Example Sheet 8

1. (a) If \( f \) is holomorphic in an open set \( U \) and \( \gamma \) a closed piecewise \( C^1 \) curve with \( \gamma \to 0 \), then
\[
\oint_{\gamma} f(z) \, dz = 0
\]

(b) If \( U \) is simply connected, any closed piecewise \( C^1 \) curve \( \gamma \) is homotopy to 0, hence \( \oint_{\gamma} f(z) \, dz = 0 \).

(c) A star region is simply connected, the above applies.

2. (1) \( f(z) = (-3z^3 + 7) + (z^7 + e^z) \)

If \( |z| = 1 \), \( |-3z^3 + 7| \geq 7 - 3 = 4 \)
\[
|z^7 + e^z| < 1 + |e^z| < 1 + e < 4.
\]

By Rouche's theorem, \( f(z) \) has same # of zeros in \( \{ z : |z| < 1 \} \) as \( -3z^3 + 7 \).

- \( -3z^3 + 7 \) has no root in \( \{ z : |z| < 1 \} \).

(2) \( f(z) = z^7 + (-3z^3 + e^z + 7) \).

On \( \{ z : |z| = 2 \} \), \( |z^7| = 128 \)
\[
|-3z^3 + e^z + 7| \leq 3 
\]

By Rouche's theorem, \( f \) has 7 zeros inside \( \{ z : |z| < 2 \} \) (same as \( z^7 = 0 \)).

\( f \) has 7 zeros inside \( \{ z : |z| < 2 \} \), no zeros on \( \{ z : |z| = 13 \} \), no \( \frac{df}{dz} \) in \( \{ z : |z| = 11 \} \).

So it has 7 zeros in \( \{ z : |z| < 2 \} \). Q.E.D.
Example sheet 8

3. Suppose \( f \) is not identically zero and \( \exists z_0 \in U \) s.t. \( f(z_0) = 0 \).

Firstly by Weierstrass' theorem, \( f \) is holomorphic on \( U \). Then there exist \( \epsilon > 0 \) s.t. if \( |z - z_0| < \epsilon \) and \( z \neq z_0 \), \( f(z) \neq 0 \).

Let \( S = \max_{|z - z_0| = \epsilon} |f(z)| \neq 0 \).

By the uniform convergence, \( \exists N > 0 \) s.t. if \( n > N \),

\[
|f_n(z) - f(z)| < \frac{\epsilon}{2}, \quad |z - z_0| = \epsilon.
\]
i.e. \( |f_n(z) - f(z)| < |f(z)| \) on \( \overline{c(z_0, \epsilon)} \).

Note \( f_n(z) = f(z) + (f(z) - f_n(z)) \).

We apply Rouché's theorem to see that the number of zero's of \( f_n \) in \( D(z_0, \epsilon) \) equals the number of zero's of \( f \) in \( D(z_0, \epsilon) \). Hence \( f_n \) has a zero, contradicting with the assumption.
4) \( f(z) = \frac{e^{\xi z}}{1 + e^z}, \  \alpha \in (0, 1). \)

**Feasibly**, \( 1 + e^z = 0 \) at
\[ z = \pi i + 2\pi ki, \ k \in \mathbb{Z}. \]

Also \( e^{\xi z} \neq 0 \) at zero's of \( 1 + e^z \).

This means \( f(z) \) has a pole at \( \pi i \) inside the rectangular. If \( k \neq 0 \), \((\pi i + 2\pi k) i \) does not lie inside the rectangular.

**Ord \( f; z_0 = 2n \pi i \):**
\[
\lim_{z \to z_0} (z - z_0)^2 f(z) \text{ exists and non-zero.}
\]

\[
\lim_{z \to \pi i} \frac{(z - \pi i)}{1 + e^z} \ e^{\xi z} = \left( \lim_{z \to \pi i} \frac{z - \pi i}{1 + e^z} \right) e^{\xi \pi i} = \frac{1}{e^{\pi i}} e^{\pi i} = -e^{\pi i}.
\]

**Hence \( \text{ord}(f; \pi i) = 1 \), \( \text{Res}(f; \pi i) = \lim_{z \to \pi i} (z - \pi i) f(z) = -e^{\pi i} \).**

**Note:**
\[
\text{Observe } e^z = e^{\pi i} + e^{(z - \pi i)} + \cdots + f(z)(z - \pi i)^2 + \cdots
\]
where \( f \) is a pole or branch function (Taylor expansion at \( \pi i \)).

Hence \( e^z + 1 = -(z - \pi i) + (z - \pi i)^2 f(z). \)
5) Let \( f(z) = \frac{e^{r^2}}{1 + e^z} \) and \( R \) as in 4).

By the residue theorem,
\[
\int_{\gamma} f(z) \, dz = 2\pi i \cdot \text{Res}(f, \pi i) \cdot \text{ind}(\gamma, \pi i)
\]
\[
= 2\pi i \cdot (-e^{-\pi i}) \cdot 1
\]
\[
= -e^{-\pi i} \cdot 2\pi i
\]

Define \( I(R) = \int_{\gamma} f(z) \, dz \).

So \( I(R) = \int_{-R}^{R} f(x) \, dx \).

\[
= -\int_{-R}^{R} \frac{e^{rx}}{1 + e^x} \, dx.
\]

\[
\int_{\gamma_3(R)} f(z) \, dz = \frac{z(t) = 2\pi it,}{\gamma_3(R)} \int_{-R}^{R} \frac{e^{r(2\pi it - t)}}{1 + e^{2\pi it - t}} (-1) \, dt
\]
\[
= e^{r\pi i} \int_{-R}^{R} \frac{e^{-rt}}{1 + e^{-t}} (-1) \, dt = -e^{r\pi i} I(R).
\]

\[
\left| \int_{\gamma_2(R)} f(z) \, dz \right| = \frac{Z(t) = it + R}{\gamma_2(R)} \left| \int_{0}^{2\pi} \frac{i e^{itR + R}}{1 + e^{itR + R}} \, dt \right|
\]
\[
\leq \frac{\pi e^R}{e^R - 1} \to 0 \quad (\text{as } R \to \infty).
\]

Similarly \( \left| \int_{\gamma_4(R)} f(z) \, dz \right| \to 0 \quad (\text{as } R \to \infty). \)

Finally \( (1 - e^{2r\pi i}) \int_{-\infty}^{\infty} f(z) \, dx = -2\pi i e^{-\pi i} \).

Answer: \( \frac{e^{r\pi i} 2\pi i}{e^{2r\pi i} - 1} = \frac{\pi i \left( e^{\pi i} - e^{-\pi i} \right)}{2} = \pi i / \sinh(\pi R). \)
6. Denote \( \{x_i\} \) the set of points on the closed curve \( \gamma \).

\[ z \in \{x_i\}, \]

\[ G_m(z) = \int_{\gamma} \frac{f(s)}{(s-z)^m} \, ds, \]

We first prove \( G_m \) is continuous on \( \mathbb{C} \setminus \{x_i\} \). For \( m \geq 1 \), \( a, b \in \mathbb{C} \),

\[ a^m - b^m = (a-b)(a^{m-1} + a^{m-2}b + \cdots + ab^{m-2} + b^{m-1}), \]

Let \( z_0 \in \mathbb{C} \setminus \{x_i\} \),

\[ \frac{1}{(z-z_0)^m} - \frac{1}{(z-z_0)^m} = (z-z_0)\left(\frac{1}{(z-z_0)(z-z_0)} + \frac{1}{(z-z_0)(z-z_0)} + \cdots + \frac{1}{(z-z_0)(z-z_0)}\right), \]

Hence

\[ \left| \frac{1}{(z-z_0)^m} - \frac{1}{(z-z_0)^m} \right| = \frac{1}{|z-z_0|^m}, \]

Since \( f \) is continuous on \( \gamma \), it is bounded by a number \( M \).

\[ |G_m(z) - G_m(z_0)| \leq |z-z_0| \int_{\gamma} \left| \frac{f(s)}{(s-z)^m} \right| \, ds \cdot M \cdot |z|, \]

If \( |z-z_0| \leq \frac{1}{2} \cdot d(z_0, \mathbb{C}) \), \( |f(s)| \) is bounded. This proves

continuity of \( f \) at \( z_0 \).

Let \( z_0, z \in \mathbb{C} \setminus \{x_i\}, z \neq z_0 \),

\[ \frac{G_m(z) - G_m(z_0)}{z - z_0} = \int_{\gamma} \frac{f(s)}{(s-z)^m} - \frac{f(s)}{(s-z_0)^m} \, ds \int_{\gamma} f(s) \, ds. \]

Each term \( \frac{1}{(s-z)^m} \) is continuous in \( z \) for all \( s \in \gamma \).

The integrals are also \( \in \mathbb{C} \).

In particular, \( \lim_{z \to z_0} \) of both sides exists and

\[ G_m'(z_0) = \int_{\gamma} \frac{f(s)}{(s-z)^m} \, ds = m \cdot G_m(z_0). \]