An efficient multigrid preconditioner for edge element discretization of Maxwell's equations in micromagnetism.

Ľubomír Baňas

Department of Mathematics
Heriot-Watt University, Edinburgh

http://www.ma.hw.ac.uk/~lubomir
L.Banas@hw.ac.uk

Joint work with A. Prohl and S. Bartels

Maxwell's equations

$$
\begin{array}{rcll}
\varepsilon_0\,\mathbf{E}_t - \nabla\times\mathbf{H} \quad +\sigma\mathbf{E} & = & -\mathbf{J} & \text{in} \quad \Omega_T := (0,T)\times\Omega\,, \\
\mathbf{B}_t + \nabla\times\mathbf{E} & = & \mathbf{0} & \text{in} \quad \Omega_T\,.
\end{array}
$$

- $\mathbf{H}$ magnetic field
- $\mathbf{B}$ magnetic induction ($\nabla\cdot\mathbf{B} = \mathbf{0}$)
- $\mathbf{E}$ electric field
- $\mathbf{J}$ electric current, ε_0 permitivity of vacuum,σ conductivity

Constitutive relations $\mathbf{B}$-$\mathbf{H}$

- vacuum $\mathbf{B} = \mu_0\mathbf{H}$
- linear materials $\mathbf{B} = \mu\mathbf{H}$, $\mu(\mathbf{x})$ is a 3×3 tensor
- ferromagnetic materials $\mathbf{B} = \mu_0(\mathbf{H}+\mathbf{m})$

Introduction

Landau-Lifshitz-Gilbert (LLG) equation

Landau-Lifshitz formulation

$$\mathbf{m}_t = \mathbf{m} \times \mathbf{H}_{\text{eff}} - \alpha\, \mathbf{m} \times (\mathbf{m} \times \mathbf{H}_{\text{eff}})\,.$$

Gilbert formulation

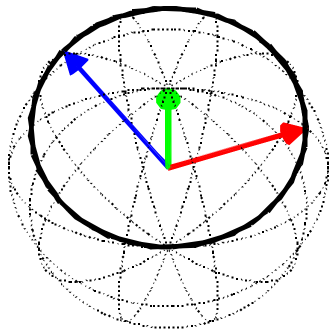
$$\mathbf{m}_t + \alpha\, \mathbf{m} \times \mathbf{m}_t = (1 + \alpha^2)\, \mathbf{m} \times \mathbf{H}_{\text{eff}}\,.$$

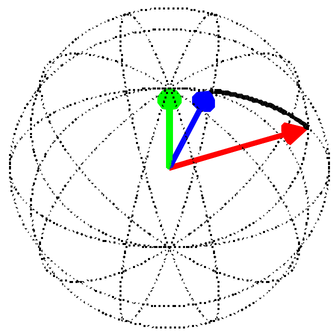
Both formulations are equivalent for smooth $\mathbf{m}$.

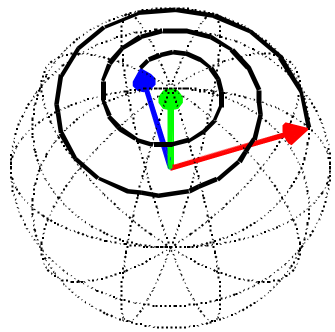
- $\mathbf{m}$ magnetization vector, $|\mathbf{m}| = 1$
- $\mathbf{H}_{\text{eff}}$ effective field (nonlinear)
- α damping constant

Exact solution

Take $\mathbf{H}_{\text{eff}} = \mathbf{h} = (0, 0, 1)$, then LLG reduces to an ODE.







- $\boldsymbol{m}_t = \boldsymbol{h} \times \boldsymbol{m}$ (precession)
- $\boldsymbol{m}_t = \boldsymbol{m} \times (\boldsymbol{h} \times \boldsymbol{m})$ (phenomenological damping)
- $\boldsymbol{m}_t = \boldsymbol{h} \times \boldsymbol{m} + \boldsymbol{m} \times (\boldsymbol{h} \times \boldsymbol{m})$ (precession and damping)

Effective (total) field

Free energy - magnetic field energy, exchange energy, Zeeman energy, anisotropy energy, magnetomechanical energy.

Effective field

$$\boldsymbol{H}_{eff} = -\frac{\partial E}{\partial \boldsymbol{m}} = \boldsymbol{H} + \boldsymbol{H}_{ex} + \boldsymbol{H}_{app} + \boldsymbol{H}_{an} + \boldsymbol{H}_{ms}$$

- Magnetic field $\mathbf{H}$ coupling with Maxwell's equations
- Exchange $\boldsymbol{H}_{ex} = \Delta \boldsymbol{m}$
- Applied field $\boldsymbol{H}_{app}$ constant in space and time
- Anisotropy $\boldsymbol{H}_{an} = (\boldsymbol{p} \cdot \boldsymbol{m})\boldsymbol{p}$
- Magnetostriction field coupling with elastodynamics

Maxwell-LLG system

$$
\begin{aligned}
\varepsilon_0\,\mathbf{E}_t - \nabla\times\mathbf{H} + \sigma\,\chi_\omega\mathbf{E} &= -\mathbf{J} && \text{in} \quad \Omega_T\,,\\
\mu_0\,\mathbf{H}_t + \nabla\times\mathbf{E} &= -\mu_0\,\chi_\omega\mathbf{m}_t && \text{in} \quad \Omega_T\,,\\
\mathbf{m}_t + \alpha\,\mathbf{m}\times\mathbf{m}_t &= (1+\alpha^2)\,\mathbf{m}\times\mathbf{H}_{\mathrm{eff}} && \text{in} \quad \omega_T := (0,T)\times\omega.
\end{aligned}
$$

Boundary conditions

$$
\partial_{\mathbf{n}}\mathbf{m} = 0 \quad \text{on } \partial\omega_T\,, \qquad \mathbf{E}\times\mathbf{n} = 0 \quad \text{on } \partial\Omega_T\,.
$$

Initial conditions $(\nabla\cdot(\mathbf{H}_0+\chi_\omega\mathbf{m}_0) = 0 = \nabla\cdot\mathbf{B} \quad \text{in } \Omega)$

$$
\mathbf{m}(0,\cdot) = \mathbf{m}_0 \quad \text{in } \omega\,, \qquad \mathbf{E}(0,\cdot) = \mathbf{E}_0\,, \quad \mathbf{H}(0,\cdot) = \mathbf{H}_0 \quad \text{in } \Omega\,.
$$

Energy $(\sigma = 0,\ \mathbf{J} = \mathbf{0})$:

$$
\frac{\mu_0}{2}\int_\omega |\,\nabla\mathbf{m}\,|^2\,\mathrm{d}\mathbf{x} - \mu_0\int_\omega(\mathbf{m}\cdot\mathbf{H}_{app})\,\mathrm{d}\mathbf{x} + \int_\Omega\Big[\frac{\mu_0}{2}|\,\mathbf{H}\,|^2 + \frac{\varepsilon_0}{2}|\,\mathbf{E}\,|^2\Big]\,\mathrm{d}\mathbf{x}.
$$

Magnetostaic formulation

Take the stationary Maxwell's equations and $\sigma = 0$, i.e.:

$$\mathbf{E}_t = \mathbf{B}_t = \mathbf{J} = \mathbf{0},$$

we get

$$\begin{aligned} \nabla \times \mathbf{H} &= \mathbf{0}, \\ \nabla \times \mathbf{E} &= \mathbf{0}, \\ \nabla \cdot \mathbf{B} &= \mathbf{0}, \\ \mathbf{B} &= \mu_0(\mathbf{H} + \mathbf{m}). \end{aligned}$$

We look for $\mathbf{H} = \nabla\phi$ (i.e. $\nabla \times (\nabla\phi) = \mathbf{0}$), then the above reduces to

$$\mu_0 \nabla \cdot (\nabla\phi + \mathbf{m}) = \mathbf{0} \qquad \text{in} \qquad \Omega,$$

with an interface condition on $\partial\omega$

$$(\nabla\phi + \mathbf{m}) \cdot \boldsymbol{n} = \mathbf{B} \cdot \boldsymbol{n} = 0.$$

Notation

- $\big(\mathbf{f}, \mathbf{g}\big) = \int_\Omega \langle \mathbf{f}, \mathbf{g} \rangle \, \mathrm{d}\mathbf{x}$
- mass lumping $\big(\boldsymbol{\phi}, \boldsymbol{\mathfrak{z}}\big)_h = \int_\omega \boldsymbol{\mathcal{I}}_{\mathbf{V_h}}\big(\langle \boldsymbol{\phi}, \boldsymbol{\mathfrak{z}} \rangle\big) \, \mathrm{d}\mathbf{x} = \sum_{\ell \in L} \beta_\ell \langle \boldsymbol{\phi}(\mathbf{x}_\ell), \boldsymbol{\mathfrak{z}}(\mathbf{x}_\ell) \rangle$.
- $d_t \varphi^j := k^{-1}\big(\varphi^j - \varphi^{j-1}\big)$
- $\overline{\varphi}^{j+1/2} := \frac{1}{2}\big(\varphi^{j+1} + \varphi^j\big)$
- $\tilde{\Delta}_h : W^{1,2}(\omega; \mathbb{R}^3) \to \mathbf{V}_h$ is a discrete Laplace operator $(-\tilde{\Delta}_h \boldsymbol{\phi}, \boldsymbol{\chi}_h)_h = (\nabla \boldsymbol{\phi}, \nabla \boldsymbol{\chi}_h) \quad \forall \boldsymbol{\chi}_h \in \mathbf{V}_h$.
- projection $\mathbf{P}_{\mathbf{V}_h} : L^2(\omega, \mathbb{R}^3) \to \mathbf{V}_h$ is defined as $(\mathbf{P}_{\mathbf{V}_h} \mathbf{u}, \boldsymbol{\varphi}_h)_h = (\mathbf{u}, \boldsymbol{\varphi}_h)$

Implicit finite element approximation

Fully discrete system nonlinear and coupled [LB, Bartels, Prohl (2008)]

$$
\begin{aligned}
&(d_t\mathbf{m}_h^{j+1},\boldsymbol{\phi}_h)_h + \alpha\,(\mathbf{m}_h^{j}\times d_t\mathbf{m}_h^{j+1},\boldsymbol{\phi}_h)_h \\
&\quad = (1+\alpha^2)\Big(\overline{\mathbf{m}}_h^{j+1/2}\times(\tilde{\Delta}_h\overline{\mathbf{m}}_h^{j+1/2}+\mathbf{P}_{\mathbf{V}_h}\overline{\mathbf{H}}_h^{j+1/2}),\boldsymbol{\phi}_h\Big)_h \qquad \forall\,\boldsymbol{\phi}_h\in\mathbf{V}_h\,,\\
&\varepsilon_0\,(d_t\mathbf{E}_h^{j+1},\boldsymbol{\varphi}_h) - (\overline{\mathbf{H}}_h^{j+1/2},\nabla\times\boldsymbol{\varphi}_h) + \sigma\,(\chi_\omega\overline{\mathbf{E}}_h^{j+1/2},\boldsymbol{\varphi}_h) = \mathbf{0} \qquad \forall\,\boldsymbol{\varphi}_h\in\mathbf{X}_h\,,\\
&\mu_0\,(d_t\mathbf{H}_h^{j+1},\boldsymbol{\mathfrak{z}}_h) + (\nabla\times\overline{\mathbf{E}}_h^{j+1/2},\boldsymbol{\mathfrak{z}}_h) = -\mu_0\,(\chi_\omega d_t\mathbf{m}_h^{j+1},\boldsymbol{\mathfrak{z}}_h) \qquad \forall\,\boldsymbol{\mathfrak{z}}_h\in\mathbf{Y}_h\,.
\end{aligned}
$$

Finite element spaces

- $\mathbf{V}_h = \big\{\boldsymbol{\phi}_h\in C(\overline{\omega};\mathbb{R}^3) : \boldsymbol{\phi}_h|_K\in\mathcal{P}_1(K;\mathbb{R}^3) \quad \forall\,K\in\mathcal{T}_h\big|_\omega\big\}\,,$
- $\mathbf{X}_h = \big\{\boldsymbol{\varphi}_h\in\mathbf{H}_0(\mathbf{curl};\Omega) : \boldsymbol{\varphi}_h|_K\in\mathcal{P}_1(K;\mathbb{R}^3) \quad \forall\,K\in\mathcal{T}_h\big\}\,,$
- $\mathbf{Y}_h = \big\{\boldsymbol{\mathfrak{z}}_h\in L^2(\Omega;\mathbb{R}^3) : \boldsymbol{\mathfrak{z}}_h|_K\in\mathcal{P}_0(K;\mathbb{R}^3) \quad \forall\,K\in\mathcal{T}_h\big\}.$

Stability

Lemma 1. *Suppose that* $|\mathbf{m}_h^0(\mathbf{x}_\ell)| = 1$ *for all* $\ell \in L$. *Then the sequence* $\{(\mathbf{m}_h^j, \mathbf{E}_h^j, \mathbf{H}_h^j)\}_{j\geq 0}$ *satisfies for all* $j \geq 0$

$$\text{(i)} \qquad |\,\mathbf{m}_h^{j+1}(\mathbf{x}_\ell)\,| = 1 \quad \forall\,\ell \in L\,,$$

$$\text{(ii)} \qquad \mathcal{E}_h\big(\{\mathbf{m}_h^{j+1}, \mathbf{H}_h^{j+1}, \mathbf{E}_h^{j+1}\}\big) + k\sum_{\ell=0}^{j} \frac{\alpha\mu_0}{1+\alpha^2}\|\,d_t\mathbf{m}_h^{\ell+1}\,\|_h^2 + \sigma\,\|\,\overline{\mathbf{E}}_h^{\ell+1/2}\,\|_{L^2(\omega)}^2$$

$$= \mathcal{E}_h\big(\{\mathbf{m}_h^0, \mathbf{H}_h^0, \mathbf{E}_h^0\}\big) - k\sum_{\ell=0}^{j}(\overline{\mathbf{J}}_h^{\ell+1/2}, \overline{\mathbf{E}}_h^{\ell+1/2})\,,$$

where

$$\mathcal{E}_h\big(\{\mathbf{m}_h^j, \mathbf{H}_h^j, \mathbf{E}_h^j\}\big) = \frac{\mu_0}{2}\int_\omega |\,\nabla\mathbf{m}_h^j\,|^2 - 2(\mathbf{m}_h^j\cdot\mathbf{H}_{app})\,\mathrm{d}\mathbf{x} + \int_\Omega\Big[\frac{\mu_0}{2}\,|\,\mathbf{H}_h^j\,|^2 + \frac{\varepsilon_0}{2}\,|\,\mathbf{E}_h^j\,|^2\Big]\,\mathrm{d}\mathbf{x}\,.$$

Convergence

Theorem 1. *[LB, Bartels, Prohl (2008)] Suppose that we have* $|\mathbf{m}_h^0(\mathbf{x}_\ell)| = 1$ *for all* $\ell \in L$ *and let* $\{(\mathbf{m}_h^j, \mathbf{E}_h^j, \mathbf{H}_h^j)\}_{j\geq 0}$ *be the finite element solution. Assume that* $\mathbf{m}_h^0 \to \mathbf{m}_0$ *in* $W^{1,2}(\omega)$ *and* $(\tilde{\mathbf{H}}_h^0, \tilde{\mathbf{E}}_h^0) \to (\mathbf{H}_0, \mathbf{E}_0)$ *in* $L^2(\Omega, \mathbb{R}^3)$ *as* $h \to 0$ *and let* $T > 0$ *be a fixed constant. As* $k, h \to 0$*, a subsequence of* $(\tilde{\mathbf{m}}, \tilde{\mathbf{H}}, \tilde{\mathbf{E}})$ *converges weakly to* $(\mathbf{m}, \mathbf{H}, \mathbf{E})$ *in* $\left[L^\infty\big(0,T; W^{1,2}(\omega, \mathbb{S}^2)\big) \cap W^{1,2}(\omega_T, \mathbb{R}^3)\right] \times \left[L^\infty\big((0,T); L^2(\Omega, \mathbb{R}^3)\big)\right]^2$*, and* $(\mathbf{m}, \mathbf{H}, \mathbf{E})$ *is a weak solution.*

Where $(\tilde{\mathbf{m}}, \tilde{\mathbf{H}}, \tilde{\mathbf{E}})$ are defined as

$$\tilde{\boldsymbol{\xi}}(t, \mathbf{x}) := \frac{t - t_j}{k} \boldsymbol{\xi}_h^{j+1}(\mathbf{x}) + \frac{t_{j+1} - t}{k} \boldsymbol{\xi}_h^j(\mathbf{x}).$$

Solution of the nonlinear system

Fixed-point iterations

$$
\begin{aligned}
&\frac{2}{k}(\mathbf{w}_h^{\ell+1},\boldsymbol{\phi}_h)_h - \frac{2\alpha}{k}(\mathbf{w}_h^{\ell+1}\times\tilde{\mathbf{m}}_h^j,\boldsymbol{\phi}_h)_h \\
&\qquad -(1+\alpha^2)\big(\mathbf{w}_h^{\ell+1}\times(\tilde{\Delta}_h\mathbf{w}_h^{\ell}+\mathbf{P}_{\mathbf{V}^h}\mathbf{G}_h^{\ell}),\boldsymbol{\phi}_h\big)_h = \frac{2}{k}(\tilde{\mathbf{m}}_h^j,\boldsymbol{\phi}_h)_h,
\end{aligned}
$$

$$
\frac{2\varepsilon_0}{k}(\mathbf{F}_h^{\ell+1},\boldsymbol{\varphi}_h)-(\mathbf{G}_h^{\ell+1},\nabla\times\boldsymbol{\varphi}_h)+\sigma(\chi_\omega\mathbf{F}_h^{\ell+1},\boldsymbol{\varphi}_h)=\frac{2\varepsilon_0}{k}(\tilde{\mathbf{E}}_h^j,\boldsymbol{\varphi}_h)-(\overline{\mathbf{J}}_h^{j+1/2},\boldsymbol{\varphi}_h),
$$

$$
\frac{2\mu_0}{k}(\mathbf{G}_h^{\ell+1},\boldsymbol{\mathfrak{z}}_h)+(\nabla\times\mathbf{F}_h^{\ell+1},\boldsymbol{\mathfrak{z}}_h)+\frac{2\mu_0}{k}(\mathbf{w}_h^{\ell+1},\boldsymbol{\mathfrak{z}}_h)=\frac{2\mu_0}{k}(\tilde{\mathbf{H}}_h^j,\boldsymbol{\mathfrak{z}}_h)+\frac{2\mu_0}{k}(\tilde{\mathbf{m}}_h^j,\boldsymbol{\mathfrak{z}}_h),
$$

where $\mathbf{w}_h := \overline{\mathbf{m}}_h^{j+1/2}$, $\mathbf{F}_h := \overline{\mathbf{E}}_h^{j+1/2}$ and $\mathbf{G}_h := \overline{\mathbf{H}}_h^{j+1/2}$.

Stopping criterion

$$
||\tilde{\Delta}_h(\mathbf{w}_h^{\ell+1}-\mathbf{w}_h^{\ell})||_h+||\mathbf{G}_h^{\ell+1}-\mathbf{G}_h^{\ell}||_{L^2}\le\varepsilon.
$$

Time step restriction $k=\mathcal{O}(h^2)$.

Solution of the discrete Maxwell system

Matrix notation

$$\begin{pmatrix} \mathbf{A} & -\mathbf{C}^T \\ \mathbf{C} & \mathbf{B} \end{pmatrix} \begin{pmatrix} \overline{\mathbf{e}} \\ \overline{\mathbf{h}} \end{pmatrix} = \begin{pmatrix} \overline{\mathbf{f}} \\ \overline{\mathbf{g}} \end{pmatrix},$$

$\overline{\mathbf{e}}$, $\overline{\mathbf{h}}$ vectors of unknown finite element coefficients.

Matrix definitions

$$\mathbf{A}_{ij} = \frac{2\varepsilon_0}{k}(\boldsymbol{\varphi}_h^i, \boldsymbol{\varphi}_h^j)\,, \quad \mathbf{B}_{ij} = \frac{2\mu_0}{k}(\boldsymbol{\mathfrak{z}}_h^i, \boldsymbol{\mathfrak{z}}_h^j)\,, \quad \mathbf{C}_{ij} = (\nabla \times \boldsymbol{\varphi}_h^j, \boldsymbol{\mathfrak{z}}_h^i)\,,$$

Preconditioned Uzawa algorithm solves Schur complement formulation

$$\left(\mathbf{C}^T\mathbf{B}^{-1}\mathbf{C} + \mathbf{A}\right)\overline{\mathbf{e}} = \overline{\mathbf{f}} + \mathbf{C}^T\mathbf{B}^{-1}\overline{\mathbf{g}}.$$

Two sub-steps:

1. $\overline{\mathbf{h}}^n = \mathbf{B}^{-1}(\overline{\mathbf{g}} - \mathbf{C}\overline{\mathbf{e}}^n)$

2. $\overline{\mathbf{e}}^{n+1} = \overline{\mathbf{e}}^n + \rho\mathbf{S}^{-1}(\overline{\mathbf{f}} + \mathbf{C}^T\overline{\mathbf{h}}^n - \mathbf{A}\overline{\mathbf{e}}^n)$

Preconditioner

Preconditioning dramatically reduces the number of Uzawa iterations. We need:

$$\mathbf{S}^{-1} \approx [\mathbf{C}^T \mathbf{B}^{-1} \mathbf{C} + \mathbf{A}]^{-1} .$$

Auxiliary problem edge elements $\mathbf{X}_h \subset \mathbf{H}_0(\mathbf{curl}; \Omega)$

$$\frac{2\varepsilon_0}{k}(\mathbf{F}_h^*, \boldsymbol{\varphi}_h) + \frac{k}{2\mu_0}(\nabla \times \mathbf{F}_h^*, \nabla \times \boldsymbol{\varphi}_h) = (\mathbf{f}_h, \boldsymbol{\varphi}_h) + \frac{k}{2\mu_0}(\nabla \times \mathbf{g}_h, \boldsymbol{\varphi}_h) \qquad \forall \boldsymbol{\varphi}_h \in \mathbf{X}_h.$$

with $\mathbf{g}_h = \sum_i \{\overline{\mathbf{g}}\}_i \boldsymbol{\varphi}_{h,i}$.

In matrix notation $\mathbf{S} = \mathbf{M} + \mathbf{R}$, with

$$\mathbf{M}_{ij} = \frac{2\varepsilon_0}{k}(\boldsymbol{\varphi}_h^i, \boldsymbol{\varphi}_h^j) , \quad \mathbf{R}_{ij} = \frac{k}{2\mu_0}(\nabla \times \boldsymbol{\varphi}_h^i, \nabla \times \boldsymbol{\varphi}_h^j) .$$

$\mathbf{S}^{-1}$ needs to be approximated effectively, e.g., by multigrid.

Multigrid algorithm

Multigrid components

- grid hierarchy
- prolongation $\mathbf{I}_{k-1}^{k}$ [eg. Bossavit, Rapetti 2005],
- restriction $\mathbf{I}_{k}^{k-1} = (\mathbf{I}_{k-1}^{k})^{T}$,
- smoother (Gauss-Seidel type iterative scheme)

Two-grid scheme

- **pre smoothing** m smoothing iterations
- **coarse grid correction** solve the coarse problem exactly
- **post smoothing** m smoothing iterations

Smoothers for edge elements

Hybrid smoother one smoothing steps consists of one loop over all edges followed by one loop over all vertices [Hiptmair (1998)]

Patch smoother smooth several edges simultaneously [Arnold, Falk, Winther (2000)]

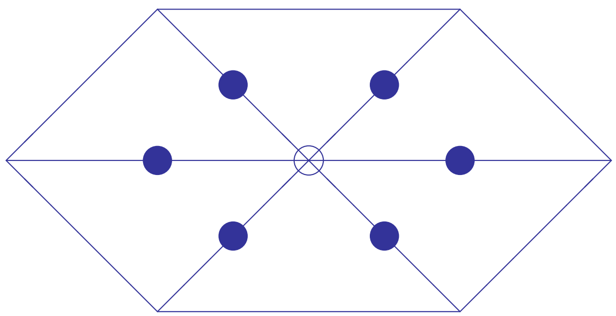


Figure 1: Local smoothing patch

Numerical experiments

Blow-up of MLLG

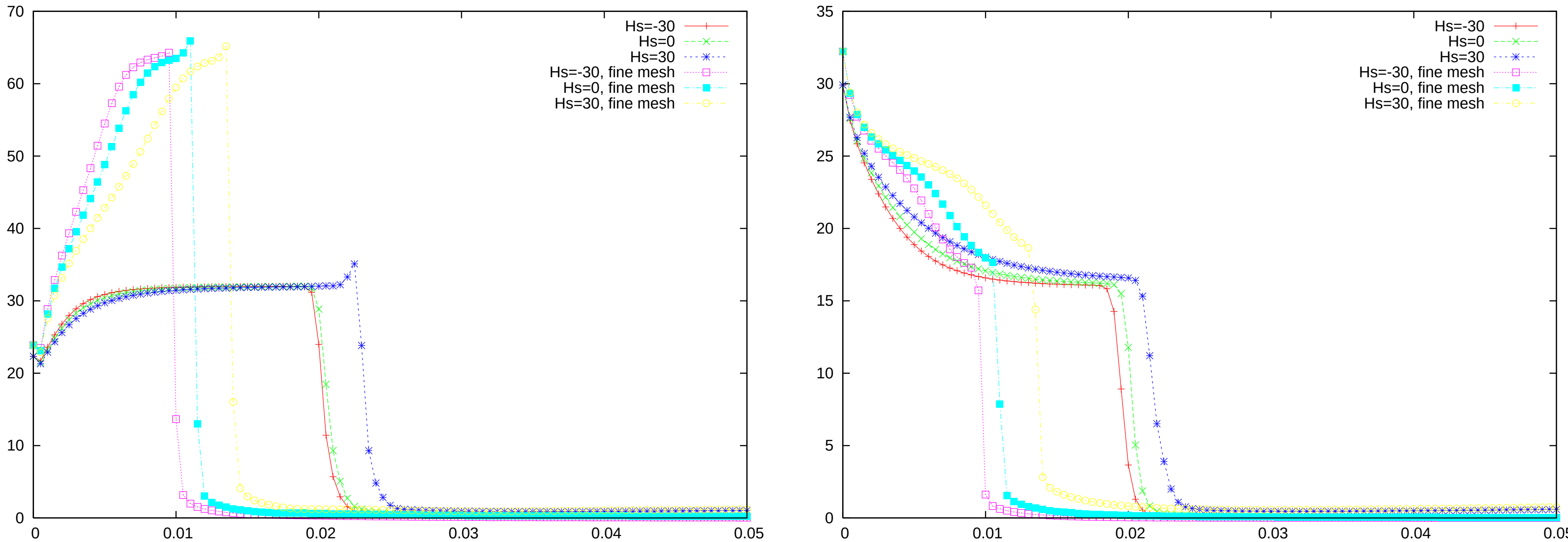


Figure 2: Plot of $t \mapsto \|\nabla \mathbf{m}_h(t,\cdot)\|_\infty$ and $t \mapsto \|\nabla \mathbf{m}_h(t,\cdot)\|_2$

Numerical experiments

Blow-up of MLLG

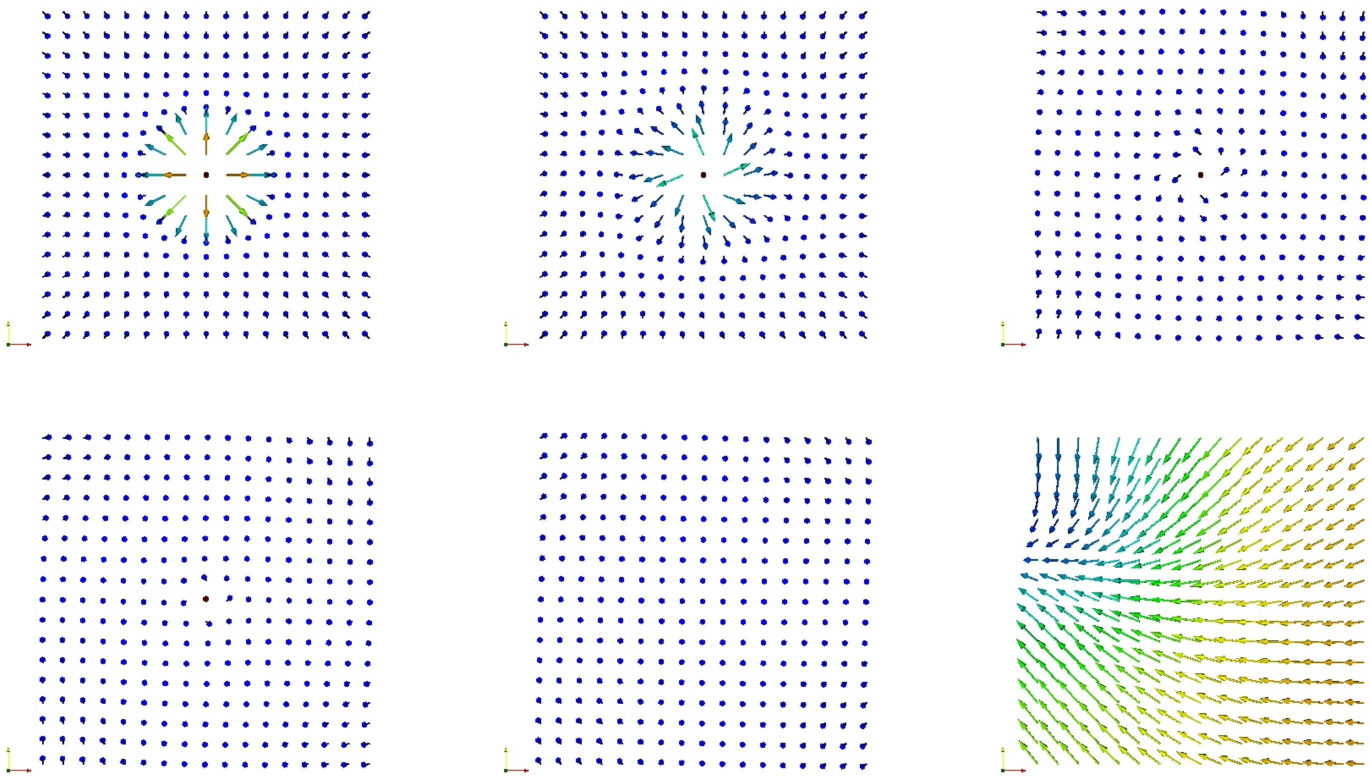


Figure 3: Magnetization at times $t = 0, 0.01, 0.015, 0.020, 0.030, 0.214$ for $h = 1/2^4$

Numerical experiments

Blow-up of MLLG

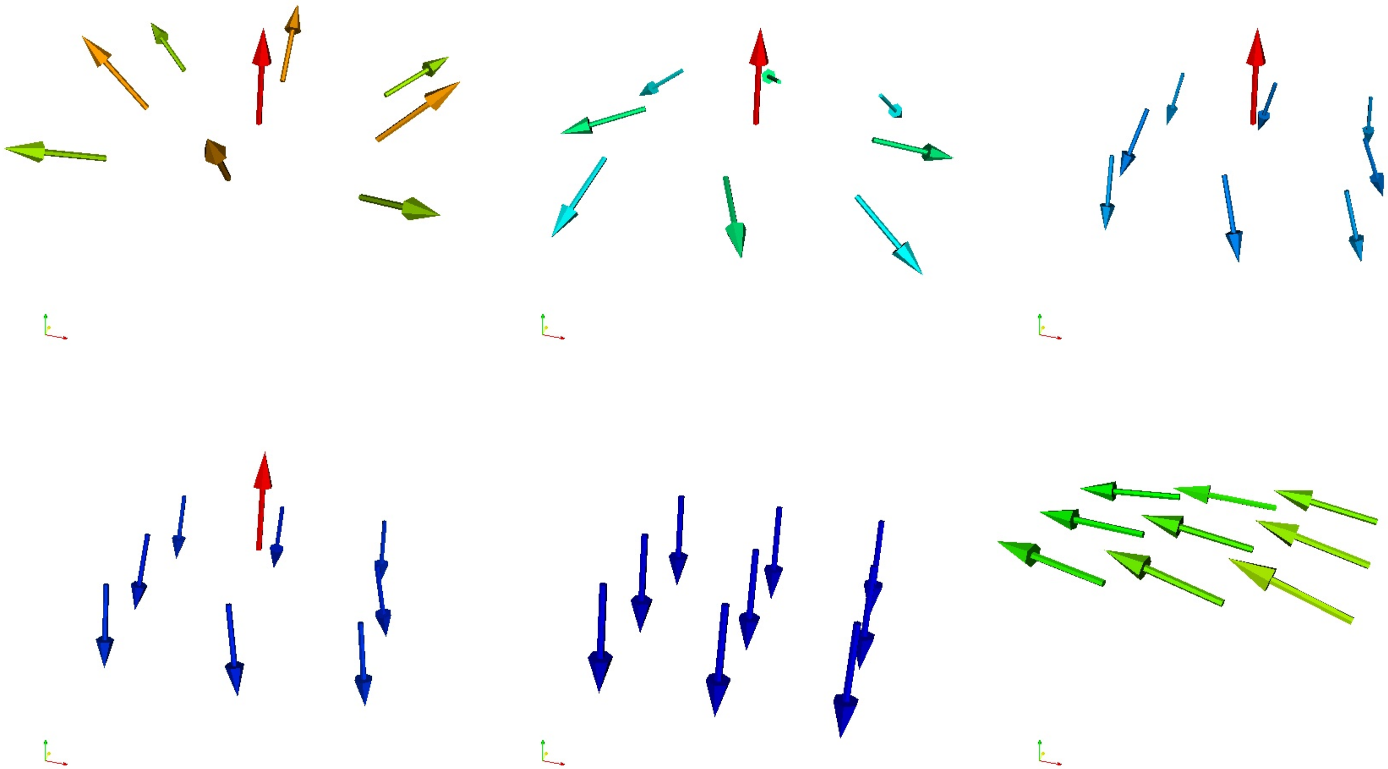


Figure 4: Details of the magnetization near the singularity times $t = 0, 0.01, 0.015, 0.020, 0.030, 0.214$ for $h = 1/2^4$.

Numerical experiments

μ-mag standard problem 4: $\mu_0 \mathbf{H}_{app} = (-24.6, 4.3, 0)$, $\omega = 500 \times 125 \times 3$ nm, $h = 3.90625 \times 10^{-9}$, $\tau = 1.13 \times 10^{-13}$ ([d'Aquino, Serpico, Miano (2005)] used $\tau = 2.5 \times 10^{-12}$).

Initial condition S-state

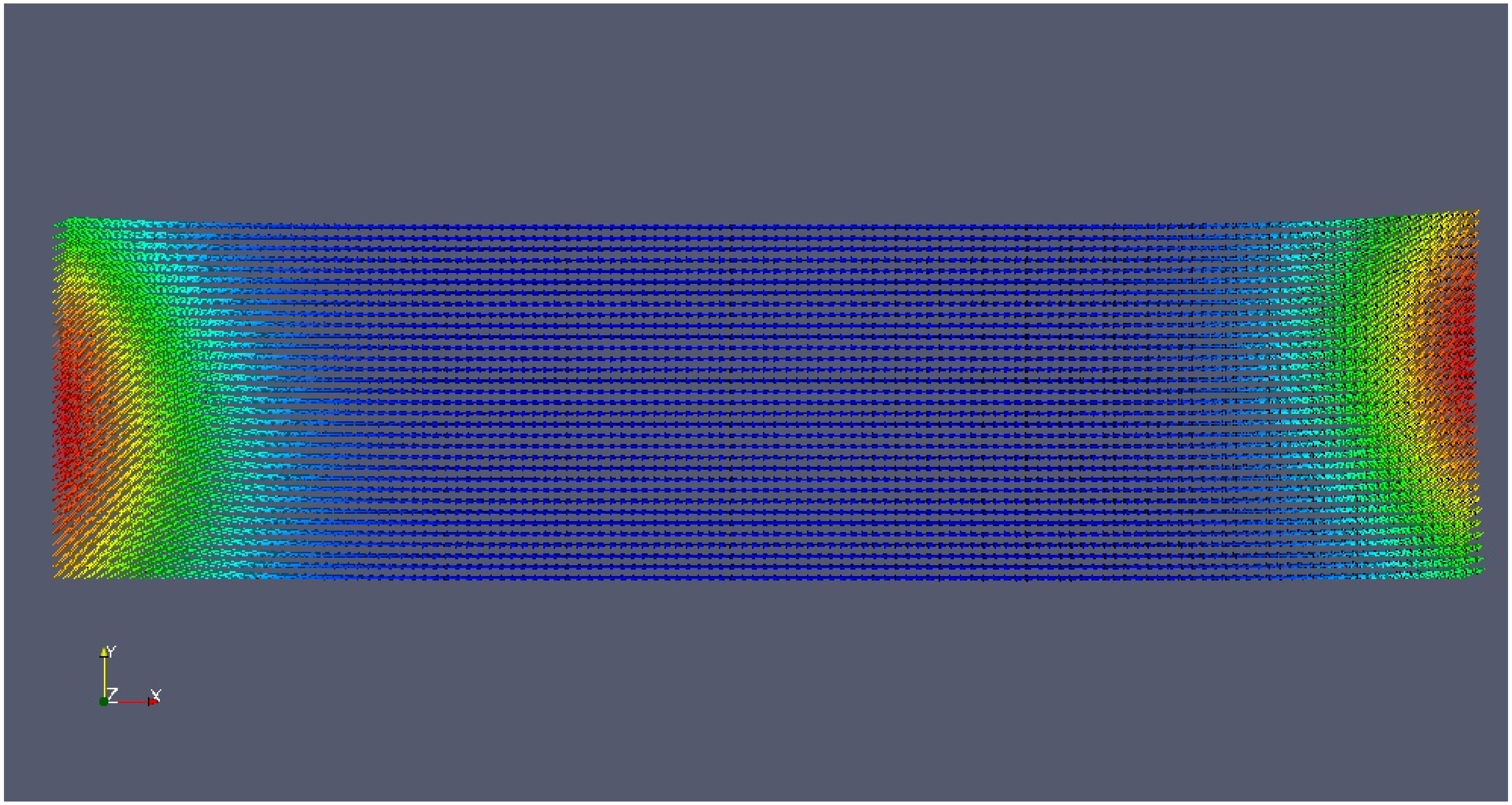


Figure 5: Initial condition $\mathbf{m}^0$.

Numerical experiments

Unstructured grid

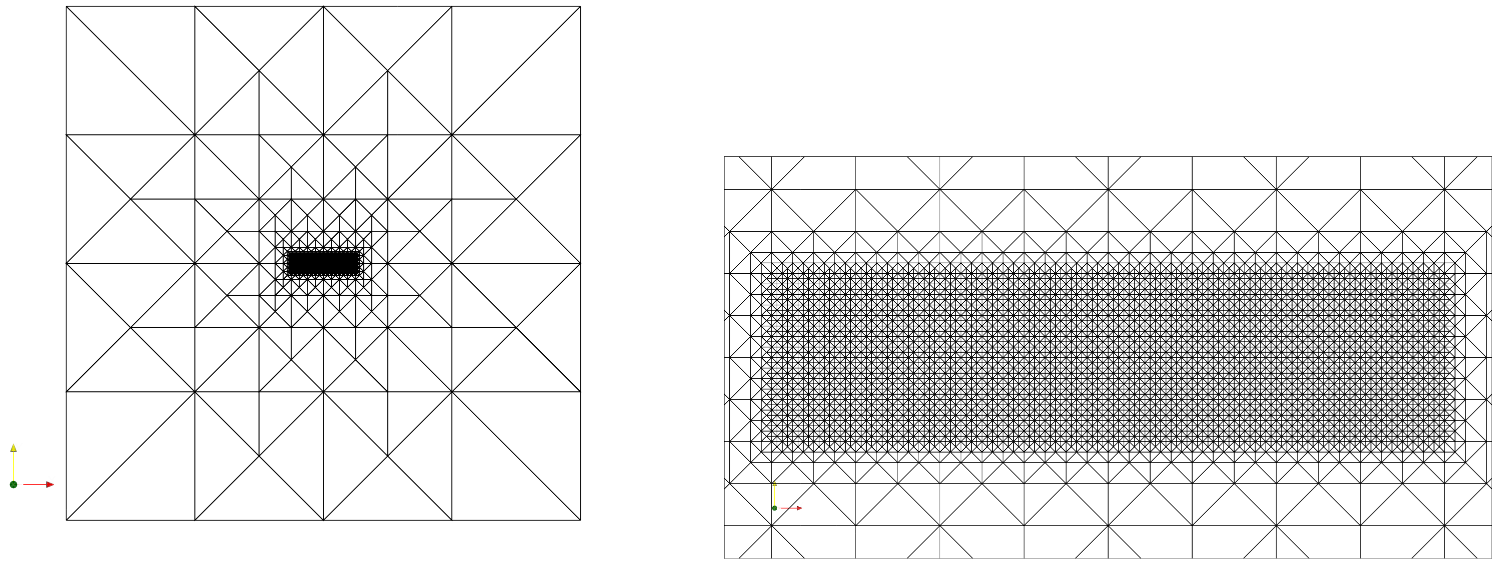


Figure 6: Mesh for the domain Ω at $x_3 = 0$ (left) and zoom at the mesh for the domain ω at $x_3 = 0$ (right).

Numerical experiments

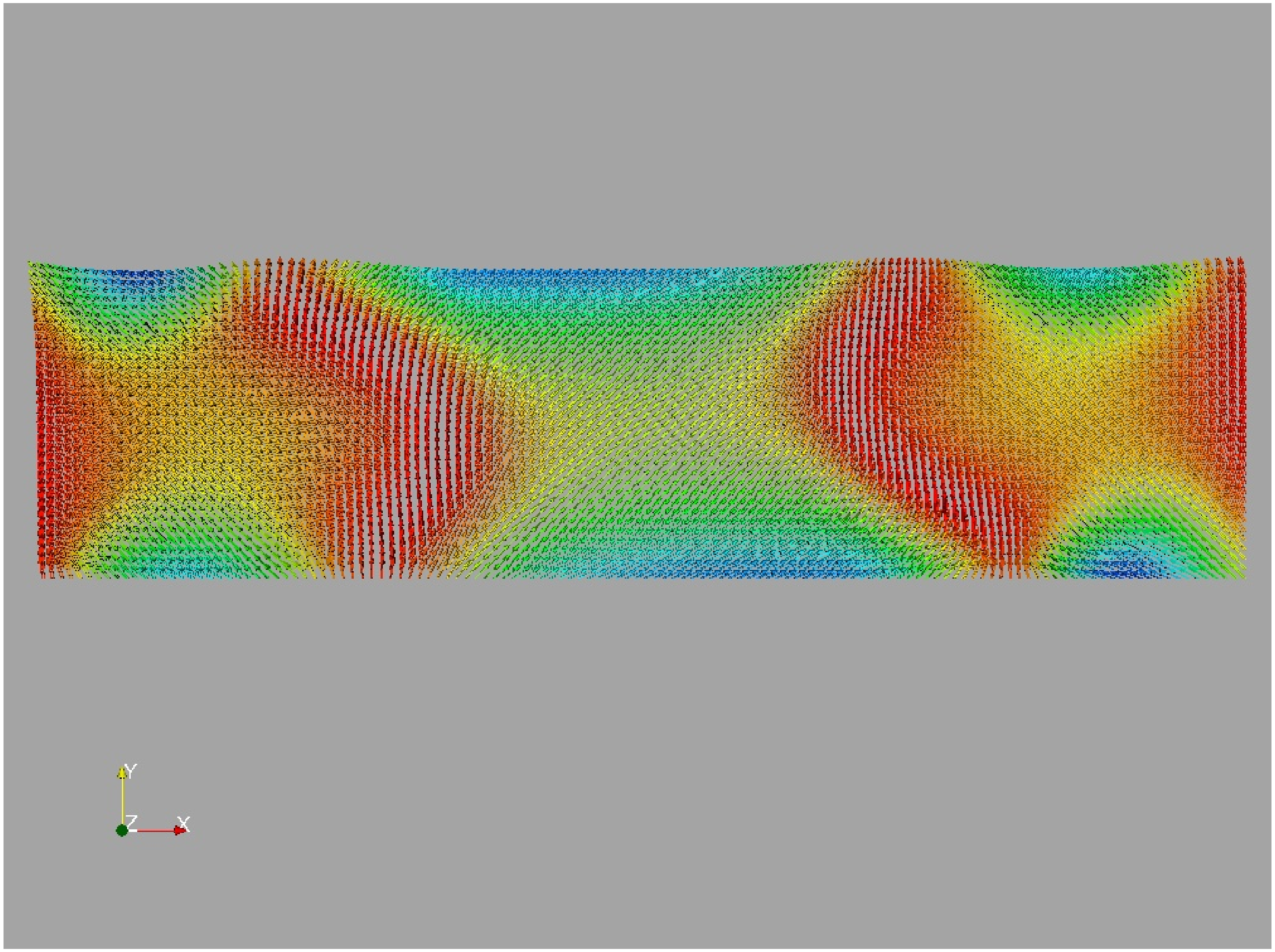


Figure 7: MLLG: solution at $|\omega|^{-1}\int_{\omega}\mathbf{m}_x = 0$.

Numerical experiments

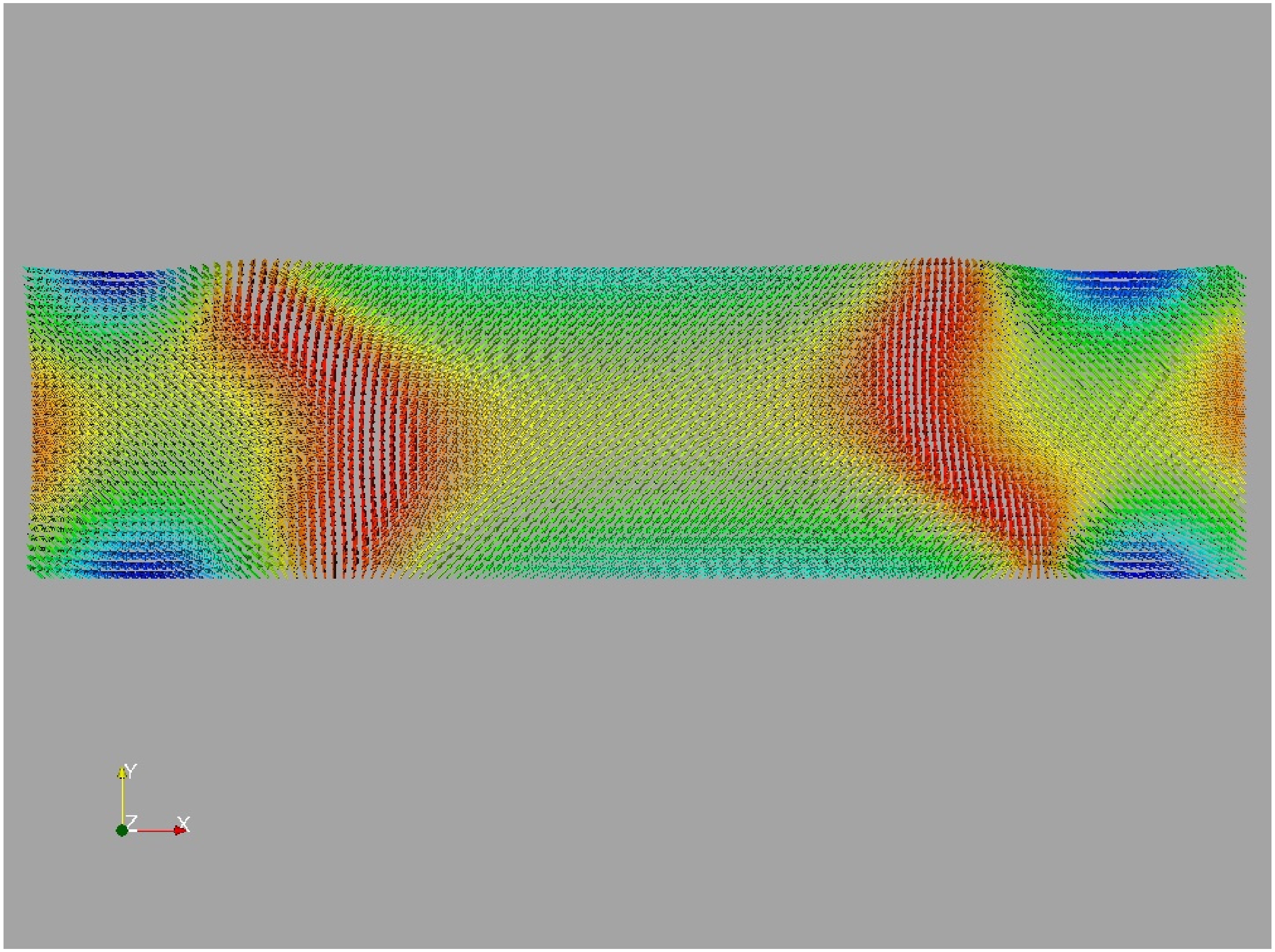


Figure 8: Magnetostatic formulation: solution at $|\omega|^{-1}\int_\omega \mathbf{m}_x = 0$.

Numerical experiments

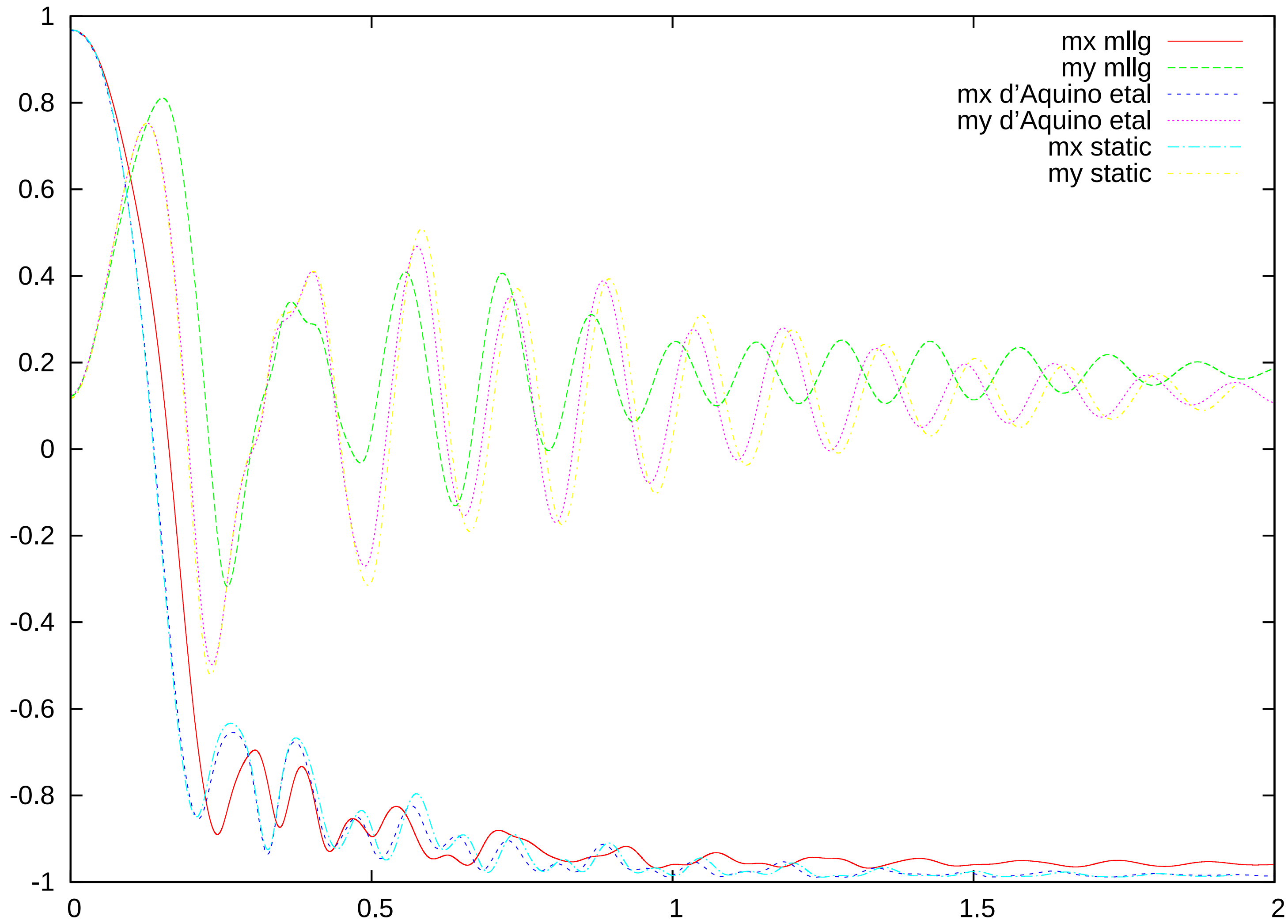


Figure 9: Evolution of $|\omega|^{-1} \int_\omega \mathbf{m}_{x(y)}$.

Numerical experiments

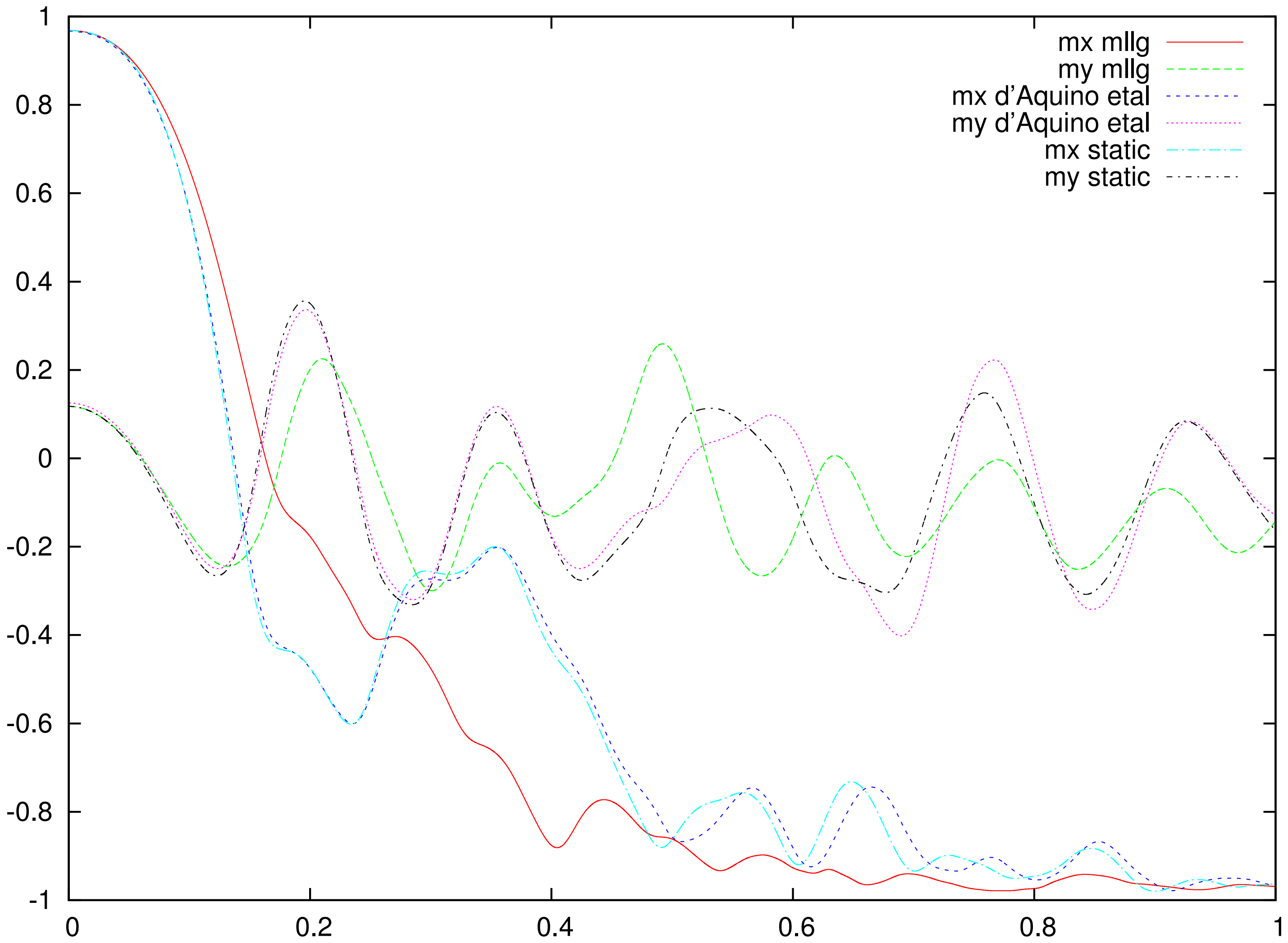


Figure 10: Evolution of $|\omega|^{-1}\int_{\omega}\mathbf{m}_{x(y)}$.

End

Thank you for your attention!