Introduction

Aim: understand SDE of Markovian type

\[ dx_t = \sum_{k=1}^{\infty} \sigma_k(x_t) \, dB_k^t + \sigma_0(x_t) \, dt \]

\[ x_t = x_0 + \sum_{k=1}^{\infty} \int_0^t \sigma_k(x_s) \, dB_k^s + \int_0^t \sigma_0(x_s) \, ds \]

Stochastic integrals (they are local martingales)

Need: stochastic integration, BM's, Ito's formula, Martingales, stochastic processes

An outline of topics around stochastic differential equations

Random perturbations of
independent inhomogeneous
\[ \sqrt{\text{Law of large numbers}} + \text{central limit theorem} \]
Gaussian noise

ODE \downarrow + \text{noise} \rightarrow SDE

Examples
linear ODE
\[ x_t = r x_t \]
\[ dx_t = \sigma \left( b x_t + r x_t dt \right) \]

linear SDE (Black-Scholes)

Hamiltonian
\[ \dot{x}_t = \frac{\partial H}{\partial y} \]
\[ \dot{y}_t = -\frac{\partial H}{\partial x} + \text{noise} \]

PDE \downarrow + \text{noise} \rightarrow SPDE

\[ \partial_t u = \frac{1}{2} \Delta u \quad \text{heat eq} \]
\[ \partial_t u = \frac{1}{2} \Delta u + \xi \]

Stochastic Navier-Stokes

\[ \frac{\partial u}{\partial t} + (u \cdot \nabla) u = -\nabla p + \nu \Delta u \]
Sample continuous
Strong Markov process
\[ \rightarrow \text{SDE} \rightarrow \text{Parabolic equation} \]
\[ \frac{\partial}{\partial t} = L \quad (\text{e.g., heat eqn}) \]
Probability measures on the space of paths

- Sample cts strong Markov process \[ \rightarrow \text{Markov generator} \]
  \[ \rightarrow \text{Markov semigroup theory} \rightarrow \text{Dirichlet form theory} \]
- Parabolic equation
  \[ L = \sum \text{sum of squares of operators} \]
  \[ L = (\sum a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum b_i \frac{\partial}{\partial x_i}) \quad \frac{\partial f(t, x)}{\partial t} = L f(t, x) \]

The matrix \((a_{ij}) \geq 0\). If \((a_{ij}) > 0\), this is elliptic theory, inverse matrix can be considered as a Riemannian metric

\[ L = \frac{1}{2} A + Z. \quad \text{Heat equation} \]

Heat kernels (density, fundamental solution)

If \((\rho_i, \ldots, \rho_k)\) satisfies Hörmander's criteria, hypoelliptic kernels (Malliavin calculus)

Constant rank operator \[ \rightarrow \text{sub-Riemannian geometry} \]
Path spaces: Sobolev calculus, & Malliavin calculus
Topyly of loop spaces.
Measure — Poincaré ineq
ESI etc.
Integration by parts, Hodge–DeRham theory.

SDE topics:
- existence of solutions (notion of solutions)
- uniqueness of solutions (notion of uniqueness)
- non-explosion problem
- global smooth flow problem (smooth dependence on initial data)
- large time behaviour
  - moment stability, Lyapunov exponents
  - invariant measure, ergodicity
- regularity of diffeomorphs
- stochastic averaging, homogenisation
- limit theorems.
Example of sDE's (on IR)

Ex. 1.1
\[ dx_t = dB_t + b(x_t) \, dt \]
\[ x_t(w) = x_0(w) + B_t(w) + \int_0^t b(x(s)) \, ds \]
No stochastic integration is involved

Ex. 1.2. Linear sDE:
\[ dx_t = \sigma \cdot x_t \, dB_t + \gamma x_t \, dt \]
\[ (x = \gamma x \Rightarrow x(t) = \gamma^t x(0)) \quad x_t = x_0 \, e^{\gamma B_t + \gamma^2 t} + \gamma t \]

Ito's formula will verify it.

Ex. 1.3. Ornstein-Uhlenbeck process
\[ dx_t = -x_t \, dt + \, dW_t \]
\[ x_t = e^{-t} x_0 + e^{-t} \int_0^t e^s \, dW_s \]
\[ \lim_{t \to \infty} x_t = 0 \]

Var\((x_t) = E\left((x_t - E(x_t))^2\right)\)
\[ \text{Ito's lemma:} \quad e^{2t} E \left( \int_0^t e^{s} \, dW_s \right)^2 = e^{2t} \int_0^t e^{2s} \, ds \]
\[ = \frac{1}{2} (1 - e^{2t}) \to \frac{1}{2} \]
\[ x_t \text{ is Gaussian process.} \]
\[ \text{Distr of } x_t \to N(0, \frac{1}{2}). \]
§2. Stochastic processes

\[(X_t, \ t \in I)\]

- **Time** \( I = [0,1] \cup [0,\infty) \cup (N, \infty) \).
- \( X_t \) is a random variable (i.e., measurable function).

- **Underlying probability space**

\[(\Omega, \mathcal{F}, \mathbb{P})\]

1. \( \Omega = [0,1] \), \( \mathcal{F} = B([0,1]) \), Borel \sigma-algebra,
\( \mathbb{P} = \text{Lebesgue measure} \)

2. \( \Omega = C([0,1]; \mathbb{R}^d) \), \( C([0,\infty); \mathbb{R}^d) \)
\( \mathcal{F} = \text{Borel \sigma-algebra (generated by open subsets)} \)
\( \mathbb{P} = \text{Wiener measure} \) (i.e., the probability distribution of a BM)

- If \( \mathbb{P} \) is a probability measure, \( \int \text{f}(w) \, d\mathbb{P}(w) = 1 \).

- **State space**: \( \mathbb{R}, \mathbb{R}^d \) for us (spheres, Lie group, manifold)
- Borel \sigma-algebra: \( (\mathbb{R}^d, B(\mathbb{R}^d)) \).

fix \( t, \omega \), \( X_t(\omega) \in \mathbb{R} \) (or \( \mathbb{R}^d \)).

fix \( t \), \( \omega \in \Omega \rightarrow X_t(\omega) \in \mathbb{R} \) is (Borel) measurable.

fix \( \omega \), \( t \rightarrow X_t(\omega) \) is a map from \( I \) to \( \mathbb{R}^d \),

i.e. belongs to \( (\mathbb{R}^d)^I \).

A stochastic process describes evolution of random variables in time.
Prop 2.1. If \((E_1, \mathcal{B}_1)\) and \((E_2, \mathcal{B}_2)\) be two measurable spaces and \(\phi : E_1 \to E_2\) is a measurable function. If \(\mu_1\) is a measure on \(E_1\), we define

\[(\phi_* \mu_1)(A) = \mu_1 \left( \{ x : \phi(x) \in A \} \right)\]

This is the pushforward measure. If \(f : E_2 \to \mathbb{R}\) is s.t. \(f \circ \phi\) is integrable, then

\[
\int_{E_1} f \circ \phi(y) \, d \mu_1(y) = \int_{E_2} f(\phi(z)) \, d(\phi_* \mu_1)(z)
\]

\(x_t : \Omega \to \mathbb{R}^d\) induces a measure on \(\mathbb{R}^d\)

\[\mu_t(A) = \{ w : x_t(w) \in A \}\]

This is the probability distribution of \(x_t\).

Similarly, \((x_{t_1} \ldots, x_{t_n})\) induces a prob measure on \((\mathbb{R}^d)^n\)?

The collection of such are finite dimensional dist of \(x_t\).

Finally, \(x_t : \Omega \to (\mathbb{R}^d)^I\) induces

also a probability measure, the distribution of the process \(X\) (the product algebra is the smallest s.t. \(\mu_t\) are measurable).

**Problem 1.** Does finite distribution determine this distribution?
Firstly, \( \mu = (x_1) \times P \),

\[
\pi_{t_1} \ldots \pi_{t_n} \times \mu(A)
\]

\[= (x_{t_1}, \ldots, x_{t_n}) \times (P)\]

\[= \mu_{t_1 \ldots t_n}(A).
\]

i.e. \( \mu \) projects to \( \mu_{t_1 \ldots t_n} \). (\( \mu \) determines fine distribution).

Consistency property

\[
\left( \pi_{t_1}, \ldots, \pi_{t_n} \right) \times \left( A_1 \times \ldots \times A_n \right)
\]

\[= \left( \pi_{t_1}, \ldots, \pi_{t_j}, \ldots, \pi_{t_n} \right) \times \mu \left( A_1 \times A_{j+1} \times \ldots \times A_n \right).
\]

**Def 2.2** A family of \( \{ \mu_F : F \in C_I, \#F \leq w \} \) of measures is consistent if

1. \( \mu_F \) is a measure on \( (\mathbb{R}^F, B(\mathbb{R}^F)) \)
2. \( (\pi_F)_* \mu = (\pi_{F_2})_* \mu_{F_2} \)

**Thm 2.3** Kolmogorov's Extension Theorem.

If \( \{ \mu_F \} \) is a consistent family of measure, there exists a unique measure \( \mu \) on \( BI \) s.t.

\[
(\pi_F)_* \mu = \mu_F.
\]

The collection of sets \( \pi_F^{-1}(A) : A \in B_F, \#F \leq w \) are cylindrical sets.
Lecture 3

Let $(E_{\alpha}, \mathcal{F}_{\alpha})$, $\alpha \in I$, be a family of measure spaces. The Cartesian product space

$$E = \{ \sigma : I \to \bigcup_{\alpha \in I} E_{\alpha} \text{ with } \sigma(\alpha) \in E_{\alpha} \}$$

has a natural $\sigma$-algebra $\mathcal{F}$, it is the smallest one s.t. all projection maps $\pi_\alpha : E \to E_{\alpha}$

$$\pi_\alpha(\sigma) = \sigma(\alpha),$$

are measurable, i.e.

$$\mathcal{F} = \sigma \{ \pi_\alpha^{-1}(A_{\alpha}) : A_{\alpha} \in \mathcal{F}_{\alpha}, \alpha \in I \}.$$  

If $I$ is countable, $\mathcal{F} = \sigma \{ \pi_\alpha^{-1}(A_{\alpha}) : A_{\alpha} \in \mathcal{F}_{\alpha}, \alpha \in I \}$.

Kolmogorov's Extension Thm: $\mathcal{B}(\mathbb{R}^d)^I = \bigotimes \mathcal{B}(\mathbb{R}^d)$.

Given a consistent family of cylindrical measures

$$\{ \mu_F : F \text{ ranges through a finite subset of } I \}$$

$\mu_F$ measures on $\otimes \mathcal{F}$.

Then there exists a unique measure $\mu$ on $\mathcal{F}$ such that

$$\mu_F = (\otimes \mu)^x.$$

This is trivial if $I$ is countable. (Proof: Parthetally)

We only need the countable version, so no proof is needed.

Observation: if $(\mu_t)_{t \in \mathbb{T}}$ are finite dimensional distributions of a stochastic process $(X_t)$, then the measure $\mu$ determined by Kolmogorov's Thm on $(\mathbb{R}^d)^I$ is the same as $(\mu_t)_t$. P.
§ 3. Markov processes with transition functions.

**Definition 3.1.** A function \( P(t, x, \tau) \) defined on \( I \times E \times B \), where \( I \subset \mathbb{R} \), \( E \) a separable metric space, \( B \) its Borel \( \sigma \)-algebra is a time homogeneous transition function if

1. For fixed \( t \in I, x \in E \),
   \[ P(t, x, \cdot) \] is a probability measure on \((E, B)\).

2. Given \( \tau \in B \), \( (t, x) \mapsto P(t, x, \tau) \) is a Borel measurable function on \( I \times E \).

3. For any \( x \in E \),
   \[ P(0, x, \cdot) = \delta_x \] the delta measure at \( x \),
   i.e., \( \delta_x(\tau) = 1 \) if \( x \in \tau \), 0 if \( x \notin \tau \).

4. Chapman-Kolmogorov equation, \( s, t \geq 0 \)
   \[ P(t+s, x, \tau) = \int_{E} P(s, y, \tau) P(t, x, dy) \]

**Definition 3.2.** A stochastic process \((X_t)\) is a Markov process with transition function \( P(t, \cdot) \) if

\[ P(X(t+s) \in A | \sigma(X_s \leq t)) = P(s, x_t, A), \quad A \in B. \]
Equivalently, if
\[ P \left( t \leq X < t+s \right) = \int_{t}^{t+s} P_c(s, x, A) \, dx \]
for any \( t \in [0, \infty) \) and \( x \in A \).

Proposition 8.3: If \( \{X_t\} \) is a Markov process with transition function \( P_t(x, y, A) \), and \( X_0 \sim \nu \), then for any \( 0 < t_1 < \ldots < t_n \), \( A_2 \in \mathcal{B}(\mathbb{R}^d) \), then Einstein's relation, as below, holds:

\[ P \left( x_0 \in A_0, X_{t_1} \in A_1, \ldots, X_{t_n} \in A_n \right) = \int \cdots \int \int P_{t_n-t_{n-1}}(y_n, x_{n-1}, A_n) \, P_{t_{n-1}-t_{n-2}}(y_n, x_{n-2}, A_{n-1}) \, \ldots \, P_{t_1-t_0}(y_0, x_0, A_0) \, \nu(dy_0). \]

Equivalently, let \( f_t : E \to \mathbb{R} \) be Borel measurable and bounded.

\[ \mathbb{E} \left( \prod_{i=0}^{n} f_t(X_{t_i}) \right) = \int \cdots \int \prod_{i=0}^{n} f_t(x_i) \prod_{i=0}^{n} P \left( X_{t_{i+1}} \in A_i \right) \, d\nu(x_0). \]

Set \( t_n = 0 \).

i.e. \( \prod_{i=0}^{n} P \left( (t_i - t_{i-1}, X_{t_i}, \theta_i) \right) \) is the probability distribution of \( \{X_t, \ldots, X_{t_n}\} \).
Proof. \( n=1 \)

\[
P(x_0 \in A_0, x_1 \in A_1)
= \int \frac{1}{\mathcal{E}} \mathbf{1}_{x_0 \in A_0} P(s, x_0, A_1) \, d\mathcal{P}
= \int \frac{1}{\mathcal{E}} \mathbf{1}_{A_0} P(s, x_0, A_1) \, d\mathcal{P} dy_0
= \int \frac{1}{\mathcal{E}} P(s, x_0, A_1) \, d\mathcal{P} dy_0.
\]

Assume conclusion for \( n \). Consider \( 0 < t_1 < \ldots < t_n < t_{n+1} \), \( A_i \in \mathcal{B} \), then by definition of transition function,

\[
\mathcal{E} \left( \prod_{i=1}^{n+1} \mathbf{1}_{x_i \in A_i} \right) = \int \prod_{i=1}^{n} \frac{1}{\mathcal{E}} \mathbf{1}_{x_i \in A_i} \cdot P(t_{n+1} - t_n, x_n, A_{n+1}) \, d\mathcal{P}
\]

induction: \[
\int \ldots \int \frac{1}{\mathcal{E}} \mathbf{1}_{x_i \in A_i} P(t_{n+1} - t_n, y_n, A_{n+1}) \cdot \prod_{i=0}^{n} P(t_{i+1} - t_i, y_i, dy_i).
\]

Conclusion. The transition function and initial distribution

determines 2-times marginal distributions. The Markov property leads to all finite dimensional distribution,

\( \Rightarrow \) get measure on \((\mathbb{R}^d)^2\) by Kolmogorov's extension theorem. \( \Rightarrow \) get a Markov process with transition function \( \mathbb{P}(\cdot|\cdot) \).
Example 3.1  Let $P_t(x, y) = \frac{1}{(2\pi t)^d} e^{-\frac{(x-y)^2}{2t}}, \ x, y \in \mathbb{R}^d$.

Define $P^t f(x) = \int_{\mathbb{R}^d} f(y) P_t(x, y) \, dy, \ f \text{ bounded measure}$.

Then $P^t f$ is sol-in to $\frac{\partial u}{\partial t} = \frac{1}{2} \Delta u(x) = \frac{1}{2} \frac{\partial^2 u}{\partial x^2}$

$u(x, 0) = f(x)$.

$P_t(x, y)$ is the heat kernel (fundamental solution to heat equation).

Set $P(t, x, y) = \int_{T} P_t(x, y) \, dy$.

Then this determines a stochastic process, which we prove later to be the Brownian motion on $\mathbb{R}^d$.

The measure $P(t, x, :)$ is precisely the Gaussian measure with mean $x$ and variance $tI$, which we denote by $\mathcal{N}(x, tI)$.

If $d = 1$, this is $\mathcal{N}(x, t)$.

Definition. A non-degenerate Gaussian measure on $\mathbb{R}^d$ is a prob measure with density (with respect to Lebesgue)

$$\frac{1}{(2\pi)^d |\det\Theta|^{1/2}} e^{-\frac{1}{2} \langle (x-a)^T \Theta^{-1} (x-a), x-a \rangle}$$

where $\Theta$ is a symmetric and non-degenerate real matrix, $a \in \mathbb{R}^d$, and $\Theta$ the definite.
Let us consider a Wiener space:

\[ W_0^d := \text{C}_0([0,1]; \mathbb{R}^d) = \{ \omega : [0,1] \to \mathbb{R}^d \text{ cts, } \omega(0) = 0 \} \]

This is a separable Banach space with supremum norm.

\[ \| \omega \| = \sup_{t \in [0,1]} |\omega(t)|. \]

\[ \text{Theorem 4.6} \] There exists a probability measure \( \mu \) on \((W_0^d, B(W_0^d))\) with the property its finite dimensional distributions are given by Einstein's relation (8):

\[ (\pi_{\xi_1}, ..., \pi_{\xi_n}) \ast (\mu)(A_1 \times ... \times A_n) = \int_{A_1} ... \int_{A_n} \prod_{k=1}^n p_{\xi_k}(x_{k_1}, x_{k_2}, ..., x_{k_{n-1}}) \, dx_{k_1} dx_{k_2} ... dx_{k_{n-1}}. \]

In particular \((\pi_{\xi_1}, \xi)\) is a BM on the probability space \((W_0^d, B(W_0^d), \mu) (\ldots, d=1)\).

\[ \text{Proof.} \] We construct a measurable map \( \overline{\omega} \) from \((\mathbb{R}^d)^\infty \to B(\mathbb{R}^d)\) to \((W_0^d, B(W_0^d))\).

If \( x : [0,1] \to \mathbb{R}^d \) is continuous, we set \( \overline{\omega}(x) \) to be the continuous function on \([0,1]\) determined by their values on \( Q \). If \( x : Q \to \mathbb{R}^d \) is not cts, we set \( \overline{\omega}(x)_t = 0 \) for \( t \). Then \( \overline{\omega} \) is measurable.

Let \( \overline{\mu} \) be the measure on \((\mathbb{R}^d)^\infty\), given by \( \otimes \), i.e.
Einstein's relation. Let $\mu = \overline{\pi}_* \tilde{\mu}$. It is clear that $\mu(W_0^d) = \tilde{\mu}(\mathbb{R}^d) = 1$. Also, by Kolmogorov's Thm, $\tilde{\mu}(x: \overline{\pi}(x) \neq x) = 0$.

Let $t_1 < t_2 < \cdots < t_n$, $n \in \mathbb{N}$, then

$$\mu(x: \Pi_{t_1}(x) \in A_1, \cdots, \Pi_{t_n}(x) \in A_n)$$

$$= \tilde{\mu}(x, \Pi_{t_1}(x) \in A_1, \cdots, \Pi_{t_n}(x) \in A_n) \geq$$

and so holds.

Remarks (1) Why don't we simply restrict $\tilde{\mu}$ to $W_0^d$?

$W_0^d$ is not a measurable set of $\mathbb{R}^{d \times d}$, the latter is determined by finite/countable evaluations, while evaluations at a countable times do not determine the continuity of the function on $[0,1]$.

(2) The above construction would induce a measure on $\mu$ from any Markov transition semigroup, so we always have a CTS process from any transition probability, the latter may not agree with the original process.
Lecture 4

8. BM and Wiener Measure

Def 4.1. A real valued stochastic process \((B_t, t \geq 0)\) is a standard (linear) Brownian motion if:

1. \(B_0 = 0\) a.e.
2. \((B_t)\) is sample cts, i.e. for a.e. \(w\),
   \[ t \mapsto B_t(w) \] is cts.
3. For \(0 \leq t_1 \leq t_2 \leq \ldots \leq t_n\),
   \[ B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}} \]
   are independent r.v.'s (independent increments)
4. For any \(s, t \geq 0\), the probability distribution of \(B_{t+s} - B_s\) is \(N(0, t)\).

Remark. We have a d-dim standard BM if \(B_{t+s} - B_s \sim N(0, tI)\).

A family of r.v.'s \(X_1, \ldots, X_n\) is independent if

\[ P(X_1 \in A_1, \ldots, X_n \in A_n) = \prod_{i=1}^n P(X_i \in A_i) \]

\(\forall A \in \mathcal{B}\).

Equivalently, the probability distributions of \((X_1, \ldots, X_n)\) are product measures of the probability distributions of \(X_i\).
Theorem 4.2. The stochastic process determined by
the transition function
\[ \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{(x-y)^2}{2t}\right) \] dy = P(t, x, dy),
and \( \xi_0 = \xi_0 \), is a standard Brownian motion (up to a modification).

Proof. Part (i) is clear from \( \xi_0 = \xi_0 \).

Part (ii). Let \( f \) be bounded measurable, real-valued, then
\[ \mathbb{E} f(x_{t_2} - x_{t_1}) \]
\[ = \iint f(y_2 - y_1) \frac{1}{\sqrt{2\pi t_1}} \exp\left(-\frac{y_1^2}{2t_1}\right) \cdot \frac{1}{\sqrt{2\pi (t_2-t)}} \exp\left(-\frac{(y_2-x)^2}{2(t_2-t)}\right) dy_1 dy_2 \]
as
\[ (B_{t_1}, B_{t_2}) \sim P_{t_1}(0, dy_1) P_{t_2-t_1}(y_1, dy_2). \]
\[ = \iint f(\zeta) \frac{1}{\sqrt{2\pi t_1}} \exp\left(-\frac{\zeta^2}{2t_1}\right) \cdot \frac{1}{\sqrt{2\pi (t_2-t)}} \exp\left(-\frac{\zeta^2}{2(t_2-t)}\right) d\zeta d\zeta \]
\[ = \int f(\zeta) \frac{1}{\sqrt{2\pi (t_2-t)}} \exp\left(-\frac{\zeta^2}{2(t_2-t)}\right) d\zeta, \]
Thus
\[ x_{t_2} - x_{t_1} \sim N(0, t_2-t_1). \]

Part (iii). Let \( f \) be bounded measurable functions.
Then
\[ \mathbb{E} f(B_{t_2} - B_{t_1}) = \iint f(\zeta_2) \frac{1}{\sqrt{2\pi t_1}} \exp\left(-\frac{\zeta_1^2}{2t_1}\right) \cdot \frac{1}{\sqrt{2\pi (t_2-t)}} \exp\left(-\frac{\zeta_2^2}{2(t_2-t)}\right) \frac{1}{\sqrt{2\pi (t_2-t)}} \exp\left(-\frac{\zeta_2^2}{2(t_2-t)}\right) d\zeta_1 d\zeta_2 \]
\[ = \iint f(\zeta_2) \frac{1}{\sqrt{2\pi t_1}} \exp\left(-\frac{\zeta_1^2}{2t_1}\right) \cdot \frac{1}{\sqrt{2\pi (t_2-t)}} \exp\left(-\frac{\zeta_2^2}{2(t_2-t)}\right) \frac{1}{\sqrt{2\pi (t_2-t)}} \exp\left(-\frac{\zeta_2^2}{2(t_2-t)}\right) d\zeta_1 d\zeta_2 = \int \mathbb{E} f(B_{t_2} - B_{t_1}) \]
This proves the independent increments property. For part (2), we first discuss Kolmogorov's city criterion, from which the city of the Markov process \( (X_t) \) follows.

**Def 4.3.**

1. Two stochastic processes \( (X_t) \) and \( (Y_t) \) on the same probability space are modifications of each other if 
   \[ P(\omega : x_t(\omega) = y_t(\omega)) = 1 \quad \text{for every } t \in I. \]

2. They are indistinguishable if 
   \[ P(\omega : x_t(\omega) = y_t(\omega), \forall t) = 1. \]

**Def 4.4.** Let \( E \) be a Banach space, with norm \( 1 \cdot 1\).

Let \( I \) be an interval, \( \alpha \in (0, 1) \). A function \( f : I \rightarrow E \) is locally Hölder continuous of exponent \( \alpha \) if for any compact subinterval \([a, b]\) of \( I\),

\[
\sup_{\substack{s,t \in [a,b] \\neq s, t \in I}} \left| \frac{f(t) - f(s)}{|t-s|^\alpha} \right| < \infty.
\]

**Exercise.** Prove Thm 4.2 for \( d > 1 \) case.
Theorem 4.5 Let \((X_t, \omega \in \Omega)\) be a stochastic process with values in a separable Banach space \(E\). Suppose that there exist positive numbers \(p, \delta\) and \(c\) s.t. for any \(s,t \in I\)

\[
E (|X_s - X_t|^p) \leq c |s-t|^\delta.
\]

Then there exists a continuous modification of \((X_t, \omega \in \Omega)\), which we denote by \((\tilde{X}_t)\) s.t. for any \(\alpha \in (0, \frac{\delta}{p})\), and \([a,b] \subset I\)

\[
E \left( \sup_{s \neq t \in [a,b]} \left( \frac{|\tilde{X}_s - \tilde{X}_t|}{|s-t|^{\alpha}} \right)^p \right) < \infty,
\]

and in particular the paths of \(\tilde{X}_t\) are locally Hölder continuous.

Remark: Continuity, uniform modulus of continuity in Hölder norms → cts.

Proof: see Revuz and Yor, Continuous Martingales and Brownian Motion, P.26, Thm(2.1). Third edition.

We return to Theorem 4.2, where \(X_t - X_s \sim N(0, t-s)\), hence

\[
E |X_t - X_s|^p = \frac{1}{2\pi(t-s)} \int e^{-\frac{1}{2} \frac{y^2}{t-s}} \frac{y^p}{t^{\frac{p}{2} + \frac{1}{2}}} dy.
\]

\[
= \frac{1}{\sqrt{2\pi(t-s)}} \tau^\frac{p}{2} \int e^{-\frac{1}{2} \frac{y^2}{t-s}} 1_{y^2 \leq \tau^2} dy.
\]

As Gaussian random variables have moments of all orders, there exists a continuous modification of \((\tilde{X}_t)\), which is a standard BM's. Note they are Hölder cts of order

\[
\alpha = \left( \frac{p-1}{p} \right) = \frac{1}{2} - \frac{1}{p}, \text{ i.e. for any } \alpha < \frac{1}{2}.
\]
Let us consider a Wiener space:

\[ W_0^d := C_0([0,1]; \mathbb{R}^d) = \{ w: [0,1] \to \mathbb{R}^d \mid \text{cts}, w(0) = 0 \} \]

can replace \([0,1]\) by \([0,T]\).

This is a separable Banach space with supremum norm.

\[ \| w \| = \sup_{t \in [0,1]} |w(t)|. \]

**Theorem 4.6** There exists a probability measure \( \mu \) on \( (\overline{W}_0^d, B(\overline{W}_0^d)) \) with the property its finite dimensional distributions are given by Einstein's relation (8.1):

\[ (\pi_1, \ldots, \pi_n) \ast (\mu)(A_1 \times \cdots \times A_n) = \int \cdots \int P_{a_k, t_k}^{-1}(A_1, A_2, \ldots, A_n) \, \mu(\cdot | A_1, A_2, \ldots, A_n) \, \mu(\cdot | A_1, A_2, \ldots, A_n) \, \ldots \, \mu(\cdot | A_1, A_2, \ldots, A_n). \]

In particular \( (\pi_t, t \leq 1) \) is a BM on the probability space \( (\overline{W}_0^d, B(\overline{W}_0^d), \mu) \) (\( \ldots, d=1 \)).

**Proof.** We construct a measurable map \( \overline{\omega} \) from \( ((\mathbb{R}^d)^I, B((\mathbb{R}^d)^I) ) \) to \( (\overline{W}_0^d, B(\overline{W}_0^d)) \).

If \( x: [0,1] \to \mathbb{R}^d \) is continuous when restricted to \( Q \), we set \( \overline{\omega}(x) \) to be the continuous function on \([0,1]\) determined by their values on \( Q \). If \( x: \mathbb{R} \to \mathbb{R}^d \) is not cts, we set \( \overline{\omega}(x)(t) = 0 \) for all \( t \). Then \( \overline{\omega} \) is measurable.

Let \( \overline{\mu} \) be the measure on \( (\mathbb{R}^d)^I \), given by (8), i.e.
Einstein's relation. Set \( \mu = \mathbb{E}_\# \tilde{\mu} \). It is clear that 
\[ \mu(W_0^d) = \tilde{\mu}(\mathbb{Q}^d \times) = 1. \] Also, by Kolmogorov's Thm, 
\[ \tilde{\mu}(x: \mathbb{E}(x) \neq x) = 0. \]

Let \( t_1 < t_2 < \cdots < t_n, n \in \mathbb{N} \), then 
\[ \mu(x: \prod_{k} (x) \in A_1, \cdots, \prod_{n} (x) \in A_n) \]
\[ = \tilde{\mu}(x, \prod_{k} (x) \in A_1, \cdots, \prod_{n} (x) \in A_n) \]
and \( \otimes \) fields.

**Remarks** (1) Why don't we simply restrict \( \tilde{\mu} \) to \( W_0^d \)?

\( W_0^d \) is not a measurable set of \( \mathbb{Q}^{BCR(d)} \), the latter is determined by finite/countable evaluations, while evaluations at a countable times do not determine the continuity of the function on \( [0,1] \).

(2) The above construction would induce a measure on \( \mu \) from a Markov transition semigroup, so we always have a cts process from any transition probability, the latter may not agree with the original process.
5.5 Conditional expectation/Martingales

The expectation of a random variable given that $A$ happened should be the average of $X$ w.r.t. the restricted measure on $A$:

$$E(X \mid A) = \int \frac{1}{P(A)} E(1_A X) \, dX.$$ 

If $\{A_i, i=1, \ldots, n\}$ is a partition of $\Omega$, $\mathcal{G} = \sigma\{A_i\}$, then $E(X \mid \mathcal{G})$ should depend on whether $\omega \in A_i$,

$$E(X \mid \mathcal{G}) = \sum_{i=1}^n 1_{A_i}(\omega) \cdot \frac{1}{P(A_i)} E(1_{A_i} X).$$

Def. 5.1 Let $\mathcal{G} \subset \mathcal{F}$ be a $\sigma$-algebra and $X$ and $Y$ r.v. The conditional expectation of $X$ given $\mathcal{G}$ is a $\mathcal{G}$-measurable r.v. $Y$ s.t. for every $\mathcal{G}$-measurable $f$,

$$\int f \, dP = \int f \, dP$$

for every $P \in \mathcal{G}$.

Thm. 5.2 The conditional expectation of an $L^1$ r.v. $X$ w.r.t. a sub $\sigma$-algebra $\mathcal{G}$ exists and is unique.

Proof. First suppose $X \geq 0$. We define $Q(A) = \int_A X \, dP$ and $Q \ll P$. Furthermore $Q(A) = \int_A \frac{dQ}{dP} \, dP = \int_A X \, dP$.

Then $\frac{dQ}{dP}$ is as required conditional expectation.

Otherwise set $X = X^+ - X^-$. We define

$$E(X \mid \mathcal{G}) = E(X^+ \mid \mathcal{G}) - E(X^- \mid \mathcal{G}).$$
To prove uniqueness. Suppose $Y_1, Y_2$ are $\mathcal{G}$-meas. and s.t. for any $\mathcal{P} \in \mathcal{G}$,
\[
\int_{\mathcal{P}} Y_1 \, d\mathcal{P} = \int_{\mathcal{P}} x \, d\mathcal{P},
\]
\[
\int_{\mathcal{P}} Y_2 \, d\mathcal{P} = \int_{\mathcal{P}} x \, d\mathcal{P}.
\]
Hence $\int_{\mathcal{P}} (Y_1 - Y_2) \, d\mathcal{P} = 0 \Rightarrow Y_1 = Y_2$ a.s.

**Proposition 5.3.**

1. \( E(ax + by \mid \mathcal{G}) = aE(x \mid \mathcal{G}) + bE(y \mid \mathcal{G}) \)
   follows from uniqueness.

2. If \( x \geq 0 \), \( E(x \mid \mathcal{G}) \geq 0 \)

3. Conditional Jensen, let \( \phi : \mathbb{R} \rightarrow \mathbb{R} \) be convex, then
   \( \phi(E(x \mid \mathcal{G})) \leq E(\phi(x) \mid \mathcal{G}) \).

4. If \( x \in \mathcal{G} \), \( E(xy \mid \mathcal{G}) = xE(y \mid \mathcal{G}) \)

5. If \( \sigma(x) \) is independent of \( \mathcal{G} \), \( E(x \mid \mathcal{G}) = E(x) \) a.e.

6. If \( \mathcal{G}_1 \) is a sub-\( \sigma \)-algebra of \( \mathcal{G} \),
   \( E(E(x \mid \mathcal{G}_1) \mid \mathcal{G}) = E(E(x \mid \mathcal{G}) \mid \mathcal{G}_1) \).

In particular, \( E(x) = E(E(x \mid \mathcal{G})) \).

Remark. \( E(x \mid \mathcal{G})(w) = \int_{\mathcal{G}} x(c) \, d\mu(c)(w) \)

This may not exist.
Example 1. Let \((X, Y)\) be integrable variable with joint distribution \(f(x,y)\) \(dx\) \(dy\). Let \(g : \mathbb{R} \to \mathbb{R}\) be bounded measurable. Then
\[
E(\Phi(x,y) \mid Y) = \Phi(Y)
\]
where \(\Phi(y) = \int \Phi(x,y) f(x,y) \, dx\).

Proof. Let \(g : \mathbb{R} \to \mathbb{R}\) be bounded Borel measurable. It suffices to prove
\[
E[g(Y) \Phi(x,y)] = E[g(Y) \Phi(y)].
\]
\[
E[g(Y) \Phi(x,y)] = \int \int g(y) \Phi(x,y) f(x,y) \, dx \, dy.
\]
The distribution of \(Y\) is:
\[
\mu_Y(A) = P(Y \in A) = \int \int 1_A(x,y) \, dx \, dy.
\]
Hence
\[
E[g(Y) \Phi(Y)]
\]
\[
= \int \int g(y) \Phi(y) \int \int \Phi(x,y) f(x,y) \, dx \, dy
\]
\[
= \int \int g(y) \int \int \Phi(x,y) f(x,y) \, dx \, dy
\]
\[
= \int \int g(y) \int \int \Phi(x,y) f(x,y) \, dx \, dy
\]
Equating
\[
\int \int g(y) \int \int \Phi(x,y) f(x,y) \, dx \, dy
\]
\[
\text{since } g, \Phi \text{ are bdd, } \int \int f(x,y) \, dx \, dy = 1.
\]
Example 2

Let \((B_t)\) be a standard BM. Set

\[ \mathcal{F}_t^B = \sigma \{ B_s : 0 \leq s \leq t \}, \]

the smallest \(\sigma\)-algebra s.t. each \(B_s\),

where \(s \leq t\), is \(\mathcal{F}_t^B\)-measurable.

Prove that

(i) \(\sigma(\mathcal{B}_t - \mathcal{B}_s)\) is independent of \(\mathcal{F}_t^B\)

for any \(s < t\).

(ii) \(E(\mathcal{B}_t \mid \mathcal{F}_s^B) = B_s\), for any \(s \leq t\).

Proof. (i) Take \(\tilde{f}_i, \tilde{g} : \mathbb{R} \to \mathbb{R}\) bounded Borel measurable.

Let \(0 < t_1 < t_2 < \cdots < t_n = s < t\). Then

\[
E \left[ \tilde{f}(B_t - B_s) \prod_{i=1}^n \tilde{f}_i(B_{t_i}) \right]
= \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} \tilde{f}(y_{n+1} - y_n) \prod_{i=1}^n \tilde{f}_i(B_{t_i}) \prod_{i=1}^n P_{t_i - t_{i-1}}(y_{i+1}, y_i) \prod_{i=1}^n P_{t_i}(-t_{i-1}(y_i, y_0)) \prod_{i=1}^n d^ny_i
= \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} \tilde{f}(y_{n+1} - y_n) P_{t_n - t_{n-1}}(y_{n+1}, y_n) \cdots \tilde{f}_i(B_{t_i}) \prod_{i=1}^n P_{t_i - t_{i-1}}(y_i, y_0) \prod_{i=1}^n d^ny_i
= E \tilde{f}(B_{t+n} - B_{t_n}) \cdot E \left[ \prod_{i=1}^n \tilde{f}_i(B_{t_i}) \right].
\]

(ii) Take \(\tilde{f}(x) = x\). \(\Rightarrow \) \(E \{ B_t - B_s \mid \mathcal{F}_s^B \} = 0\).
**Definition** Let \((X_t, \mathcal{F}_t)\) be a stochastic process. For \(s \in I\), define

\[ \mathcal{F}_{s}^X = \sigma \{ X_r : r \geq s \}. \]

We say \(\mathcal{F}_{s}^X, s \in I\) is the natural filtration of \((X_t, \mathcal{F}_t)\).

**Note** \(\mathcal{F}_{s}^X \subseteq \mathcal{F}_{t}^X\) if \(s \leq t\).

**Definition** A family of \(\sigma\)-algebras \((\mathcal{F}_s, s \in I)\) is said to be a filtration if for any \(s \leq t\), \(\mathcal{F}_s \subseteq \mathcal{F}_t\) and each \(\mathcal{F}_s\) is a \(\sigma\)-algebra with \(\mathcal{F}_s \subseteq \mathcal{F}_t\).

**Usual conditions:**

Let \(\mathcal{F}_s^+ = \bigwedge_{s \leq r} \mathcal{F}_r\), (1) \(\mathcal{F}_s^+ = \mathcal{F}_s\) (right continuity).

(2) each \(\sigma\)-algebra \(\mathcal{F}_s\) is complete, i.e.

every subset of a measure zero set is contained in \(\mathcal{F}_s\).
Definition: A stochastic process \((X_t, t \in \mathcal{I})\) with each
\(x_t \in \mathcal{L}\) is a \((\mathcal{F}_t, t \in \mathcal{I})\) martingale if
\[\mathbb{E}(X_t | \mathcal{F}_s) = X_s\]
for any \(s \leq t\).

It is a sub-martingale if
\[\mathbb{E}(X_t | \mathcal{F}_s) \geq X_s\]
for any \(s \leq t\).

It is a super-martingale if
\[\mathbb{E}(X_t | \mathcal{F}_s) \leq X_s\]
for any \(s \leq t\).

\(\bigcirc\) implies \(\mathbb{E}(X_t) = \mathbb{E}(X_s)\) for all \(s, t \in \mathcal{I}\).

\(\bigcirc\bigcirc\) \(\mathbb{E}(X_t) \geq \mathbb{E}(X_s)\) for all \(s \leq t, s, t \in \mathcal{I}\).

\(\bigcirc\bigcirc\bigcirc\) \(\mathbb{E}(X_t) \leq \mathbb{E}(X_s)\)

The converse is not true in general.

Corollary 1: A Brownian motion on \(\mathbb{R}^d\) is a martingale.

1. If \(X_t \in \mathcal{L}\), \(X_t = \mathbb{E}(X | \mathcal{F}_t)\) is a martingale.
Limit Thms

(1) **Monotone Convergence Thm.**: If \( \{f_n\} \) are non-negative measurable functions s.t. \( f_n(x) \) increases to \( f(x) \) for a.e. \( x \), then \( \lim_{n \to \infty} \int f_n \, dp = \int f \, dp \).

(1') If \( 0 \leq x_n \leq x \), \( \mathbb{E}(x_n | G) \leq \mathbb{E}(x | G) \) a.e.

(2) **Fatou's lemma**: If \( f_n \) are non-negative function,

\[
\int \liminf_{n \to \infty} f_n \, dp \leq \liminf_{n \to \infty} \int f_n \, dp.
\]

(since \( \inf_{k \geq n} f_k \leq f_n \) each \( n \))

(2') If \( x_n \geq 0 \), \( \mathbb{E}(\lim_{n \to \infty} x_n | G) \leq \lim_{n \to \infty} \mathbb{E}(x_n | G) \).

(3) **Dominated Convergence Thm.**.

Let \( f_n \in L^1 \), s.t. \( f_n \to f \) a.e. and

\( \exists \) an \( L^1 \) function \( g \) s.t. \( |f_n| \leq g \) a.e. for all \( n \).

Then \( f \in L^1 \), and \( \int \lim f_n = \lim \int f_n \).

(3') \( X_n \in L^1 \), \( X_n \to X \) a.e., \( |X_n| \leq |Y| \), \( Y \in L^1 \). Then \( \mathbb{E}(\lim_{n \to \infty} X_n | G) = \mathbb{E}(X | G) \).
If \( f^+ \) and \( f^- \) are integrable, \( f \) is integrable, i.e., in \( L^1 \).

**Theorem 4 (Basic approximation)**

1. If \( f: \mathbb{R} \to [0, \infty] \) is measurable, there exists a sequence of simple functions \( f_n \) s.t. for all \( \omega \),
   \[
   0 \leq f_1 \leq f_2 \leq \cdots \leq f_n \leq \cdots \quad \& \quad f_n \to f.
   \]
   Also, on any subset of \( \mathbb{R} \) on which \( f \) is bdd, \( f_n \) converges to \( f \) uniformly.

2. If \( f: \mathbb{R} \to \mathbb{R} \) is integrable, \( \exists \) a sequence of simple functions \( f_n \) s.t.
   \[
   0 \leq f_1 \leq f_2 \leq \cdots \leq f_n \leq \cdots \quad \& \quad f_n \to f \text{ pointwise, and s.t. } f_n \to f \text{ uniformly on any subset of } \mathbb{R} \text{ on which } f \text{ is bdd}.
   \]

3. If \( f \in L^1 \), there exist a sequence of simple functions \( f_n \) s.t. \( f_n \) converges to \( f \) in \( L^1 \).

- If \( f \geq 0 \).
  
  Let \( n = 0, 1, 2, \ldots \), let
  \[
  f_n(\omega) = 2^n \cdot 1_{[0, 1]} f^-( [2^n, \infty]) + \sum_{k=0}^{\lfloor \frac{1}{2^n} \rfloor} 1_{\left[ \frac{k}{2^n}, \frac{k+1}{2^n} \right]} f( \frac{k}{2^n}, \frac{k+1}{2^n} ).
  \]
  On \( [0, 2^n] \),
  \[
  |f_n(\omega) - f(\omega)| \leq \frac{1}{2^n}.
  \]

- Otherwise,
  \[
  f_n(\omega) = (f^+)_n(\omega) - (f^-)_n(\omega).
  \]
  \[
  |f_n(\omega) - f(\omega)| \leq |f_n(\omega) - (f^+)_n(\omega)| + |f_n(\omega) - (f^-)_n(\omega)|
  \leq \frac{3}{2^n}.
  \]

**Note**
If \( f = f^+ + f^- \), \( f \in L^1 \) \( \Rightarrow \mu( f^+ > \varepsilon ) \cdot 2^n \to 0.\)
Let \((E, \mathcal{A}, \mu)\) be a measure space, \(f: E \to \mathbb{R}\) measurable.

**Lemma 2.** Let \(f \in L^1\). Then

1. \[\int f \, d\mu \xrightarrow{(c \to \infty)} 0; \quad \{1f > c\}\]

2. For all \(\varepsilon > 0\), \(\exists \delta > 0\), s.t. for \(A \in \mathcal{A}\) with \(\mu(A) < \delta\), \[\int_A |f| \, d\mu < \varepsilon.\]

**Proof.** (1) We may assume \(f \geq 0\). Then
\[
\sum_{n=0}^{\infty} \int_E \mathbf{1}_{f \in [n, n+1)} \, d\mu = \int f \, d\mu < 0.
\]

Thus,
\[
\sum_{n=1}^{\infty} \int_E \mathbf{1}_{f \geq n} \, d\mu = \int \mathbf{1}_{\{f \geq 1\}} \, d\mu \xrightarrow{(n \to \infty)} 0.
\]

(2) Suppose not, \(\exists \varepsilon_0 > 0\), and a sequence of \(A_n\) with \(\mu(A_n) < \frac{1}{2^n}\). Set \(A = \lim_{n \to \infty} A_n = \bigcup_{n=1}^{\infty} A_n\). Then \(\mu(A) = 0\); \(\lim_{n \to \infty} \mu(\bigcup_{k=1}^{n} A_k) = 0\).

Which implies \(\int_A \mathbf{1}_{f} \, d\mu = 0\).

But \[\int_A |f| \, d\mu \geq \lim_{n \to \infty} \int_{A_n} |f| \, d\mu \geq \varepsilon_0,\] giving a contradiction.
Definition 3 (1) A family of $L^p$ functions $\{f_\alpha, \alpha \in \Lambda\}$ is uniformly integrable (U.I.) if

$$\sup_{\alpha \in \Lambda} \int_{\{f_\alpha > c\}} 1_{|f_\alpha| > c} \, d\mu \to 0 \quad (c \to \infty)$$

(2) It is $L^p$ bounded if $\sup_{\alpha \in \Lambda} \int |f_\alpha|^p \, d\mu < \infty$.

(3) It is uniformly absolutely continuous if for every $\varepsilon > 0$, $\exists \delta > 0$, s.t. for all $A \in \mathcal{A}$ with $\mu(A) < \delta$, $\int |f_\alpha| \, d\mu < \varepsilon$ for every $\alpha \in \Lambda$.

Lemma

Suppose $\mu$ is a finite measure.

(a) If $\{f_\alpha\}$ is U.I., then it is $L^1$ bounded.

(b) If $\{f_\alpha\}$ is $L^p$ bounded for $p > 1$, it is U.I.

Proof

We may assume $\mu$ is a probability measure.

(1) $E|f_\alpha| = E\left[1_{|f_\alpha| < 1} + 1_{|f_\alpha| \geq 1}\right] = E[1_{|f_\alpha| < 1} + 1_{|f_\alpha| \geq 1}]$.

By U.I., we choose $C$ s.t. $E[1_{|f_\alpha| \geq 1}] \leq 1 + C$, for all $\alpha$.

$E(|f_\alpha| \leq 1 + C$.

(2) $\int 1_{|f_\alpha| > C} \, d\mu \leq \int \frac{|f_\alpha|^p}{C^{p-1}} \, d\mu \leq \frac{1}{C^{p-1}} \sup_{\alpha} \int |f_\alpha|^p \, d\mu \to 0 \quad (p \to 1)$.
Theorem 5. Let \((X_n)\) be a sequence r.u. on a prob.

Then the following are equivalent.

(1) \(X_n \to x\) in \(L^1\) (i.e. \(\mathbb{E}|X_n - x|^1 < \infty\))

(2) \((X_n)\) is u.i. and \(X_n \to x\) in measure.

Proof. Suppose \((X_n)\) is u.i. and \(X_n \to x\). Then \(x \in \mathbb{L}^1\).

Choose an a.s. convergent subsequence \(X_{n_k}\), \(E|X_{n_k}| = E\lim_{n \to \infty} |X_n| \leq \lim_{n \to \infty} E|X_{n_k}| < \infty\).

Define \(Y_c(x) = \{x, x \leq c\} \cup \{K, x > c\}\).

\[
\int |4c(x_n) - x_n| \leq \int |x_n| dp \quad (x_n \geq c)
\]

For \(\varepsilon > 0\), choose \(c(\varepsilon) > 0\), so that

\[
\int |4c(x_n) - x_n| dp < \frac{\varepsilon}{3}, \quad \int |4c(x) - x| dp < \frac{\varepsilon}{3}.
\]

Then

\[
\int |x_n - x| \leq \int |4c(x_n) - 4c(x)| + \int |4c(x_n) - x_n| + \int |4c(x) - x| dp
\]

\[
\leq \frac{2\varepsilon}{3} + \int |4c(x_n) - 4c(x)|.
\]

By the DCT, \(\int |4c(x_n) - 4c(x)| \to 0\).

Choose \(N\) s.t. if \(n > N\), \(\int |4c(x_n) - 4c(x)| < \frac{\varepsilon}{3}\).

and for \(n\), \(\int |x_n - x| < \varepsilon\).
Proposition: Let $x \in L^1$. Then the family of r.v.
\[ \{ E\subseteq \Omega : \mathcal{A} \text{ sub-

algebra of } \mathcal{F} \} \]

is uniformly integrable.
§7 Stopping time.

**Definition** A map \( T : \Omega \rightarrow [0, \infty] \) is a \((\mathcal{F}_t, t \in \mathbb{T})\) stopping time (s.t.), if for every \( t \in \mathbb{T} \),

\[
\{ T \leq t \} = \{ \omega : T(\omega) \leq t \} \in \mathcal{F}_t.
\]

(i.e. \( 1_{T \leq t} \) as a process is adapted to \( \mathcal{F}_t \)).

**Example** \( T \equiv \infty \), \( T \equiv t_0 \), \( t_0 \) a constant.

\[
\mathcal{F}_t = V_{s \leq t} \mathcal{F}_s, \quad \mathcal{F}_{t+} = \bigwedge_{s > t} \mathcal{F}_s
\]

\[
\mathcal{F}_\infty = \bigvee_{t \geq 0} \mathcal{F}_t = \bigvee_{t \geq 0} \mathcal{F}_{t-} = \bigvee_{t \geq 0} \mathcal{F}_{t+}.
\]

**Proposition 7.2** A map \( T : \Omega \rightarrow \mathbb{T} \) is an \( \mathcal{F}_{t+} \) stopping time if and only if for every \( t \), \( \mathcal{F}_t \) is \( \mathcal{F}_{t+} \) s.t.

\[
\Rightarrow \{ T < t \} = \bigcap_{n=1}^{\infty} \{ T < t + \frac{1}{n} \} \in \mathcal{F}_t.
\]

**Proof.** \( \Rightarrow \{ T < t \} = \bigcap_{n=1}^{\infty} \{ T < t + \frac{1}{n} \} < \mathcal{F}_t \), if \( T \) is \( \mathcal{F}_{t+} \) s.t.

**Corollary 7.3** If \( \mathcal{F}_t \) is right continuous, then \( T \) is an \( \mathcal{F}_t \) stopping time if and only if \( \{ T < t \} \in \mathcal{F}_t \).

Observe the enlarged filtration \( \mathcal{F}_t^+ \) is always right cts.

If \( (X_t) \) is a stochastic process, \( B \) a measurable set in its state space, the hitting time \( B \) by \( (X_t) \) is:

\[
T_B(\omega) = \inf \{ t > 0 : X_t(\omega) \in B \}, \quad \inf \phi = 0.
\]
Example 2. Let \((X_n, n \in \mathbb{N})\) be an \((\mathcal{F}_n, n \in \mathbb{N})\) adapted process, then \(T_B\) is a stopping time.

Just note \(\bigcap_{n \geq 1} \bigcup_{k=1}^n X_k(w) \in B^3\).

Proposition 7.4 Suppose that \((X_t)\) is continuous and \(B\) is a closed set. Then \(T_B\) is a stopping time w.r.t. the natural filtration \(\mathcal{F}^0_t\) of \((X_t)\).

Proof. \(\{T_B \leq t\} = \bigcup_{n=1}^\infty \{d(X_{t_n}, B) \leq \frac{t}{n}\}\).

If \(x_0 \in B\), there exist a sequence \(t_n \in \mathbb{Q} \cap [0, t_0]\), s.t. \(d(X_{t_n}, B) \to 0\).

For open set, this is different. We define \(T_B\) by infimum.

Let \(T_a = \inf \{t > 0, X_t = a\}, X_t \in \mathbb{R}\).

If \(x < a\) for all \(s \leq t\), \(x_0 = a\), just after \(t_0\).

\(X_t\) may be greater than \(a\) (in which case \(T_a = t_0\)), if \(x < a, T_a > t_0\).

If \(T_a \leq t_0\), it cannot be determined by \(x\) at \(t_0\).
Prop. 7.5 Let $U$ be an open set. Suppose $(x_t)$ is right cts, then $T_U$ is a $\mathcal{F}_t$ stopping time.

Proof. \[ \{ T_U < t \} = \bigcup \{ x_t \in U \} \in \mathcal{F}_t. \]

If $x_t \in U$, $\exists \varepsilon > 0$ st. $x_s \in U$ for $s \in [t, t+\varepsilon]$.

Example (Brownian filtration)

Let $(B_t)$ be a 1-dim BM with $B_0 = 0$. Then $\mathcal{F}_t = \{ \emptyset, \Omega \}$. Let $T := \inf \{ t > 0 : B_t > 0 \}$.

\[ S = \inf \{ t > 0 : B_t < 0 \}. \]

\[ P(T < t) = P(B_t > 0) = \frac{1}{2} \quad \text{for any } t > 0. \]

\[ P(T = 0) = P\left( \bigcap_{n=1}^{\infty} \{ T \leq \frac{1}{n} \} \right) = \lim_{n \to \infty} P(T \leq \frac{1}{n}) \geq \frac{1}{2}. \]

\[ P(S = 0) \geq \frac{1}{2}. \]

BM takes +ve and -ve values in any interval $[0, \varepsilon]$.

$\{ T = 0 \} \in \mathcal{F}_0$. 

\[ 7-3 \]
Let us use the enlarged filtration.

Prop. 7.6. For every $s \geq 0$, the stochastic process
\[
\{B_{t+s} - B_s, \, t \geq 0\}
\]
is independent of $F_s$.

Proof. Take an increasing sequence $S_0 < S_1 < \ldots < S_n$. Let
\[
ost, \, t_1 < t_2 \ldots < t_n.
\]
Then,
\[
\{B_{t_1+s_n} - B_{s_n}, \ldots, B_{t_n+s_n} - B_{s_n}\}
\]
is independent of $F_{s_n}$.

Taking $n \to \infty$, get $\{B_t - B_s, \ldots, B_T - B_s\}$ is independent.

In particular $F_0^+$ consists of sets of full measure or measure zero.

An $(F_t)$ adapted $(\mathbb{F}_t)$ is Strong Markov process if
\[
E\{X_{t+T} | F_t\} = E\{X_{t+T} | \mathbb{F}_T\}.
\]

For every finite stopping time $T$, any $t > 0$.
\[
B(T+t) - B_t \text{ is independent of } F_{T+t}.
\]

Def. 7.7. If $T$ is a stopping time, we define the
$\sigma$-algebra $F_T$ by:
\[
F_T = \{ A \in F_\infty : A \cap \{T \leq t\} \in F_t \, \forall t \in T \}.
\]
Def. 7.8 A stochastic process \((X_t, t \geq 0)\) on a state space is progressively measurable if for every \(t > 0\), then
\[
(s, \omega) \mapsto X_s(\omega)
\]
is measurable as a map from \([0, t] \times \Omega, B([0, t]) \times \mathcal{F}_t\) to \((\Omega, \mathcal{B}(\Omega))\).

Example Let \(0 = t_0 < t_1 < \ldots < t_n < \ldots\).
Let \(H_0 \in \mathcal{F}_0, H_i \in \mathcal{F}_{t_i}\).

\[
H_t(\omega) = H_0(\omega) \mathbb{1}_{S_{0, t}}(t) + \sum_{i=1}^{\infty} H_i(\omega) \mathbb{1}_{(t_i, t_{i+1}]}(t)
\]
is called an elementary process.
Every elementary process is progressively measurable.

Prop. 7.9 An adapted process which is \(\ldots\) left cts (or right cts) is progressively measurable.

Sketch Proof Suppose \((X_t)\) is left cts. Define
\[
X_t^{(n)}(\omega) = X_0(\omega) \mathbb{1}_{S_{0, t}}(s) + \sum_{j=0}^{2^n} X_{t_j}(\omega) \mathbb{1}_{(t_j, t_{j+1}]}(s),
\]
\[
X_t(\omega) \rightarrow X_t(\omega)
\]
The limit of a sequence of progressively measurable functions is measurable.

Prop. 8.0 If \((X_t)\) is \(\ldots\) left cts (or progressively meas.) and adapted process, there exists a sequence of elementary processes \(X_t^{(n)}\) s.t.
\[
\lim_{n \to \infty} \int_0^t X_s - X_s^{(n)} \, ds = 0.
\]
We define $X_T(w) = X_t(w)$ if $T(w) = t$.
So $X_T$ is defined on $\{ T < \infty \}$.

**Proposition 8.1** If $(X_t)$ is progressively meas. and $T$ a stopping time, then $X_T \in \mathcal{F}_T$ on $\{ T < \infty \}$.

**Proof.** This means $\{ X_T \in \mathcal{B} \} \cap \{ T < \infty \} \in \mathcal{F}_T \forall t \geq 0$.
Sufficient to prove $\{ X_{T} \in \mathcal{B} \} \in \mathcal{F}_T$.
'' '' '' for any $s \leq t$. Let $s \in \mathcal{T}$, $\{ X_s \in \mathcal{B} \} \in \mathcal{F}_t$.

Let $X : \mathbb{R} \times [0, t] \to \mathbb{R}$

$$X(w, t) = X_t(w)$$

$X$ is measurable.

$\Psi : (\mathbb{R}, \mathcal{F}_t) \to \mathbb{R} \times [0, t]$, $\mathcal{F}_t \times \mathcal{B}(0, t)$

$w \mapsto (w, T(w))$

Then $\Psi$ is measurable, and $X_T = X \circ \Psi \in \mathcal{F}_T$.

$$\{ w : (w, T(w)) \in A \times (s_1, s_2) \} = A \cap \bigcup_{s_1 \leq t \leq s_2} \mathcal{F}_t \in \mathcal{B}(0, t) \in \mathcal{F}_t.$$
**Martingale Convergence Theorem.**

**Thm 8.1** Let \((X_n)\) be a supermartingale, \(n \in \mathbb{N}\).

1. Suppose \(\sup_n X_n < \infty\), then \(X_\infty = \lim_{n \to \infty} X_n\) exists.
2. If \(\sup_n E|X_n| < \infty\), then \(X_\infty\) is in \(L^1\). (Fatou's lemma)

**Sketch** This is due to an upper crossing theorem (which is simple to prove).

\[
(b-a) E\left[\sup_{t \leq a} (X_t - a)\right] \leq E( X_a - a )
\]

\# of crossings made by \(X_t\) from \(a\) to \(b\)

**Betting strategy:** Pay \(\$1\) when \(X_t < a\)

If \(\lim \sup X_n \neq \lim\inf X_n\), stops playing when \(X_t > b\)

Net gain \((b-a)\), stops playing when \(X_t > b\) often enough.

There exist two numbers \(a, b \in \mathbb{R}\) s.t. \(X_n (\omega)\) crosses \(a \to b\) \(n\)-times.

**Thm 8.2** (continuous version) Let \((X_t)\) be a right \(ctd\) stochastic process. Then \(\lim_{t \to T} X_t\) exists a.s. if

one of the following conditions hold:

1. \((X_t)\) is a supermartingale with \(\sup_{t \leq T} E(X_t^-) < \infty\)
2. \(\ldots \sup_{t \leq T} E(X_t^+) < \infty\).

**Note**

\((X_t, t \in [0,1])\), \(\lim_{t \to 1} X_t\).

\((X_t, t \in (0,\infty))\), \(\lim_{t \to \infty} X_t\).

Can define \(X_T (\omega) = \{X_t (\omega), \forall t \leq T\} \cup \{X_\infty (\omega), T(\omega) < \infty\}\)