$ and $(r_t, t \le 1)$ is a continuous semi-martingale. Furthermore for any $p > 1$, $\mathbf{E}[r(\tilde{x}_t)^p] \le Ct^{\frac{p}{2}}$. If $K_t$ denotes the standard Gaussian kernel on $\mathbb{R}^n$ then for $z_1, z_2 \in \mathbb{R}^n$ with $|z_1 - z_2| = \rho(x_0, y_0)$,

$$
\mathbf{E}[r(\tilde{x}_t, y_0)^p] = \frac{1}{K_1(z_1, z_2)}\int_{\mathbb{R}^n} |z - z_2|^p K_t(z_1, z)K_{1-t}(z, z_2)dz \le C|z - z_1|^{\frac{p}{2}}.
$$

We also know that $\mathbf{E}[e^{2ar_t^2}] < \infty$ for some $a$ and $t \le 1$, invoking condition C2.

We show below that (2.1) has a limit as $t \to 1$. Firstly, since $|d\Phi| \le ce^{ar^2}$,

$$
\lim_{t\to 1}\mathbf{E}\left[\int_t^1 \langle \nabla \Phi, \tilde{u}_s h_s\rangle\, ds\right]^2 = 0.
$$

We work with the first term on the right hand side. By the definition of $k_{1-t}$,

$$\langle \nabla\ \log k_{1-t}(\cdot, y_0), \tilde{u}_t h_t\rangle = \frac{r(\tilde{x}_t)\langle \nabla r(\tilde{x}_t), \tilde{u}_t h_t\rangle}{1-t} + \langle \nabla \log J_{\tilde{x}_t}^{-\frac{1}{2}}, \tilde{u}_t h_t\rangle.$$

Since $|d(\log J_x^{-\frac{1}{2}})| \le ce^{ar(x)^2}$, $\lim_{t\to 1}\langle \nabla \log J_{\tilde{x}_t}^{-\frac{1}{2}}, \tilde{u}_t h_t\rangle$ converges in $L^2(\Omega)$. Thus

$$\lim_{t\uparrow 1} \mathbf{E}\left|\langle \nabla \log J^{-\frac{1}{2}}(\tilde{x}_t), \tilde{u}_t h_t\rangle\right|^2 = 0, \tag{2.3}$$

using the fact that $h_t \to 1$. Also, by the symmetry of the Euclidean bridge, $\mathbf{E}[r^2(\tilde{x}_t, y_0)] \le C\,(t \wedge (1-t))$ and hence

$$\mathbf{E}\left|\frac{r(\tilde{x}_t)\,\langle \nabla r(\tilde{x}_t), \tilde{u}_t h_t\rangle}{1-t}\right|^2 \le C\frac{|h_t|^2}{1-t}.$$

Since $h_1 = 0$, and $h \in H$,

$$\frac{|h_t|^2}{1-t} = \frac{1}{1-t}\left|\int_t^1 \dot{h}_s dr\right|^2 \le \int_t^1 |\dot{h}_s|^2 ds \to 0,$$

as $t \to 1$, using the fact that $h \in H$. We conclude that

$$\lim_{t\to 1} \mathbf{E}\left[\langle \nabla\ \log k_{1-t}(\cdot), \tilde{u}_t h_t\rangle\right]^2 = 0.$$

For the final term we observe that

$$\nabla d\left(\log k_{1-s}\right)(\tilde{u}_s e_i, \tilde{u}_s h_s) = -\frac{\nabla r(\tilde{u}_s e_i)\nabla r(\tilde{u}_s h_s)}{1-s} - \frac{r\nabla dr(\tilde{u}_s e_i, \tilde{u}_s h_s)}{1-s}.$$

We further observe that the Frobenius norm of the Hessian of the distance function satisfies:

$$\|\nabla dr\|_F := \left(\sum_{i,j=1}^n \langle \nabla_{E_i}\partial r, E_j\rangle\right)^{\frac{1}{2}} \le \frac{1}{\sqrt{n-1}}\Delta r \le \frac{1}{\sqrt{n-1}}\left(\frac{n-1}{r} + \langle \nabla r, \nabla \log J\rangle\right).$$

Since $|\nabla \log J| \le ce^{ar^2}$, for some constant $C$, which may depend on $n$,

$$\mathbf{E}\left[\sum_{i=1}^n \int_0^t \nabla d\left(\log k_{1-s}\right)(\tilde{u}_s e_i, \tilde{u}_s h_s)\, dB_s^i\right]^2$$
$$\le C\int_0^t \frac{|h_s|^2}{(1-s)^2} ds \le C\frac{|h_t|^2}{1-t} + 4C\int_0^t |\dot{h}_s|^2 ds.$$

This follows from the following standard computation,

$$\int_0^t \frac{|h_s|^2}{(1-s)^2} ds = \frac{|h_t|^2}{1-t} - \int_0^t \frac{\langle h_s, 2\dot{h}_s\rangle}{1-s} ds \le \frac{|h_t|^2}{1-t} + 2\int_0^t |\dot{h}_s|^2 ds + \frac{1}{2}\int_0^t \frac{|h_s|^2}{(1-s)^2} ds.$$

This concludes the proof of the convergence of the integral. The required identity follows from the formula, given earlier, for $\alpha_t$. □

Let $u_t$ be the solution to the equation $du_t = \sum_{i=1}^n H_i(u_t) \circ dB_t^i$ with initial value $u_0 \in \pi^{-1}(x_0)$. Then $x_t := \pi(u_t)$ is a Brownian motion on $M$ starting at $x_0$ and the integration by parts formula holds on $L^2(C_{x_0}M; \mu)$. For any $F, G \in$ Cyl, and $h \in H(T_{x_0}M)$ with $h(0) = 0$, $d$ is the differential on $L^2(C_{x_0}M)$ with respect to the Brownian motion measure:

$$\mathbf{E}[dF(u.h.)G] = -\mathbf{E}[FdG(u.h.)] + \mathbf{E}\left[FG\int_0^1 \langle \dot{h}_s + \frac{1}{2}u_s^{-1}\mathrm{Ric}^\sharp(u_s h_s), dB_s\rangle\right]. \tag{2.4}$$

If $M$ is compact, see e.g. B. Driver [5]. This is also known to hold if the Ricci curvature is bounded from below. The divergence of $u.h.$ is

$$\mathrm{div}(u.h.) = \int_0^1 \langle \dot{h}_s + \frac{1}{2}u_s^{-1}\mathrm{Ric}^\sharp_{u_s}(u_s h_s), dB_s\rangle.$$

The following lemma completes the proof of Theorem 1.

**Lemma 2** *Suppose stochastic completeness,* **C2-C4***, and suppose that the integration by parts formula (2.4) holds for the Brownian motion measure. Then the conclusion of Theorem 1 holds.*

Let $h \in H^0$. Our plan is to pass the integration on the path space to the pinned path space by a Girsanov transform. We first observe that if $F \in \mathrm{Dom}(d)$, adapted to $\mathcal{G}_t$ where $t < 1$, then

$$\mathbf{E}[dF(\tilde{u}h.)] = \mathbf{E}\left[dF(uh.)\frac{k_{1-t}(x_t)}{k_1(x_0)}e^{-\int_0^t \Phi(x_s)ds}\right].$$

In fact, the formula for the probability density between the original probability measure, on $\mathcal{G}_t$, and the one for which $B_t - \int_0^t \langle u_s dB_s, \nabla \log k_{1-s}(x_s)\rangle$ is a Brownian motion, is:

$$M_t = \exp\left[\sum_{i=1}^m \int_0^t \langle \nabla \log k_{1-s}(x_s, y_0), u_s e_i\rangle dB_s^i - \frac{1}{2}\int_0^t |\nabla \log k_{1-s}(x_s, y_0)|^2 ds\right].$$

By an application of Itô's formula, and identities (2.2) in the proof of Lemma 1,

$$M_t = \frac{k_{1-t}(x_t, y_0)}{k_1(x_0, y_0)} \exp\left(-\int_0^t \Phi(x_s)ds\right).$$

Since the Brownian motion and the semi-classical bridge are conservative, then $(M_s, s \leq t)$ is a martingale for any $t < 1$.

Since $\Phi$ is bounded from below and has bounded derivative, $e^{-\int_0^t \Phi(\tilde{x}_s)ds}$ can be approximated by smooth cylindrical functions in the domain of $d$. Next we observe that

$$\nabla k_{1-s}(\cdot, y_0) = 2\pi(1-s)^{-\frac{n}{2}} e^{-\frac{r^2}{2(1-s)}} J^{-\frac{1}{2}}\left(-\frac{r\nabla r}{1-s} + \nabla \log J^{-\frac{1}{2}}\right),$$

is bounded and smooth, so $\frac{k_{1-t}(x_t, y_0)}{k_1(x_0, y_0)}e^{-\int_0^t \Phi(x_s)ds}$ belongs to the domain of $d$. Consequently, for $F, G$ measurable with respect to the canonical filtration up to time $t < 1$, we apply (2.4) to see

$$\mathbf{E}[GdF(\tilde{u}.h.] = \mathbf{E}\left[dF(u.h.)G(\tilde{x}.)M_t\right]$$

$$
\begin{aligned}
&= \mathbf{E}\left[(FG)(x.)M_t \,\mathrm{div}(u.h.)\right] - \mathbf{E}\left[(FG)(x.)dM_t(u.h.)\right] - \mathbf{E}[F(x.)dG(uh)M_t]\\
=&\mathbf{E}\left[F(x.)M_t\int_0^t \langle \dot h_s + \frac{1}{2}u_s^{-1}\mathrm{Ric}^\sharp_{u_s}(h_s), dB_s\rangle\right] - \mathbf{E}[F(\tilde x.)dG(\tilde u h)]\\
&- \mathbf{E}\left[F(x.)M_t d\left(\log k_{1-t}(x_t,y_0) - \int_0^t \Phi(x_s)ds\right)(u.h.)\right]\\
=&\mathbf{E}\left[F(\tilde x.)\int_0^t \langle \dot h_s + \frac{1}{2}\tilde u_s^{-1}\mathrm{Ric}^\sharp_{\tilde u_s}(h_s), d\tilde B_s\rangle\right] - \mathbf{E}[F(\tilde x.)dG(\tilde u h)]\\
&- \mathbf{E}\left[F(\tilde x.)\langle \nabla \log k_{1-t}(\tilde x_t,y_0), \tilde u_t h_t\rangle - F(\tilde x.)\int_0^t d\Phi(u_s h_s)ds\right].
\end{aligned}
$$

We take $t \uparrow 1$, by (2.3) and Lemma 1, $\lim_{t\uparrow 1}\langle \nabla \log k_{1-t}(\tilde x_t, y_0), \tilde u_t h_t\rangle = 0$ in $L^2$,

$$
\begin{aligned}
&\mathbf{E}[GdF(\tilde u.h.] + \mathbf{E}[F(\tilde x.)dG(\tilde u h)]\\
=&\mathbf{E}\left[(FG)(\tilde x)\int_0^1 \langle \dot h_s + \frac{1}{2}\tilde u_s^{-1}\mathrm{Ric}^\sharp(\tilde u_s h_s), d\tilde B_s\rangle\right] + \mathbf{E}\left[(FG)(\tilde x)\left(\int_0^1 d\Phi(\tilde u_s h_s)ds\right)\right].
\end{aligned}
$$

In particular, $\mathrm{Dom}(d^*) \supset \mathrm{Cyl}$,

$$
d^*1 = \int_0^1 \langle \dot h_s + \frac{1}{2}\tilde u_s^{-1}\mathrm{Ric}^\sharp(\tilde u_s h_s), d\tilde B_s\rangle + \left(\int_0^1 d\Phi(\tilde u_s h_s)ds\right),
$$

and $d^*$ is a closable operator. This completes the proof of the Lemma.

### 2.1 Comment

Let us consider briefly for which manifolds our assumptions on $\Phi$ hold. Denote by $\partial r$ the radial curvature which, evaluated at $x \in M$, is the unit vector field tangent to the normal geodesic between $x$ and the pole pointing away from the pole. The Hessian of $r$ describes the change of the Riemannian tensor in the radial directions, while the change of the volume form in the radial direction is associated to the Laplacian of $r$. More precisely we have:

$$
L_{\partial r} g = 2\,\mathrm{Hess}(r), \qquad L_{\partial r} d\mathrm{vol} = \Delta r d\mathrm{vol}, \qquad \Delta r = \frac{n-1}{r} + dr(\nabla \log J),
$$

indicating how the Jacobian determinant adjusts the speed of the convergence so that the semi-classical bridge behaves exactly like the Euclidean Brownian bridge.

For the Hyperbolic space, $\Phi$ is bounded from the formula below, $\Phi = -\frac{1}{8}(n-1)^2c^2 + \frac{1}{8}(n-1)(n-3)\left(\frac{1}{r^2} - c^2\sinh^{-2}(rc)\right)$. If $(N,o)$ is a model space, its Riemannian metric in the geodesic polar coordinates takes the form $g = dr^2 + f^2(r)d\theta^2$, then on $N \setminus \{o\}$, $\mathrm{Hess}(r) = \frac{f'(r)}{f(r)}(g - dr\otimes dr)$. For the hyperbolic space of constant sectional curvature $-c^2$, the Riemannian metric is $g = dr^2 + (\frac{1}{c}\sinh(cr))^2 d\theta^2$. Also $\mathrm{Hess}(r^2) = 2dr\otimes dr + 2cr\coth(cr)(g - dr\otimes dr)$. Furthermore its Jacobian determinant is $J = (\frac{\sinh(cr)}{cr})^{(n-1)}$.

For manifolds of non-constant curvature we may use the Hessian comparison theorem. The radial curvature at a point $x \in M$ is the sectional curvature in a plane at $T_xM$ containing the radial vector field $\partial_r$. Let us recall a comparison theorem from [19, R. E. Greene and H. Wu]: let $(N,o)$ be another Riemannian manifolds with a pole which we denote by $o$. Suppose that $(\gamma(t), t\in[0,b])$ is a normal geodesic in $M$ with the initial point $y_0$ and $(\gamma_2(t) : t \in [0,b])$ a normal geodesic in $N$ from $o$. We suppose that the radial curvature at $\gamma_2(t)$ is greater than or equal to the radial curvatures at $\gamma(t)$. By this we mean the curvature operator $\mathcal{R}$ on $M$ and $\mathcal{R}_2$ on $N$ satisfy the relation $\langle \mathcal{R}(w,\dot\gamma)w,\dot\gamma\rangle \le \langle \mathcal{R}_2(w_2,\dot\gamma_2)w_2,\dot\gamma_2\rangle$ for any unit vectors $w \in ST_{\gamma(t)}M$ and $w_2 \in ST_{\gamma_2(t)}N$, satisfying the relation $\langle w, \partial_r\rangle = \langle w_2, \partial_r\rangle$ where $\partial_r$ denotes the radial vector fields for both manifolds. Then for any nondecreasing function $\alpha : \mathbb{R}_+ \to \mathbb{R}$, Hess$(\alpha \circ r_2)(\gamma_2(t)) \le$ Hess$(\alpha \circ r)(\gamma(t))$, where $r_2$ is the Riemannian distance function on $N$ from $o$.

## 3 Conclusion

We have proved an integration by parts formula on $L^2(C_{x_0,y_0}, \nu)$ where $\nu$ is the probability measure induced by the semi-classical bridge. A probability measure $\mu$ on the path space is said to satisfy the Poincaré inequality if there exists a constant $c$ such that

$$\int \left(F - \int F d\mu\right)^2 d\mu \le c \int \left(|\nabla F|_{\mathcal{H}}\right)^2 d\mu$$

for all $F \in \mathcal{D}om(d)$ and the inner product on $\mathcal{H}$ can be defined either by stochastic parallel translation or by damped stochastic parallel translation.

**Conjecture.** A Poincaré inequality holds for the semi-classical bridge measure on a class of Cartan-Hadamard manifolds. Of course it is reasonable to assume growth conditions on $J$, $J^{-1}$ and suitable conditions on the range of the sectional curvature.

We remark that, for the Brownian bridge measure the question whether the Poincaré inequality holds is not solved satisfactorily. The spectral gap inequality is known to hold for Gaussian measure on $\mathbb{R}^n$ by L. Gross [20], who also made a conjecture on its validity. The spectral gal inequality has been proven to hold on the hyperbolic space [4], see also [1, 17, 2, 15, 11]. A counter example exists [8], see also the more recent articles [21, 18].

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