Solutions to the Exercises in Stochastic Analysis

Lecturer: Xue-Mei Li

Problem Sheet 1

In these solution I avoid using conditional expectations. But do try to give alternative proofs once we learnt conditional expectations.

Exercise 1 For $x, y \in \mathbb{R}^d$ define $p_t(x, y) = (2\pi t)^{-\frac{d}{2}} e^{-\frac{|x-y|^2}{2t}}$. Prove that $P_t(x, dy) = p_t(x, y) dy$ satisfies the Chapman-Kolmogorov equation: for $\Gamma$ Borel subset of $\mathbb{R}^d$,

$$P_{t+s}(x, \Gamma) = \int_{\mathbb{R}^d} P_s(y, \Gamma) P_t(x, dy).$$

Solution: One one hand

$$P_{t+s}(x, \Gamma) = (2\pi (t+s))^{-\frac{d}{2}} \int_{\Gamma} e^{-\frac{|y-x|^2}{2(t+s)}} dz.$$

On the other hand,

$$\int_{\mathbb{R}^d} P_s(y, \Gamma) P_t(x, dy) = (2\pi t)^{-\frac{d}{2}} (2\pi s)^{-\frac{d}{2}} \int_{\mathbb{R}^d} \int_{\Gamma} e^{-\frac{|y-z|^2}{2s}} e^{-\frac{|y-x|^2}{2t}} dz dy.$$

We now complete the squares in $y$.

$$-\frac{|y-z|^2}{2s} - \frac{|y-x|^2}{2t} = -\frac{1}{2} \frac{t+s}{st} \left| y - tx + sy \right|^2 - \frac{1}{2} \frac{|x-z|^2}{t-s}.$$
next we change the variable $y - \frac{t+sx}{t+s}$ to $\tilde{y}$, then

$$\int_{\mathbb{R}^d} P_s(y, \Gamma)P_t(x, dy)$$

$$= (2\pi t)^{-\frac{d}{2}} (2\pi s)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-\frac{1}{2} \frac{t+s}{s} |\tilde{y}|^2} e^{-\frac{1}{2} \frac{t-s}{t-s} |x-z|^2} dz d\tilde{y}$$

$$= (2\pi t)^{-\frac{d}{2}} (2\pi s)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-\frac{1}{2} \frac{t+s}{s} |\tilde{y}|^2} d\tilde{y} \int_{\mathbb{R}^d} e^{-\frac{1}{2} \frac{t-s}{t-s} |x-z|^2} dz$$

$$= (2\pi t)^{-\frac{d}{2}} (2\pi s)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-\frac{1}{2} \frac{|x-z|^2}{2(t-s)}} dz.$$

□

**Exercise 2** Let $(X_t, t \geq 0)$ be a Markov process with $X_0 = 0$ and transition function $p_t(x, y)dy$ where $p_t(x, y)$ is the heat kernel. Prove the following statements.

1. For any $s < t$, $X_t - X_s \sim P_{t-s}(0, dy)$;
2. Prove that $(X_t)$ has independent increments.
3. For every number $p > 0$ there exists a constant $c(p)$ such that

   $$\mathbb{E}|X_t - X_s|^p = c(p)|t - s|^\frac{p}{2}.$$

4. State Kolomogorov’s continuity theorem and conclude that for almost surely all $\omega$, $X_t(\omega)$ is locally Hölder continuous with exponent $\alpha$ for any number $\alpha < 1/2$.

5. Prove that this is a Brownian motion on $\mathbb{R}^d$.

**Solution:** Let $f$ be a bounded measurable function.

1. Since $(x_s, x_t) \sim P_s(0, dx)P_{t-s}(x, dy)$,

   $$\mathbb{E}f(x_t - x_s) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(y-x)p_s(0,x)p_{t-s}(x,y)dxdy$$

   $$= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(z)p_s(0,x)p_{t-s}(0,z)dxdz$$

   $$= \int_{\mathbb{R}^d} f(z)p_{t-s}(0,z)dz \int_{\mathbb{R}^d} p_s(0,x)dxdz = \int_{\mathbb{R}^d} f(z)p_{t-s}(0,z)dz.$$

Hence $x_t - x_s \sim P_{t-s}(0, dz)$. 
(2) Let us fix \(0 = t_0 < t_1 < t_2 < \ldots < t_n\) and Borel sets \(A_i \in B(\mathbb{R})\), \(i = 1, \ldots, n\). Let \(f_i(x) = 1_{x \in A_i}\) where \(A_i\) are Borel measurable set. Then we obtain
\[
\mathbb{P}(X_{t_1} \in A_1, \ldots, X_{t_n} - X_{t_{n-1}} \in A_n) = \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} f_1(x_1)f_2(x_2 - x_1) \cdots f_n(x_n - x_{n-1})
\]
\[
\times p_{t_1}(0, x_1)p_{t_2 - t_1}(x_1, x_2) \cdots p_{t_n - t_{n-1}}(x_{n-1}, x_n) \, dx_n \cdots dx_1,
\]
where in the last line we have used the identity (ii). Introducing new variables:
\[
y_1 = x_1, \quad y_2 = x_2 - x_1, \ldots, y_n = x_{n-1} - x_n,
\]
we obtain
\[
\mathbb{P}(X_{t_1} \in A_1, \ldots, X_{t_n} - X_{t_{n-1}} \in A_n) = \prod_{i=1}^{n} \int_{\mathbb{R}} f_i(y_i) p_{t_i - t_{i-1}}(0, y_i) \, dy_i.
\]
This means that \(\{X_{t_i - t_{i-1}}, i = 1, \ldots, n\}\) are independent random variables.

(3)
\[
\mathbb{E}|x_t - x_s|^p = \frac{1}{(2\pi(t-s))^d/2} \int_{\mathbb{R}^d} |z|^p e^{-|z|^2/(4(t-s))} \, dz
\]
\[
= \frac{1}{(2\pi(t-s))^{d/2}} \int_{\mathbb{R}^d} (t-s)^{d/2} |y|^p e^{-|y|^2/2(t-s)} \, dy
\]
\[
= (t-s)^{d/2} \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} |y|^p e^{-|y|^2/2} \, dy.
\]
The integral is finite. Let
\[
c(p) = \int_{\mathbb{R}^d} |y|^p p_1(0, y) \, dy,
\]
which is the \(p\)th moment of a \(N(0, I_{d \times d})\) distributed variable.

(4) and (5) are straight forward application of Kolmogorov’s theorem. □

**Exercise 3** If \((B_t)\) is a Brownian motion prove that \((B_t)\) is a Markov process with transition function \(p_t(x, y)\).
Solution: Let us denote for simplicity $f_i(x) = 1_{x \in A_i}$. Furthermore, we define the random variables $Y_i = X_{t_i} - X_{t_{i-1}}$, for $i = 1, \ldots, n$, where we have postulated $t_0 = 0$. From the properties of the Brownian motion we obtain that the random variables $\{Y_i\}$ are independent and moreover $Y_i \sim N(0, t_i - t_{i-1})$. Thus, we have

\[
P[ X_{t_1} \in A_1, \ldots, X_{t_n} \in A_n ] = \mathbb{E} \prod_{i=1}^{n} f_i(X_{t_i})
\]

\[
= \mathbb{E} [ f_1(Y_1) f_2(Y_2 + Y_1) \ldots f_n(Y_n + \ldots + Y_1) ]
\]

\[
= \int_{\mathbb{R}} \ldots \int_{\mathbb{R}} f_1(y_1) f_2(y_2 + y_1) \ldots f_n(y_n + \ldots + y_1)
\]

\[
\times p_{t_1}(0, y_1) p_{t_2 - t_1}(0, y_2) \ldots p_{t_n - t_{n-1}}(0, y_n) dy_n \ldots dy_1.
\]

Now we introduce new variables: $x_1 = y_1, x_2 = y_2 + y_1, \ldots, x_n = y_n + \ldots + y_1$, and obtain that the last integral equals

\[
\int_{\mathbb{R}} \ldots \int_{\mathbb{R}} f_1(x_1) f_2(x_2) \ldots f_n(x_n)
\]

\[
\times p_{t_1}(0, x_1) p_{t_2 - t_1}(0, x_2 - x_1) \ldots p_{t_n - t_{n-1}}(0, x_n - x_{n-1}) dx_n \ldots dx_1.
\]

Noticing that $p_{t}(0, y - x) = p_{t}(x, y)$ and recalling the definition of the functions $f_i$, so the finite dimensional distribution agrees with that of the Markov process determined by the heat kernels. The two processes must agree.

Exercise 4 Let $(X_t, t \geq 0)$ be a continuous real-valued stochastic process with $X_0 = 0$ and let $p_t(x, y)$ be the heat kernel on $\mathbb{R}$. Prove that the following statements are equivalent:

(i) $(X_t, t \geq 0)$ is a one dimensional Brownian motion.

(ii) For any number $n \in \mathbb{N}$, any sets $A_i \in \mathcal{B}(\mathbb{R})$, $i = 1, \ldots, n$, and any $0 < t_1 < t_2 < \ldots < t_n$,

\[
P[ X_{t_1} \in A_1, \ldots, X_{t_n} \in A_n ]
\]

\[
= \int_{A_1} \ldots \int_{A_n} p_{t_1}(0, y_1) p_{t_2 - t_1}(y_1, y_2) \ldots p_{t_n - t_{n-1}}(y_{n-1}, y_n) dy_n \ldots dy_1.
\]

Solution: This follows from the previous exercises.
Exercise 5  A zero mean Gaussian process $B^H_t$ is a fractional Brownian motion of Hurst parameter $H$, $H \in (0, 1)$, if its covariance is

$$\mathbb{E}(B^H_t B^H_s) = \frac{1}{2} (t^{2H} + s^{2H} - |t - s|^{2H}) .$$

Then $\mathbb{E}|B^H_t - B^H_s|^p = C|t - s|^{pH}$. If $H = 1/2$ this is Brownian motion (Otherwise this process is not even a semi-martingale). Show that $(B^H_t)$ has a continuous modification whose sample paths are Hölder continuous of order $\alpha < H$.

Solution: Since $\mathbb{E}|B^H_t - B^H_s|^p = C|t - s|^{pH} = C|t - s|^{1+(pH-1)}$, we can apply the Kolmogorov continuity criterion to obtain that $B^H_t$ has a modification whose sample paths are Hölder continuous of order $\alpha < (pH - 1)/p$. This means that for any $\alpha < H$ we can take $p$ large enough to have $\alpha < (pH - 1)/p$. This finishes the proof. □

Exercise 6  Let $(B_t)$ be a Brownian motion on $\mathbb{R}^d$. Let $T$ be a positive number. For $t \in [0, T]$ set $Y_t = B_t - \frac{t}{T} B_T$. Compute the probability distribution of $Y_t$.

Solution:

$$\mathbb{E}f(B_t - \frac{t}{T} B_T) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f \left( x - \frac{t}{T} y \right) (2\pi T)^{-d/2} e^{-\frac{|x|^2}{2T}} (2\pi (T - t))^{-d/2} e^{-\frac{|y|^2}{2(T-t)}} dxdy.$$

We observe that

$$|x|^2 = \left| x - \frac{t}{T} y \right|^2 + \frac{2t}{T} \langle x, y \rangle - \frac{t^2}{T^2} |y|^2$$

and that

$$|x - y|^2 = \left| x - \frac{t}{T} y \right|^2 - 2 \frac{(T-t)}{T} \langle x, y \rangle + \frac{2t(T-t)}{T^2} |y|^2 + \frac{(T-t)^2}{T^2} |y|^2 .$$

Thus,

$$\frac{|x|^2}{t} + \frac{|y - x|^2}{(T-t)} = \frac{1}{t} \left| x - \frac{t}{T} y \right|^2 + \frac{1}{T-t} \left| x - \frac{t}{T} y \right|^2 + \frac{1}{T} |y|^2 .$$
Finally

\[ \mathbb{E} f (B_t - \frac{t}{T} B_T) = (2\pi t)^{-\frac{d}{2}} (2\pi (T - t))^{-\frac{d}{2}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f \left( x - \frac{t}{T} y \right) e^{-\frac{1}{2t} |x - \frac{t}{T} y|^2} e^{-\frac{1}{2 (T - t)} |x - \frac{t}{T} y|^2} e^{-\frac{|y|^2}{2T}} \, dx \, dy \]

\[ = (2\pi t)^{-\frac{d}{2}} (2\pi (T - t))^{-\frac{d}{2}} \int_{\mathbb{R}^d} f(z) e^{-\frac{1}{2t} |z|^2} e^{-\frac{1}{2 (T - t)} |z|^2} \, dz \int_{\mathbb{R}^d} e^{-\frac{|y|^2}{2T}} \, dy \]

\[ = \int_{\mathbb{R}^d} f(z) p_t(0, z)p_{T-t}(z, 0) \, dz \cdot (2\pi T)^{\frac{d}{2}}. \]

Finally we see

\[ B_t - \frac{t}{T} B_T \sim \frac{p_t(0, z)p_{T-t}(z, 0)}{p_T(0, 0)} \, dz. \]
Problem Sheet 2. Brownian Motion, Conditional Expectation, and Uniform Integrability

Exercise 7 Let \((B_t)\) be a standard Brownian motion. Prove that

(a) (i) for any \(t > 0\), \(\mathbb{E}[B_t] = 0\) and \(\mathbb{E}[B_t^2] = t\);
   (ii) for any \(s, t \geq 0\), \(\mathbb{E}[B_s B_t] = s \wedge t\), where \(s \wedge t = \min(s, t)\).

(b) (scaling invariance) For any \(a > 0\), \(\frac{1}{\sqrt{a}} B_{at}\) is a Brownian motion;

(c) (Translation Invariance) For any \(t_0 \geq 0\), \(B_{t_0+t} - B_{t_0}\) is a standard Brownian motion;

(d) If \(X_t = B_t - tB_1\), \(0 \leq t \leq 1\), then \(\mathbb{E}(X_s X_t) = s(1-t)\) for \(s \leq t\). Compute the probability distribution of \(X_t\).

   Hint: break \(X_t\) down as the sum of two independent Gaussian random variables, then compute its characteristic function).

Solution: (a) (i) Since the distribution of \(B_t\) is \(\mathcal{N}(0, t)\), we have \(\mathbb{E}[B_t] = 0\) and \(\mathbb{E}[B_t^2] = t\).

(a) (ii) We fix any \(t \geq s \geq 0\). Then we have

\[
\mathbb{E}[B_s B_t] = \mathbb{E}[(B_t - B_s) B_s] + \mathbb{E}[B_s^2].
\]

Since the Brownian motion has independent increments, the random variables \(B_t - B_s\) and \(B_s\) are independent and we have

\[
\mathbb{E}[(B_t - B_s) B_s] = \mathbb{E}[B_t - B_s] \mathbb{E}[B_s] = 0.
\]

Furthermore, from (i) we know that \(\mathbb{E}[B_s^2] = s\). Hence, we conclude

\[
\mathbb{E}[B_s B_t] = s,
\]

which is the required identity.

(b) Let us denote \(W_t = \frac{1}{\sqrt{a}} B_{at}\). Then \(W\) has continuous sample paths and independent increments, which follows from the same properties of \(B\). Furthermore, for any \(t > s \geq 0\), we have

\[
W_t - W_s = \frac{1}{\sqrt{a}} (B_{at} - B_{as}) \sim \mathcal{N}(0, \frac{a(t-s)}{a}) = \mathcal{N}(0, t-s),
\]

which finishes the proof.
(c) If we denote $W_t = B_{t_0+t} - B_{t_0}$, then $W$ has continuous sample paths and independent increments, which follows from the same properties of $B$. Moreover, for any $t > s \geq 0$, we have

$$W_t - W_s = B_{t_0+t} - B_{t_0+s} \sim \mathcal{N}(0, (t_0 + t) - (t_0 + s)) = \mathcal{N}(0, t - s) ,$$

which finishes the proof.

(d) For $t = 0$ or $t = 1$ we have $X_t = 0$. For $1 > t \geq s > 0$ we have

$$\mathbb{E}[X_t X_s] = \mathbb{E}[(B_t - tB_1)(B_s - sB_1)]
= \mathbb{E}[B_t B_s] - s\mathbb{E}[B_t B_1] - t\mathbb{E}[B_1 B_s] + st\mathbb{E}[B_1 B_1]
= s - st - st + st = s(1 - t) .$$

Let us take $t \in (0, 1)$. Then we have $X_t = (1 - t)B_t - t(B_1 - B_t)$. Since the random variables $B_t$ and $B_1 - B_t$ are independent, the distribution of $X_t$ is $\mathcal{N}(0, (1-t)t^2 + t^2(1-t)) = \mathcal{N}(0, t(1-t))$.

Exercise 8 Let $X \in L^1(\Omega, \mathcal{F}, \mathbb{P})$. Prove that the family of random variable $\{\mathbb{E}\{X|\mathcal{G}\} : \mathcal{G} \subset \mathcal{F}\}$ is $L^1$ bounded, i.e. $\sup_{\mathcal{G} \subset \mathcal{F}} \mathbb{E}(|\mathbb{E}\{X|\mathcal{G}\}|) < \infty$.

Solution: Let us take any $\mathcal{G} \subset \mathcal{F}$. Then using the Jensen’s inequality we have

$$\mathbb{E}(|\mathbb{E}\{X|\mathcal{G}\}|) \leq \mathbb{E}(\mathbb{E}(|X||\mathcal{G}|)) = \mathbb{E}|X| < \infty ,$$

which proves the claim.

Exercise 9 Let $X \in L^1(\Omega; \mathbb{R})$. Prove that the family of functions

$$\{\mathbb{E}\{X|\mathcal{G}\} : \mathcal{G} \text{ is a sub } \sigma\text{-algebra of } \mathcal{F}\}$$

is uniformly integrable.

Solution: We note that, if a set of measurable sets $A_C$ satisfies $\lim_{C \to \infty} \mathbb{P}[A_C] = 0$, then $\lim_{C \to \infty} \mathbb{E}[1_{A_C}|X|] = 0$ (since $X \in L^1$, dominated convergence).

Let us take any $\mathcal{G} \subset \mathcal{F}$ and consider the family of events

$$A(C, \mathcal{G}) = \{\omega : |\mathbb{E}[X|\mathcal{G}|](\omega)| \geq C\} .$$

Applying the Markov’s and Jensen’s inequalities we obtain

$$\mathbb{P}(A(C, \mathcal{G})) \leq C^{-1}\mathbb{E}(|\mathbb{E}[X|\mathcal{G}|]|) \leq C^{-1}\mathbb{E}[\mathbb{E}[|X||\mathcal{G}|]] = C^{-1}\mathbb{E}|X| \to 0 ,$$

as $C \to \infty$ (since $\mathbb{E}|X| < \infty$).
For any $\varepsilon > 0$, there exist a $\delta > 0$ such that if $P[A] < \delta$, then $E[1_A | X] < \varepsilon$.

For this $\delta$, take $C > \frac{E[X]}{\delta}$, then $P[A(C, G)] < \delta$ for any $G \subset F$, which implies

$$\sup_{G \subset F} E[1_A(C, G)|X] < \varepsilon.$$ 

Finally, we conclude

$$\sup_{G \subset F} E[1_A(C, G)|E[X|G]] \leq \sup_{G \subset F} E[1_A(C, G)|X] < \varepsilon,$$

which proves the claim. $\square$

**Exercise 10** Let $(G_t, t \geq 0), (F_t, t \geq 0)$ be filtrations with the property that $G_t \subset F_t$ for each $t \geq 0$. Suppose that $(X_t)$ is adapted to $(G_t)$. If $(X_t)$ is an $(F_t)$-martingale prove that $(X_t)$ is an $(G_t)$-martingale.

**Solution:** The fact that $(X_t)$ is $(G_t)$-adapted, follows from the inclusion $G_t \subset F_t$ for each $t \geq 0$. Furthermore, for any $t > s \geq 0$, using the tower property of the conditional expectation, we obtain

$$E[X_t|G_s] = E[E[X_t|F_s]|G_s] = E[X_s|G_s] = X_s.$$

This shows that $(X_t)$ is an $(G_t)$-martingale. $\square$

**Exercise 11 (Elementary processes)** Let $0 = t_0 < \cdots < t_n < t_{n+1}$, $H_i$ be bounded $F_t$-measurable functions, and

$$H_t(\omega) = H_0(\omega)1_{(0)}(t) + \sum_{i=1}^n H_i(\omega)1_{(t_i,t_{i+1})}.$$ 

(1)

Prove that $H : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ is Borel measurable. Define the stochastic integral

$$I_t \equiv \int_0^t H_s dM_s \equiv \sum_{i=1}^n H_i(M_{t_{i+1} \land t} - M_{t_i \land t})$$

and prove that

$$E \left[ \int_0^t H_s dB_s \right] = 0, \quad E \left[ \int_0^t H_s dB_s \right]^2 = E \left[ \int_0^t (H_s)^2 ds \right].$$ 

(2)
**Solution:** First, we will prove that the function (1) is Borel-measurable. To this end, we take any Borel set \( A \in \mathcal{B}(\mathbb{R}) \) and we have to show that the set \( \{(t, \omega) : H(t, \omega) \in A\} \) is measurable in the product space \((\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+)) \times (\Omega, \mathcal{F})\). We can rewrite this set in the following way:

\[
\{(t, \omega) : H(t, \omega) \in A\} = \{(0) \times \{\omega : H_0(\omega) \in A\}\} \cup \bigcup_{i=1}^{n} ((t_i, t_{i+1}] \times \{\omega : H_i(\omega) \in A\}) .
\]

Since the sets \(\{0\}\) and \((t_i, t_{i+1}], i = 1, \ldots, n, \) belong to \(\mathcal{B}(\mathbb{R}_+)\), and \(\{\omega : H_i(\omega) \in A\} \in \mathcal{F}_{t_i} \subset \mathcal{F}\), the claim now follows from the fact that the product of two measurable sets is measurable in the product space.

Next, we will show the identities (2). Let us denote \(I_t = \int_0^t H_s dB_s\). Then we have

\[
\mathbb{E}[I_t] = \sum_{i=1}^{n} \mathbb{E} [H_i(B_{t_{i+1} \wedge t} - B_{t_i \wedge t})] = \sum_{i=1}^{n} \mathbb{E} [H_i] \mathbb{E} [B_{t_{i+1} \wedge t} - B_{t_i \wedge t}] = 0 ,
\]

where in the second equality we have used the independence of \(B_{t_{i+1} \wedge t} - B_{t_i \wedge t}\) from \(\mathcal{F}_{t_i}\), which follows from the properties of the Brownian motion and the fact that \(B_{t_{i+1} \wedge t} - B_{t_i \wedge t} = 0\) if \(t_i \geq t\). Furthermore, in the last equality we have used \(\mathbb{E} [B_{t_{i+1} \wedge t} - B_{t_i \wedge t}] = 0\).

For the variance of the stochastic integral we have

\[
\mathbb{E}[I_t^2] = \sum_{i=1}^{n} \mathbb{E} [H_i^2(B_{t_{i+1} \wedge t} - B_{t_i \wedge t})^2] + \sum_{i \neq j} \mathbb{E} [H_i H_j(B_{t_{i+1} \wedge t} - B_{t_i \wedge t})(B_{t_{j+1} \wedge t} - B_{t_j \wedge t})] .
\]

In the same way as before, using the independence of the increments of the Brownian motion, we obtain that the second sum is 0. Thus, we have

\[
\mathbb{E}[I_t^2] = \sum_{i=1}^{n} \mathbb{E} [H_i^2(B_{t_{i+1} \wedge t} - B_{t_i \wedge t})^2] = \sum_{i=1}^{n} \mathbb{E} [H_i^2] \mathbb{E} [(B_{t_{i+1} \wedge t} - B_{t_i \wedge t})^2]
\]

\[
= \sum_{i=1}^{n} \mathbb{E} [H_i^2] \mathbb{E} \left[ (t_{i+1} \wedge t - t_i \wedge t) \right] = \mathbb{E} \left[ \int_0^t (H_s)^2 ds \right] ,
\]

where in the second line we have used the fact that \(H_i^2\) and \((B_{t_{i+1} \wedge t} - B_{t_i \wedge t})^2\) are independent. \(\square\)