Stochastic integral equation:

\[ \{ B^1_t, \ldots, B^m_t \} \text{ BM's on } \mathbb{R}^m, \]
\[ \sigma_i : \mathbb{R}^d \rightarrow \mathbb{R}^d, \quad i = 0, 1, \ldots, m. \]
\[ \sigma_i = (\sigma^1_i, \ldots, \sigma^d_i), \quad \sigma^i : \mathbb{R} \rightarrow \mathbb{R}, \]

Borel measurable.

Consider the equation on \( \mathbb{R}^d \)

(1) \[ x_t = x_0 + \sum_{k=1}^{m} \int_0^t \sigma^k(x_s) \, dB^k_s + \int_0^t \sigma^0(x_s) \, ds \]

In components:

\[ x^i_t = x^i_0 + \sum_{k=1}^{m} \int_0^t \sigma^{i,k}(x_s) \, dB^k_s + \int_0^t \sigma^{i,0}(x_s) \, ds \quad i = 1, \ldots, d. \]

A solution \((x_t, t \leq T)\) is an adapted stochastic process satisfying (1), where \( T \) is a random time.

The life time of \( x_t \) is the maximal time for which \( x_t, t \leq T \) exists, and \( T = \lim_{n \to 0} R \) where \( R = \inf_{t \geq 0} \{ |x_t| > 2 \} \).

Example: \( x_t = \frac{1}{2 - B_t}, \quad t < \inf_{s \geq 0} B_s \geq 2 \), \( x_T = 0 \) for \( t > T \).

is a maximal solution to

\[ x_t = \frac{1}{2} + \int_0^t (x_s)^2 \, dB_s + \int_0^t (x_s)^3 \, ds. \]

We say add a coffee state \( A \), and \( x_t = x \) for \( t > T \).

Then \((x_t, t \geq 0)\) is defined for all \( t \).
Proof. Let \( T = T_{k+1} \), \( B_k \geq L \). Let \( T < T \).

Take \( f(x) = \frac{1}{2-x} \). Then \( f(x) = \frac{1}{2-x}, f'(x) = \frac{2}{(2-x)^2} \).

\[
f'(x_0) = \frac{1}{2} + \int_{T_{k-1}}^{T_k} \frac{2}{(2-x_0)^2} dB_s + \int_{T_{k-1}}^{T_k} \frac{-1}{(2-x_0)^3} ds
\]

So \( x_k = \frac{1}{2} + \int_{T_{k-1}}^{T_k} (x_0)^2 dB_s + \int_{T_{k-1}}^{T_k} (x_0)^3 ds \), on \( \{ T < T \} \).

Since \( \lim_{T \to 2} x_T = \infty \), \( T \) is the life line of solution to the equation with initial value \( x_0 \).

Definition 1. A solution to the stochastic differential equation

\[
dx_t = \sum_{k=1}^{N_t} \sigma_k(x_t) dB_t^k + \sigma_0(x_t) dt
\]

with initial value \( x_0 \) is an adapted, continuous, stochastic process on a filtered probability space \((\Omega, \mathcal{F}, \mathcal{F}_t, P)\) satisfying the usual conditions and a driving Brownian motion \( B_t = (B_t^1, \ldots, B_t^d) \) with values in \( \mathbb{R}^d \) s.t. for any stopping time \( T < T \),

\[
x_T = x_0 + \sum_{k=1}^{N_T} \int_{0}^{T} \sigma_k(s, x_s) dB_s^k + \int_{0}^{T} \sigma_0(s, x_s) ds
\]

If we can choose \( T = \infty \) a.e., then the SDE is said to have a global solution from \( x_0 \).

If for any \( x_0 \in \mathbb{R}^d \), the solutions to the SDE are global, we say the SDE is complete (is conservative, does not explode).

Standard results. Lipschitz \( \to \) global solution
Local Lipschitz \( \to \) uniqueness, existence.
Example 1. \( dx_t = a x_t \, dB_t + b x_t \, dt, \ a, b \in \mathbb{R} \nabla \) 

We have seen that if \( B_t \) is a \( \mathcal{B} \), \( x_0 \) is a solution with initial value \( x_0 \). For any 
\( t : (0, \infty) \to \mathbb{R} \) 
\( F(t, x_0, t) = x_0 e^{at - \frac{bt^2}{2}} \) 
\( F(t, x_0, t) : \mathcal{W} \to \mathbb{R} \) 
\((B, F(t, x_0, B))\) is a solution.

Example 2. Consider Tanaka's equation:

\( d x_t = \text{sgn}(t) \, dB_t \). 

Let \((W_t)\) be a \( \mathcal{B} \) on a probability space. 

Set \( B_t = \int_0^t \text{sgn}(s) \, dB_s \) 

Then \( W_t = \int_0^t \left( \int_0^s \text{sgn}(w) \, dw \right) \, dw = \int_0^t \text{sgn}(w) \, dw \). 

So \( (W_t, \int_0^t \text{sgn}(w) \, dw) \) is a solution
from \( x_0 \). We have seen that \( W_t \) is also a solution to \( W_t = \int_0^t \text{sgn}(w) \, dw \). 
And the drift of \((W_t)\) is that of a \( \mathcal{B} \).
§15 Notion of solutions & uniqueness

Def 1. A solution \((\xi_t)\) with driving BM \((B_t)\) is a strong solution if \((\xi_t)\) is adapted to the filtration generated by \((B_t)\) and satisfying the usual assumption. Otherwise a solution is weak.

Solution in Example 1, Section 14 is strong.

For Tanaka's SDE, \(d\xi_t = g_0(\xi_t) dB_t\) any BM \((\xi_t)\) on any probability space is a solution with driving Brownian motion \(B_t = \int_0^t g_0(\xi_s) \, ds\).

It is known that \((\xi_t)\) is not adapted to the filtration of \((B_t)\).

Def 2. We say pathwise uniqueness holds for the SDE if whenever \((\xi_t)\) and \((\eta_t)\) are solutions to the SDE on the same probability space and same driving Brownian motion and \(\xi_0 = \eta_0\) a.e., we have \(\xi_t = \eta_t\) a.e.

We say uniqueness in law holds if any two solutions of the SDE with the same initial value have the same probability distributions.

Example 2. Pathwise uniqueness fails for Tanaka's equation.

If \(d\xi_t = \xi_t \, dB_t, d\xi_t \, dB_t, x_t \in \mathbb{R}\), pathwise uniqueness.

(For \(0 < \xi \) \(d\xi_t = \xi_t \, dt, x < 1, \xi_0 = 0\). Uniqueness fails.)
Prop 4. If pathwise uniqueness holds, then any solution is a strong solution and uniqueness in law holds.

Thm 5 (The Yamada–Watanabe Thm).

If for each initial probability distribution, there is a weak solution to the SDE, and suppose pathwise uniqueness holds, then there exists a Borel measurable map \( F : \mathbb{R}^d \times \mathcal{W}_0 \to \mathcal{W} \) such that:

1) For any BM \((B_t)\) on \(\mathbb{R}^m\) and any \(\omega\),
\[ F_t(x_0, B_t) \] is a solution to the equation with driving noise \((B_t)\).

2) If \((x_t)\) is a solution with a diagonal noise \((B_t)\), then \(x_t = F_t(x_0, B_t)\).

Example
\[
\begin{align*}
\text{d}x_t &= x_t \, dB_t \\
x_t &= x_0 \, e^{{\frac{1}{2}\sigma_t}} \\
F_t(x_0, \sigma) &= x_0 \, e^{{\frac{1}{2}\sigma_t}}.
\end{align*}
\]
Lemma 6

Let $f, g$ be locally bounded predictable processes, (i.e. they are measurable w.r.t. the filtration generated by left cts processes), let $(B_t, \mathcal{F}_t)$ be continuous semi-martingales.

If $(f, B)$ and $(g, W)$ have the same distribution, then so do

$$(f, B, \int_0^t f \, \text{d}B_s) \text{ and } (g, W, \int_0^t g \, \text{d}W_s).$$

Sketch proof for Thm 5.

- If $(X_t, B_t)$ is a solution, denote by $\mu$ its distribution on $W_0^d \times W_0^m$. Then the canonical process $(X_t, \mathcal{F}_t)$ on $(W_0^d \times W_0^m, \mu)$ has the same law as $(X_t, B_t)$ on $(\mathcal{F}_t, \mathbb{P})$.

And $(X_t, B_t)$ is a solution to the SDE.

- Given two solutions $(X^1_t, B^1_t)$, $(X^2_t, B^2_t)$ with distributions $\mathbb{Q}^1, \mathbb{Q}^2$ we build them on the same prob. space:\n
  Write $\mathbb{Q}^i (\text{d}W, \text{d}W') = \mathbb{Q}^i (\text{d}W, \text{d}W') \mathbb{Q}(\text{d}\mathbb{W})$.

  Then $\mathcal{Q}^1 (W, \text{d}W') \times \mathcal{Q}^2 (W, \text{d}W') \mathbb{Q}(\text{d}W)$. Then $\mathbb{W} = \mathbb{W}'$ by pathwise uniqueness, they are independent (and $\mathbb{W}$)

  Thus $X^i_t = X^i_t = F_t(x, W)$, for some constant $F_t(x, W) : W \rightarrow \mathbb{R}^d$. 
§16 Existence and uniqueness theorem

**Theorem (Banach fixed point theorem)**

Let $(X, d)$ be a complete metric space and $\Phi : X \to X$ a contraction mapping, i.e., there exists a number $C \in (0,1)$ s.t.

$$d(\Phi(x), \Phi(y)) \leq C d(x, y) \quad \text{for all } x, y \in X,$$

then there exists exactly one $x \in X$ s.t. $\Phi(x) = x$.

Furthermore, for any $x_0 \in X$, the sequence defined by

$$x_1 = \Phi x_0, \quad \ldots, \quad x_{n+1} = \Phi x_n (= \Phi^2 x_0)$$

converges to $x$ (as $n \to \infty$).

**Proof.** Let $x_{n+1} = \Phi x_n$, some $x_0 \in X$. Then

$$d(x_{n+1}, x_n) \leq C d(x_n, x_{n+1}) \leq \cdots \leq C^n d(x_0, x_0)$$

Hence

$$\sum_{n=0}^{\infty} d(x_{n+1}, x_n) \leq \frac{C}{1-C} d(x_0, x_0) < \infty,$$

Thus

$$d(x_n, x_m) \leq \sum_{k=n}^{m-1} d(x_{k+1}, x_k) \to 0, \quad n, m \to \infty$$

and $(x_n)$ is a Cauchy sequence and has a limit $x$ in $X$.

Take $n \to \infty$ in $x_n = x_{n+1}$, by the continuity of $\Phi$,

$$\Phi x = x.$$

If $x$ and $y$ are two solutions, then

$$d(x, y) = d(\Phi^\infty x, \Phi^\infty y) \leq C d(x, y) < d(x, y).$$

This is impossible unless $C = 1$.

Q.E.D.
Gronwall's lemma. Suppose that \( c, k \) are two constants, \( f \) is a positive locally bounded function on \( \mathbb{R}^+ \) s.t.

\[
f(t) \leq a + b \int_0^t f(s) \, ds, \quad \forall t.
\]

then for every \( t \), \( f(t) \leq a e^{bt} \).

Proof.

\[
f(t) \leq a + b \int_0^t \left( a + b \int_0^s f(r) \, dr \right) \, ds
\]

\[
\leq a + abt + b^2 \int_0^t \left( t-r \right) f(r) \, dr
\]

\[
\leq a + abt + b^2 \int_0^t f(r) \, dr,
\]

\[
\leq a + abt + ab^2 t^2 \left( 1 + \frac{1}{2!} + \frac{1}{3!} + \ldots \right)
\]

\[
+ b^3 \left[ \sum_{n=2}^{\infty} \left( \frac{1}{2!} \right) \frac{1}{n!} \int_0^t \int_0^s f(r) \, dr \, ds \right]
\]

\[
\leq b^3 \int_0^t f(r) \left( 1 + \frac{1}{2!} + \frac{1}{3!} + \ldots \right) \, dr
\]

\[
\leq \frac{e^{bt} b^3}{(n+1)!} \int_0^t f(r) \, dr \to 0.
\]

\[
f(t) \leq a e^{bt}.
\]
Theorem 2

Suppose \( v_k \), \( k = 0, 1, \ldots, m \), are (globally) Lipschitz continuous vector fields on \( \mathbb{R}^d \). Then for each \( x_0 \in \mathbb{R}^d \), there exists a unique global solution to the SDE

\[
\frac{dx_t}{dt} = \sum_{k=1}^{m} v_k(x_t) \, dB^k_t + \sigma_0(x_t) dt,
\]

with initial value \( x_0 \).

Proof. Let \( E \) denote the space of progressively measurable stochastic processes \( (x_t) \), (on a given probability space), with values in \( \mathbb{R}^d \) and \( x_t \).

For any \( T > 0 \),

\[
\mathbb{E} \left( \int_0^T x_t^2 \, dt \right) = \mathbb{E} \left( \int_0^T \left( \sum_{k=1}^{m} v_k(x_t) \, dB^k_t + \sigma_0(x_t) dt \right)^2 \, dt \right) < \infty,
\]

where \( K \) is a common Lipschitz constant for \( v \) and \( \sigma \) to be chosen later. Note \( x \in L^2((0,T)) \)

Let \( (B_t) \) be an \( \mathbb{R}^m \)-valued BM. Define

\[
\Phi(x_0)(t) = x_0 + \sum_{k=1}^{m} \int_0^t v_k(x_s) \, dB^k_s + \int_0^t \sigma_0(x_s) ds.
\]

Note \( \mathbb{E} \left( \int_0^T |v_k(x_s)|^2 \, ds \right) \leq \sigma_0(x_0) \), so integral is well defined, \( \sigma_0(x_0) \in L_{\mathbb{R}^2}^2 \).

We prove below

\[
\mathbb{E} \left( (x_t - y_t)^2 \right) \leq \frac{1}{2} \| x - y \|^2_{E^2}
\]

for any \( x, y \in E \). Since the constant process

\( \mathbb{E}(0) = \mathbb{E}, \text{ and } \Phi(0) \in \mathbb{E} \), implies \( \Phi(x) \in E \) if \( x \in E \).
\[ E(\varphi(x) - \varphi(y)) \leq E \int_0^T e^{-\delta k^2 t} \left[ \frac{1}{2} \left( \int_0^t \varphi_k(x_s) \, dB^k_s - \int_0^t \varphi_k(y_s) \, dB^k_s \right)^2 + \int_0^t \varphi_0(x_s) \, ds - \int_0^t \varphi_0(y_s) \, ds \right] \, dt \]

\[ \leq C(m) \int_0^T e^{-\delta k^2 t} \left( \frac{m}{2} E \int_0^t (\varphi_k(x_s) - \varphi_k(y_s))^2 \, ds \right) \, dt \]

\[ \leq C(m) K^2 \int_0^T e^{-\delta k^2 t} \int_0^t E |x_s - y_s|^2 \, ds \, dt \]

\[ \leq (m+1) C(m) K^2 \int_0^T E |x_s - y_s|^2 \left( \int_0^T e^{-\delta k^2 t} \, dt \right) \, ds \]

\[ \leq \frac{(m+1) C(m) K^2}{\delta k^2} \int_0^T \left( e^{-\delta k^2 t} + e^{-\delta k^2 s} \right) E |x_s - y_s|^2 \, ds \]

\[ \leq \frac{1}{2} (|x - y| E)^2 \]

if we choose \( \delta = \frac{(m+1) C(m)}{4} \).

By Brouwer's fixed point theorem, \( \varphi(x_t) \in \mathcal{F} \) a.s.

By Brouwer's fixed point theorem, \( \varphi(x_t) \in \mathcal{F} \) a.s.

\[ x_t^0 = x_0 + \int_0^t \varphi_0(x_s) \, ds \]

\[ x_t = x_t^0 + \int_0^t \varphi_k(x_s) \, dB^k_s + \int_0^t \varphi_0(x_s) \, ds \]

Since R.H.S is a continuous process, so is \( x_t \).

Note each \( \varphi_k(x) \) is adapted to \( \mathcal{F}_k \), so is its limit \( x_t \), and \( x_t \) is a strong solution.
We also proved there exists a unique solution in $E$.
We now prove every solution belongs to $E$. It is sufficient to show
\[ E \int_0^T e^{\delta s} |E(t_0)|^2 ds < \infty. \]
Let $T_n$ be the first exit time of $t_0 < 2n$.
\[
\int_0^T E \left| x_s \right|^2 e^{\delta s} ds \\
\leq C |x_0|^2 + C \int_0^T E \left| x_s \right|^2 \left( \int_0^s e^{\delta r} \frac{\partial x_r}{\partial r} dr \right) \frac{e^{\frac{\delta s}{2}}}{\delta} ds \\
\leq C |x_0|^2 + C \int_0^T E \int_0^s \left( 1 + |x_r|^2 \right) d r d s \\
\leq C |x_0|^2 + C \int_0^T e^{\delta s} \left( 1 + E \left| x_s \right|^2 \right) d r d s \\
\leq C |x_0|^2 + C \int_0^T \left( 1 + E \left| x_s \right|^2 \right) \left( e^{-\delta s} - e^{-\delta T} \right) ds \\
\leq C |x_0|^2 + \frac{C}{\delta} + \frac{C}{\delta} \int_0^T E \left| x_s \right|^2 e^{-\delta s} ds \\
\text{choose } \delta \text{ small } \frac{\delta}{2} < \frac{1}{2}, \text{ since } \frac{C}{\delta} < \frac{1}{2} \\
\int_0^T E \left| x_s \right|^2 e^{-\delta s} ds \leq 2 C (x_0 + \frac{C}{\delta}).
\]
Fatous lemma applied to show $\| x(t) \|_E$. 

Uniqueness holds: if $\| x_1 \|_E$ and $\| x_2 \|_E$, solves $SDE \implies x_1 = x_2 \in E$. 
That (Existence, local Lipschitz case).

Suppose that \( (\Omega_k) \) are locally Lipschitz case.

For each \( x_0 \), there exist a unique strong maximal solution \( (x_k, t < \mathcal{C}) \).

Furthermore, \( \lim_{t \uparrow \mathcal{C}} x_k = \infty \) on \( \{ t < \mathcal{C} \} \).

Proof: We set \( \sigma_k^N = \begin{cases} \sigma_k(x), & \text{if } x \leq N \\ \sigma_k(N, \theta) & \text{in polar coordinates.} \end{cases} \)

Then \( \sigma_k^N \) is globally Lipschitz case.

By the previous lemma, \( \exists \) a solution \( (x^N_k) \),

the SDE with coefficients \( \sigma_k^N \).

Furthermore, \( x^N_k = x^{N+1}_k \) on \( t < \mathcal{C} \), the first exit time for \( \mathcal{B}(0, N) \).

Consequently, we can patch up a solution \( (x_k) \)

s.t. \( x_k = x^N_k \) on \( t < \mathcal{C} \). And

\( \lim_{t \uparrow \mathcal{C}} x_k = \infty \).
Lemmas: Gronwall's inequality

Suppose \( f : [0, T] \to \mathbb{R}^+ \) is locally \( \text{log Lip} \) measurable and s.t. \( f, k \in L^1 \mathbb{R} \),

\[
f(t) \leq C + k \int_0^t f(s) \, ds, \quad \forall t \in [0, T].
\]

Then \( f(t) \leq Ce^{kt} \). In particular, \( f=0 \) if \( C=0 \) for all \( t \in [0, T] \).

Theorem (Uniqueness) Suppose that each \( \sigma_t \) is locally \( \text{Lipschitz} \). Then pathwise uniqueness holds.

Proof. Suppose that \( (x_t), (\tilde{x}_t) \) are two solutions.

Let \( T_N = \inf \{ t \geq 0 \mid |x_t - \tilde{x}_t| > N \} \)

\( T_N = \inf \{ t \geq 0 \mid |\tilde{x}_t| > N \} \)

Let \( \tilde{T}_N = T_N \wedge T_N \).

Then

\[
\mathbb{E} \left| x_{t \wedge \tilde{T}_N} - \tilde{x}_{t \wedge \tilde{T}_N} \right|^2 \leq C(t) \int_0^t \mathbb{E} \left| x_{s \wedge \tilde{T}_N} - \tilde{x}_{s \wedge \tilde{T}_N} \right| \, ds
\]

for some constant \( C(t) \).

(exercise: fill in details).

Thus \( \mathbb{E} (x_{t \wedge \tilde{T}_N} - \tilde{x}_{t \wedge \tilde{T}_N})^2 = 0 \).

So \( x_{t \wedge \tilde{T}_N} = \tilde{x}_{t \wedge \tilde{T}_N} \Rightarrow x_t = \tilde{x}_t \) on \( t \wedge \tilde{T}_N \).

\( \Rightarrow T_N = \tilde{T}_N \) and \( x_t = \tilde{x}_t \) a.e.
Theorem. Suppose that \( \varphi_k \) are \( C^1 \) and suppose that \( \varphi_k \) grows at most linearly, i.e.
\[
|\varphi_k(x)| \leq C + C|x|.
\]
Then every solution of the SDE has finite lifetime (we say the SDE does not explode).

Proof. Let \( (x_t) \) be a solution with lifetime \( T \).

Let \( T = \sup_{t \geq 0} \{ |x_t| \geq 1 \} \).

Then
\[
E \left[ x_{t+\tau_1}^2 \right] \leq 2|x_t|^2 + 2E \left[ \sum_{k=1}^{\tau_1} \varphi_k(x_t) dB_t + \int_0^{\tau_1} \varphi(x_t) \, ds \right]
\]
\[
\leq 2|x_t|^2 + C \sum_{k=1}^{\tau_1} E \int_0^t \varphi_k(x_s) \, ds + \left( E \int_0^{\tau_1} \varphi(x_t) \, ds \right)^2
\]
\[
\leq 2|x_t|^2 + C E \int_0^t \varphi(x_s) \, ds.
\]

This implies
\[
E \left[ x_{t+\tau_1}^2 \right] \leq \left( 2|x_t|^2 + C T \right) e^{C T}
\]
\[
\Rightarrow \quad E \left[ x_{t+\tau_1}^2 \right] \leq 2 \left( C T + C T \right) e^{C T}
\]
\[
\Rightarrow \quad |x_t|^2 < \infty, \text{ a.e.}.
\]
We say an SDE is complete if for any initial value \( x_0 \), it has a global solution with \( x_0 \) as its initial value.

Let \( f \) be \( \mathcal{B} \)-measurable.

Let us suppose that the SDE is complete. Set \( P_t f(x) = \mathbb{E} f(X_t) \)

\( P_t(x,\cdot) = P(X_t \in \cdot \mid X_0 = x) \)

**Theorem 1.** If \( \sigma_k \) is Lipschitz cts, then \( P_t \) defines a family of Markov transition probabilities.

In particular \( \mathbb{E} f(X_{t+s} \in \cdot \mid \mathcal{F}_s) = P_t f(x_s) \).

Let \( f: \mathbb{R}^d \to \mathbb{R} \) be a \( C^2 \) function, then by Itô's formula,

\[
    f(x_t) = f(x_s) + \int_0^t df(\sigma_k(x_s)) dB_k^s + \int_0^t \sigma_k(x_s) dB_k^s,
\]

where

\[
    df(\sigma_k(x)) = \frac{\partial}{\partial x_k} \frac{\partial}{\partial x_k} f(x) - \frac{\partial}{\partial x_k} f(x) \sigma_k(x)
\]

\[
    L f(x) = \frac{d}{dx} \left( \sum_{k=1}^m \sigma_k^2(x) \frac{d}{dx} f(x) \right) + df(\sigma_0(x))
\]

Observe if \( f \in C_c^\infty \), \( \sigma_k \) are cts, then \( L f \)

is bounded, and \( f(x_t) - f(x_0) - \int_0^t L f(x_s) ds \)

is a martingale. (It is a local martingale for \( f \in C^2 \).)
Take expectation to see:

\[ \mathbb{E} f(x_t) = f(x_0) + \int_0^t (\mathbb{E} f(x_s)) \, ds \]

\[ \lim_{t \to 0} \frac{\mathbb{E} f(x_t) - f(x_0)}{t} = L f(x_0) \]

i.e.

\[ \frac{\partial}{\partial t} \mathbb{P} e^{tf} x_0 = L \mathbb{P} e^{tf} x_0, \]

i.e. \( L \) is \textit{infinitesimal generator} of \( \mathbb{P}_t \) (also called the generator of the semigroup).

In particular, \( \mathbb{P}_t f \) solves the Cauchy problem:

\[
\begin{cases}
\frac{\partial u(t,x)}{\partial t} = L u(t,x) \\
u(0,x) = f(x)
\end{cases}
\]

\textbf{Theorem 2}. Suppose \( \Phi \) is a \( \mathcal{C}^2 \) function s.t.

\[ \lim_{x \to \infty} \Phi(x) = \infty \quad \text{and} \quad L f \leq C_1 f + C_2 \]

where \( C_1 \) and \( C_2 \) are constants. Then the SDE is complete.
Proof. By Itô's formula, letting $\tau_n = \inf \{ \tau \geq 0 \mid \Phi(x_\tau) \leq n \},$

$$\Phi(x_{\tau_n}) = \Phi(x_0) - \int_0^{\tau_n} \mathbb{L} \Phi(x_s) \, ds$$

is a martingale. In particular

$$\mathbb{E} \Phi(x_{\tau_n}) = \mathbb{E} \Phi(x_0) + \int_0^{\tau_n} \mathbb{E} \mathbb{L} \Phi(x_s) \, ds$$

$$\leq \mathbb{E} \Phi(x_0) + \int_0^{\tau_n} \left( c_1 \mathbb{E} \Phi(x_s) + c_2 \right) ds$$

$$\leq \mathbb{E} \Phi(x_0) + c_1 \int_0^{\tau_n} \mathbb{E} \Phi(x_{s \wedge \tau_n}) ds$$

$$+ c_2 \tau_n.$$  

By Gronwall's lemma (if $f$ is the bounded measure and $f(t) \leq c_1 + c_2 \int_0^t f(s) \, ds$ for all $t$, then $f(t) \leq c_1 e^{c_2 t}$),

then

$$\mathbb{E} \Phi(x_{\tau_n}) \leq (\mathbb{E} \Phi(x_0) + c_2 \tau_n) e^{c_1 \tau_n}.$$  

i.e.

$$\mathbb{E} \Phi(x_t) 1_{t < \tau_n} + \mathbb{E} \Phi(x_{\tau_n}) 1_{t \geq \tau_n} \leq (\Phi(x_0) + c_2 \tau_n) e^{c_1 \tau_n}.$$  

i.e.

$$\mathbb{P}(t > \tau_n) \leq (\Phi(x_0) + c_2 \tau_n) e^{c_1 \tau_n}.$$  

$$\lim_{n \to \infty} \mathbb{P}(t > \tau_n) = 0.$$  

i.e. $\mathbb{P}(t > \lim_{n \to \infty} \tau_n) = 0.$

i.e. The lifetime $= \infty.$
Example Suppose \( \sigma \) are \( L^1 \), and \( \exists \) constants \( c_1, c_2 \) s.t. 
\[
\langle \sigma_0(x), x \rangle_{\mathbb{R}^d} \leq c_1 \|x\|^2 + c_2. 
\]
\[
|\sigma_k(x)| \leq c_1 \|x\| + c_2, 
\]
Then the SDE does not explode.

Proof \( L(1|x|^2) = L(\frac{d}{dx} (|x|^2)) \)
\[
= \frac{d}{dx} \sum_{i=1}^d x_i \sigma_i(x) 
\]
\[
+ \sum_{k \neq k} \frac{\sigma_k(x)}{k} \sigma_i(x) 
\]
\[
\leq 2 \langle x, \sigma_0(x) \rangle_{\mathbb{R}^d} + \sum_{k=1}^m |\sigma_k(x)|^2 
\]
\[
\leq 2c_1 \|x\|^2 + 2c_2 + m(c_1 \|x\| + c_2)^2 
\]
\[
\leq \alpha + \beta \|x\|^2 
\]
for some number \( \alpha, \beta \). Apply Lyapunov test to conclude.
**Girsanov Transform.**

Let \((\Omega, \mathcal{F}, \mathbb{F}_t, P)\) be a filtered probability space with standard assumptions. If \(Q\) is a probability measure equivalent to \(P\), and \((M_t)\) a martingale w.r.t. \(P\), then \((M_t)\) is in general not a (local) martingale w.r.t. \(Q\).

\[
\int_A M_t \, dP = \int_A M_s \, dP, \quad \forall A \in \mathcal{F}_s
\]

\[
\int_A M_t \, d\alpha = \int_A M_s \, d\alpha.
\]

**Lemma 1.** Suppose that for each \(t \geq 0\), on the \(\sigma\)-algebra \(\mathcal{F}_t\), \(Q\) is absolutely cts w.r.t. \(P\) with Radon-Nikodym derivative \(\frac{dQ}{dP} = \mathcal{F}_t\). Then \((\mathcal{F}_t)\) is a martingale w.r.t. \(P\).

**Proof.** By the definition \(\mathcal{F}_t \in \mathcal{F}_t\), and is unique a.s. w.r.t. \(P\). Let \(s < t\), \(A \in \mathcal{F}_s\). Then

\[
Q(A) = \int_A d\alpha = \int_A \frac{dQ}{dP} \, dP \Rightarrow P_s = \mathbf{E}^P \left[ \frac{dQ}{dP} \bigg| \mathcal{F}_s \right].
\]

Where \(\mathbf{E}^P\) indicates conditional expectation w.r.t. \(P\).
let \((N_t)\) be a continuous local martingale w.r.t. \(P\). We define a measure \(\mathcal{Q}\) on \(\mathcal{F}_t\) by
\[
\frac{d\mathcal{Q}}{dP} = e^{-\frac{1}{2} <N,N>_t}.
\]
If \(E(e^{\frac{1}{2} <N,N>_t}) = 1\), then \(\mathcal{Q}\) is a probability measure.

**Definition.** The stochastic process \(e^{\frac{1}{2} <N,N>_t}\) is called the exponential martingale of \((N_t)\).

**Lemma.** Novikov's condition is \(E(e^{\frac{1}{2} <N,N>_t}) < \infty\).

If Novikov's condition holds, the exponential martingale \(e^{\frac{1}{2} <N,N>_t}\) is a martingale.

**Remark.** The exponential martingale solves
\[
dX_t = X_t \, dN_t,
\]
and is therefore always a local martingale if \((N_t)\) is a cts local martingale. Positive local martingales are supermartingales. It is a true martingale if and only if
\[
E ( e^{\frac{1}{2} <N,N>_t} ) = 1.
\]
(2) Let \((f_t)\) be a strict positive local martingale with \(f_0 = 1\)
then there exists a cts local martingale \((N_t)\) s.t.
\[
P_t = e^{N_t - \frac{1}{2} \langle N, N \rangle_t}
\]

\[\log P_t = \ldots\]

Exercise. Let \(P_t = e^{N_t - \frac{1}{2} \langle N, N \rangle_t}\) where \((N_t)\) is a continuous local martingale.
Let \((M_t)\) be a continuous local martingale. Prove that
\[
\langle M, N \rangle_t = \int_0^t \frac{1}{f_s} \, d <M, N>_s
\]
It is standard to assume \(F_0 = [\mathcal{P}, \phi]\), completed, so any two means on \(F_0\) agree.

Theorem 1. (Girsanov Theorem). Let \(P\) and \(Q\)
be two equivalent probability measures on \(\mathcal{F}_\infty\) with \(f = \frac{dQ}{dP}\). Assume that \(F_t \equiv E[f^{\infty} | F_t]\)
is continuous and we write \(P_t = e^{N_t - \frac{1}{2} \langle N, N \rangle_t}\).
Let \((M_t)\) be a cts local martingale w.r.t. \(P\).
Then \(\tilde{M}_t := M_t - \langle M, N \rangle_t\)
is a local martingale w.r.t. \(Q\).
**Proof.** Take $T_n$ to be the first time that
\[ \max \left( 1, 1 / t, 1 / t^2, \langle N, N \rangle_t, \langle M, N \rangle_t \right) \] reaches $n$. We prove below that $M^n_t$ is a martingale for each $n$. Since the stochastic processes concerned are continuous, $T_n(n) \to \infty$ a.s. For simplicity we assume all the processes are bounded.

Let $s \leq t$, $A \in \mathcal{F}_s$, we show that $(M^n_t)_s$ is a martingale, i.e.
\[ \int_A \tilde{M}^n_t \, dA = \int_A \tilde{M}^n_s \, dA, \quad \forall A \in \mathcal{F}_s. \]

i.e.
\[ \int_A \tilde{M}^n_t \, dp = \int_A \tilde{M}^n_s \, dp. \]

We may assume $M^0 = 0$ for simplicity.

Note that
\[ M^n_t = M^n_t - \langle M, N \rangle_t \frac{f_t}{\langle M \rangle_t}, \]
\[ = \frac{M^n_t - \langle M \rangle_t f_t}{\langle M \rangle_t} + \langle M, f \rangle_t - \langle M, f \rangle_t \langle N, f \rangle_t. \]

\[ \text{P-martingale.} \quad \text{D}_t. \]

Note $f_t$ solves $d\tilde{f}_t = f_t \, dN_t$, hence
\[ \langle M, f \rangle_t = \int_0^t \tilde{f}_s \, d\langle M, N \rangle_s = \int_0^t f_s \langle M, N \rangle_s - \int_0^t \langle M, N \rangle_s \, df_s. \]

Hence
\[ D_t = \langle M, f \rangle_t - f_t \langle M, N \rangle_t = -\int_0^t \langle M, N \rangle_s \, df_s. \]

is a martingale, $\Rightarrow \tilde{f}^n_t$ is a $\mathcal{F}$-martingale $\Rightarrow \tilde{M}^n_t$ is a $\mathcal{F}$-martingale.

\[ \text{Q.E.D.} \]
Corollary 8

If $B_t = (B_t^1, \ldots, B_t^m)$ is an $(\mathcal{F}_t, \mathbb{P})$ Brownian motion, we define $\tilde{B}_t^i = B_t^i - \langle N, B_t^i \rangle_t$.

Then $(\tilde{B}_t^i)$ is a BM on $(\mathbb{R}, \mathcal{F}_t, \mathbb{Q})$.

Proof. By Garsia's Thm., each $(\tilde{B}_t^i)$ is a 1-dim local martingale w.r.t. $\mathbb{Q}$.

Furthermore $\langle \tilde{B}_t^i, \tilde{B}_t^j \rangle_t = \langle B_t^i - \langle N, B_t^i \rangle, B_t^j - \langle N, B_t^j \rangle \rangle_t$

$= \langle B_t^i, B_t^j \rangle_t = \delta_{ij} t$.

By Levy's characteristic theorem, $\tilde{B}_t^i = (\tilde{B}_t^i)$ is a BM w.r.t. $\mathbb{Q}$.

Q.E.D.

Example. Let $T > 0$ and $h: [0, T] \times \mathbb{R} \to \mathbb{R}$ be an $L^2$ progressively measurable function s.t. $h_0 = 0$ and $E(e^{\int_0^T |h_s|^2 \, ds}) < \infty$. Define $\mathbb{Q}$ by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = e^{\int_0^T h_s \, dB_s - \frac{1}{2} \int_0^T h_s^2 \, ds}$$

Then $B_t - \int_0^T h_s \, dB_s$ is a BM w.r.t. $\mathbb{Q}$.

Proof. By Novikov's condi., $\mathbb{Q}$ is a prob. measure, $\mathbb{Q} \mathbb{P}$.

Also, $\langle B, \int_0^T h_s \, dB_s \rangle_t = \int_0^T h_s \, ds$. Apply Garsia.
Example: Let $\sigma, b : \mathbb{R} \to \mathbb{R}$ be bounded functions with bounded first derivatives. Let $\Phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be also a bounded function.

1. \[ \int_{x_0} \sigma(x) \, dB_x + b(x) \, dx \]

2. \[ y_0 = x_0 \]

Then, \[ E \Phi(x_T) = E \Phi(x_0) e^{\int_0^T \Phi(s, x_s) \, ds} \]
\[ E \Phi(y_T) = E \Phi(x_T) e^{\int_0^T \Phi(s, x_s) \, ds} \]

**Proof:** Pathwise uniqueness & global existence hold for ODE.

Define $A$ by \[ \frac{dA}{dp} |_{x_T} = \Phi(x_T, B_t) \]

Thus, \[ \dot{y}_T = \sigma(x_T) \, dB_t + b(x_T) \, dx_t \]
\[ y_T = F_T(x_0, B_T) \]

Since \[ \int_{x_0} \sigma(x) \, dB_x + b(x) \, dx \]
By uniqueness is an law, for any $\Phi$ bounded measurable.
\[ E \Phi(F_T(x_0, B_T)) = E \Phi(F_T(x_T, B_T)) = \int \Phi(x_T, M_T) \, dp \]

Proving the first identity. The second one can be proved similarly.
1. Einstein's model for Brownian motion, colloidal pollen suspended in water.

Denote the probability of finding tagged pollen at time $t$ at a location $x$ by $p(t,x)dx$. Then

$$\frac{\partial p}{\partial t} = D \frac{\partial^2 p}{\partial x^2} + D \cdot \Delta p.$$ 

If the particle is at 0 at time 0, then the initial measure is the delta measure $\delta(0)$, and the solution is Gaussian

$$(\frac{1}{2\pi D t})^{\frac{1}{2}} e^{-\frac{x^2}{2Dt}},$$

This is verified as following.

$$D = \frac{kT}{m\beta}, \quad m\beta = 6\pi\eta a \quad \text{(Stokes' law)}$$

Where $k$ is Boltzmann constant, $T$ temperature, $\eta$ is the viscosity of the fluid, $\beta$ is the friction coefficient for a spherical particle of radius $a$ and mass $m$.

$D$, $T$, $D$, $\eta$ (hence $\beta$), $m$ and $a$ are measurable.

This gives a measurement for $k$ (per minute) given $k$ to precision $20\%$ ($k = 1.3 \times 10^{-23}$).
The deduction for \( D = \frac{KT}{m\beta} \) is obtained as follows.

At equilibrium, \( \dot{W} = 0 \), the force and friction balance out: \( K = m\beta \cdot v \)

It is believed \( K = KT \frac{\text{grad}p}{p} \)

\( \text{flux} = \text{velocity} \cdot p = \frac{Kn}{m\beta} \)

\( \text{flux from diffusion equation is} \quad D \cdot \text{grad}p \)

Hence \( D \cdot \text{grad}p = \frac{Kn}{m\beta} = \frac{KT \text{grad}p}{m\beta^2} \)

2. The Ornstein–Uhlenbeck theory.

\[
\begin{align*}
\dot{X}(t) &= \dot{V}(t) \\
\dot{V}(t) &= -\beta V(t) dt + \sigma dB_t
\end{align*}
\]

We first determine \( \sigma \), in fact \( \sigma^2 = 2\beta^2 D \).

Note \( V(t) = e^{-\beta t} V(0) + \beta \int_0^t e^{\beta s} dB_s \)

\[
\text{law}(V(t)) = N(e^{-\beta t} V(0), \frac{\sigma^2}{2\beta} (1 - e^{-2\beta t}))
\]

Fix \( \beta \). Take \( t \to \infty \), we obtain equilibrium distribution \( N(0, \frac{\sigma^2}{2\beta}) \).

We typically see the state in equilibrium.

We sample at rate \( t = \frac{1}{10^3} \) (Kilometers/hour).

\( m = 10^{-12}, \quad \alpha = 10^{-6}, \quad N \text{ ferrom.} \sim 10^{-3} \text{ at } 25^\circ C \)

\( m, \alpha \text{ per second} \)
Orstein-Uhlenbeck's theory for Brownian particles is to consider the particles to have a physical velocity (diff. w.r.t. time) and solve the O-U eq:

\[ \dot{x} = -\frac{1}{2}m \frac{\nu^2}{kT} \]

We now use a principle from statistical mechanics. If a physical system is at thermal equilibrium at a temperature \( T \) (Klein), the probability of seeing it at a given state is given proportional to the Boltzman distribution:

\[ e^{-\frac{\text{energy}}{kT}} = e^{-\frac{1}{2}m \frac{\nu^2}{kT}} \]

This is Gaussian with variance \( \frac{kT}{m} \).

The equilibrium distance we found is \( \sigma^2 = \frac{2\beta kT}{m} \)

\[ \sigma^2 = 2\beta^2 D \quad (D = \frac{kt}{m \nu}, \text{ Einstein's theory}) \]

Let us hold \( D \) as a constant and take \( \beta \rightarrow \infty \).

\[ x(t) = N \left( x_0 + \frac{1-e^{-\beta t}}{\beta} \nu_0, 2Dt + \frac{D(-3+4e-e^2)}{\beta^3} \right) \]

\[ x(t) \sim N(0, 2Dt), \text{ finite displacement c.w. and to Brownian with general } D(t) \]
Lecture 30

The OU process is \( \dot{x}(t) = \mu(x(t), t) \), \( \ddot{x}(t) = -\beta(x(t)) + \sum \beta dW_t \).

In Amos's book Prob. 44 (1), 544-566 (2016) we gave another model. The model is for manifold. On flat space, it is:

\[
\begin{align*}
\dot{z}_t^\varepsilon &= g_t^\varepsilon \, e_0 \\
\frac{dg_t^\varepsilon}{g_t^\varepsilon} &= \frac{1}{\varepsilon} \sum_k g_t^\varepsilon A_k d\beta_t^k + g_t^\varepsilon A_0 dt \\
g_0 &= I
\end{align*}
\]

Here \( g_t^\varepsilon \) takes values in \( SO(n) \), \( e_0 E \mathbb{R}^n \).

This means if a particle has unit speed, with director field moving/rotating very fast on a large interval we see a Brownian motion.

(3) Smoluchowski model:

\[
d\dot{x}(t) = \sqrt{2D} dB_t + \frac{K(x(t))}{\beta} dt.
\]

This is close to OU model, assume \( K \) varies so slowly that \( K(x(t)) \) is approximately a constant for time of order \( \beta \). Then

\[
\frac{1}{\beta} \frac{d\dot{x}(t)}{dt} = \frac{1}{\beta} \left( K(x(t), t) dt - \beta \dot{x}(t) dt + \sqrt{2D} dB_t \right)
\]

So \( \dot{x} = \dot{v} dt = \frac{K(x(t))}{\beta} dt + \sqrt{2D} dB_t \).