# Cell centered Galerkin methods

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# An essential bibliography I

- Multi-point Finite Volume methods [Aavatsmark *et al.*], [Edwards *et al.*]
- Mimetic Finite Difference methods [Brezzi, Lipnikov, Shashkov, Simoncini, Beirão da Vega, Boffi, Buffa, Cangiani, Kuznetsov, Manzini, Russo *et al.*]
- Variational Finite Volume methods [Eymard, Gallouët, Herbin, Agélas, DP, Droniou *et al.*]
- Cell centered Galerkin methods [DP, 2010]
- And, for inspiration, [Houston & Süli, 2001], [Burman & Zunino, 2006], [DP, Ern & Guermond, 2008], [Buffa & Ortner, 2009], [DP & Ern, 2010], [Agélas, Droniou, & DP, 2010]

# Outline

The L-construction to recover gradients and traces

The S$\omega$IP-ccG method

The $\mathcal{S}_h$-ccG method

Numerical examples

# Model problem

- Let $\Omega$ be a poly{gonal,hedral} domain in $\mathbb{R}^d$
- Let $\kappa$ be a uniformly positive, piecewise constant tensor field
- For $f \in L^2$, consider the problem

$$\begin{aligned} -\nabla\cdot(\kappa\nabla u) &= f && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega \end{aligned}$$

- Denote by $\mathcal{T}_h$ a poly{gonal,hedral}, possibly nonconforming mesh of $\Omega$ with (hyper)planar faces

# Cell centered Galerkin methods

- Fix the algebraic space of DOFs, in our case

$$\mathbb{V} \stackrel{\text{def}}{=} \mathbb{R}^{\mathcal{T}_h}$$

- Reconstruct a gradient ($\mathcal{S}_h = \mathcal{T}_h$ or submesh of $\mathcal{T}_h$)

$$\boxed{\mathfrak{G}_h : \mathbb{R}^{\mathcal{T}_h} \to [\mathbb{P}_d^0(\mathcal{S}_h)]^d}$$

- Let $\boxed{\mathfrak{I}_h : \mathbb{R}^{\mathcal{T}_h} \to \mathbb{P}_d^1(\mathcal{S}_h)}$ be s.t., for all $\mathbf{v} \in \mathbb{R}^{\mathcal{T}_h}$,

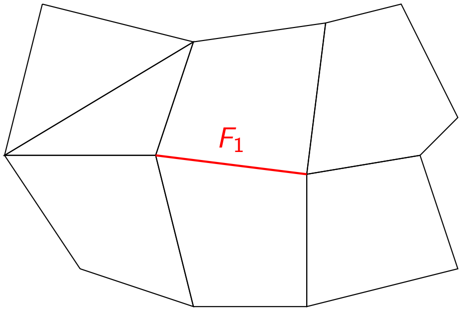
$$\forall S \in \mathcal{S}_h,\ S \subset T_S \in \mathcal{T}_h, \quad \mathfrak{I}_h(\mathbf{v})|_{T_S}(x) = v_{T_S} + \mathfrak{G}_h(\mathbf{v})|_{T_S} \cdot (x - x_{T_S})$$

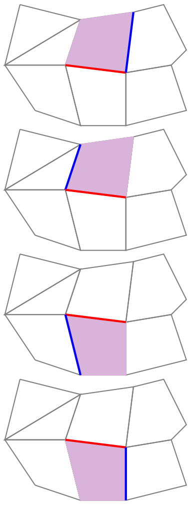
- Define the incomplete polynomial space

$$\boxed{V_h \stackrel{\text{def}}{=} \mathfrak{I}_h(\mathbb{R}^{\mathcal{T}_h}) \subset \mathbb{P}_d^1(\mathcal{S}_h)}$$

- Use $V_h$ as a trial/test space

# The L-construction I

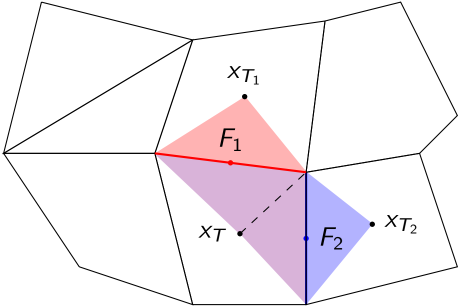


$\mathcal{T}_h$ general poly{gonal,hedral} mesh
$\mathcal{F}_h$ set of faces, $\mathcal{F}_h = \mathcal{F}_h^i \cup \mathcal{F}_h^b$
$\mathcal{F}_T$ faces of $T \in \mathcal{T}_h$
$\mathcal{T}_F$ elements sharing $F \in \mathcal{F}_h$

# The L-construction II

$$\boxed{\mathbf{v} \in \mathbb{R}^{\mathcal{T}_h} \to \xi_{\mathbf{v}} \text{ s.t. } \xi_{\mathbf{v}}(x_T) = v_T \text{ and } \xi_{\mathbf{v}}(x_{T_i}) = v_{T_i},\ i \in \{1,2\}}$$




$$\begin{cases} \kappa_T \nabla\xi_{\mathbf{v}}|_T \cdot \mathrm{n}_{F_1} = \kappa_{T_1} \nabla\xi_{\mathbf{v}}|_{T_1} \cdot \mathrm{n}_{F_1} \\ \xi_{\mathbf{v}}|_T(x) = \xi_{\mathbf{v}}|_{T_1}(x) \quad \forall x \in F \end{cases} \qquad \begin{cases} \kappa_T \nabla\xi_{\mathbf{v}}|_T \cdot \mathrm{n}_{F_2} = \kappa_{T_2} \nabla\xi_{\mathbf{v}}|_{T_2} \cdot \mathrm{n}_{F_2} \\ \xi_{\mathbf{v}}|_T(x) = \xi_{\mathbf{v}}|_{T_2}(x) \quad \forall x \in F \end{cases}$$

$$x_T \stackrel{\text{def}}{=} \int_T x/|T|,\ T \text{ star-shaped w.r.t } x_T$$

# Gradient reconstruction

- For all $F \in \mathcal{F}_h$, let $x_F \stackrel{\text{def}}{=} \int_F x / |F|$
- We can infer an injective trace reconstruction operator $\mathfrak{T}$

$$\mathfrak{T} : \mathbb{R}^{\mathcal{T}_h} \to \mathbb{R}^{\mathcal{F}_h}$$
$$\mathbb{R}^{\mathcal{T}_h} \ni \mathbf{v} \mapsto \left(v_{T_F} + G_{T_F}(\mathbf{v}) \cdot (x_F - x_{T_F})\right)_{F \in \mathcal{F}_h} \in \mathbb{R}^{\mathcal{F}_h}$$

- The gradient reconstruction operator $\mathfrak{G}_h$ is then defined as

$$\mathfrak{G}_h : \mathbb{R}^{\mathcal{T}_h} \to [\mathbb{P}_d^0(\mathcal{T}_h)]^d, \qquad \mathbb{R}^{\mathcal{T}_h} \ni \mathbf{v} \mapsto \mathfrak{G}_h(\mathbf{v}) \in [\mathbb{P}_d^0(\mathcal{T}_h)]^d,$$

where, setting $(v_F)_{F \in \mathcal{F}_h} = \mathfrak{T}(\mathbf{v})$,

$$\boxed{\forall T \in \mathcal{T}_h, \qquad \mathfrak{G}_h(\mathbf{v})|_T = \frac{1}{|T|} \sum_{F \in \mathcal{F}_T} |F| \, (v_F - v_T) \mathrm{n}_{F,T}}$$

# The discrete space I

- Define the injective operator

$$\mathfrak{I}_h : \mathbb{R}^{\mathcal{T}_h} \to \mathbb{P}^1_d(\mathcal{T}_h), \qquad \mathbb{R}^{\mathcal{T}_h} \ni \mathbf{v} \mapsto v_h \in \mathbb{P}^1_d(\mathcal{T}_h)$$

where, for all $T \in \mathcal{T}_h$ and all $x \in T$,

$$v_h|_T(x) = v_T + \mathfrak{G}_h(\mathbf{v})|_T \cdot (x - x_T)$$

- The discrete space is defined as

$$\boxed{V_h = \mathfrak{I}_h(\mathbb{R}^{\mathcal{T}_h}) \subset \mathbb{P}^1_d(\mathcal{T}_h)}$$

# S$\omega$IP-ccG method

- Introduce the spaces

$$V_\dagger \overset{\text{def}}{=} H^1_0(\Omega) \cap H^2(\mathcal{T}_h), \qquad V_{\dagger h} \overset{\text{def}}{=} V_h + V_\dagger$$

- Define on $V_{\dagger h} \times V_h$

$$a_h(w, v_h) = \int_\Omega \kappa \nabla_h w \cdot \nabla_h v_h + \sum_{F \in \mathcal{F}_h} \frac{\gamma_F}{h_F} \eta \int_F [\![w]\!][\![v_h]\!]$$
$$- \sum_{F \in \mathcal{F}_h} \int_F [\{\kappa \nabla_h w\}_\omega \cdot \mathrm{n}_F [\![v_h]\!] + [\![w]\!] \{\kappa \nabla_h v_h\}_\omega \cdot \mathrm{n}_F]$$

- Average operators and stabilization term are tailored to have coercivity/boundedness constants independent of the heterogeity of $\kappa$

# Implementation

- All the integrals in $a_h$ can be evaluated using barycenters
- The general bilinear term $\mathfrak{T}$ can be recast into the form

$$\mathfrak{T} = \chi \mathcal{A}_{\mathfrak{T}}(\mathbf{u}) \mathcal{L}_{\mathfrak{T}}(\mathbf{v}), \qquad \mathbf{u}, \mathbf{v} \in \mathbb{R}^{\mathcal{T}_h},$$

  where $\chi$ is a real coefficient and

$$\mathcal{A}_{\mathfrak{T}}(\mathbf{u}) = \alpha_0 + \sum_{T \in \mathcal{T}_{\mathcal{A}}} \alpha_T u_T, \qquad \mathcal{L}_{\mathfrak{T}}(\mathbf{v}) = \sum_{T \in \mathcal{T}_{\mathcal{L}}} \lambda_T v_T$$

- The corresponding local contribution is

$$\mathbf{M}_{\mathfrak{T}} = \chi \left[\lambda_T \alpha_{T'}\right]_{T \in \mathcal{T}_{\mathcal{L}},\, T' \in \mathcal{T}_{\mathcal{A}}}, \qquad \mathbf{r}_{\mathfrak{T}} = \chi \left(\lambda_T \alpha_0\right)_{T \in \mathcal{T}_{\mathcal{L}}}$$

# Convergence to smooth solutions

**Theorem (Error estimate for smooth exact solutions)**

*There holds*

$$|||u - u_h|||_\kappa \leq \left(1 + \frac{C_{bnd}}{C_{sta}}\right) \inf_{w_h \in V_h} |||u - w_h|||_{\kappa,*}.$$

**Corollary (Convergence rate)**

*Assuming further regularity on u*[1], *there holds*

$$|||u - u_h|||_\kappa \leq Ch,$$

*with $C$ independent of $h$ and of $\kappa$.*

[1] See [Agélas, DP & Droniou 2010]

# Convergence to solutions in $H_0^1(\Omega)$

## Theorem (Convergence to minimal regularity solutions)

*Let* $(u_h)_{h\in\mathcal{H}}$ *denote the sequence of discrete solutions on the admissible mesh family* $(\mathcal{T}_h)_{h\in\mathcal{H}}$*. Then, there exists* $u \in H_0^1$ *solution of the continuous problem s.t.*

$$\begin{aligned} u_h &\to u && \textit{strongly in } L^2(\Omega), \\ \nabla_h u_h &\to \nabla u && \textit{strongly in } [L^2(\Omega)]^d, \\ |u_h|_{J,\kappa} &\to 0. \end{aligned}$$

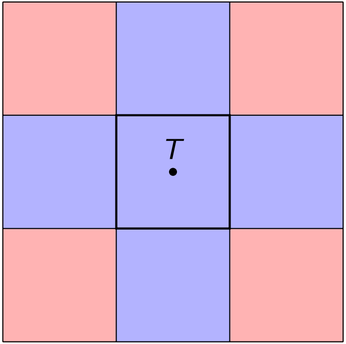
## Proof.

*A fortiori* from [DP & Ern,2010]. □

# A first balance

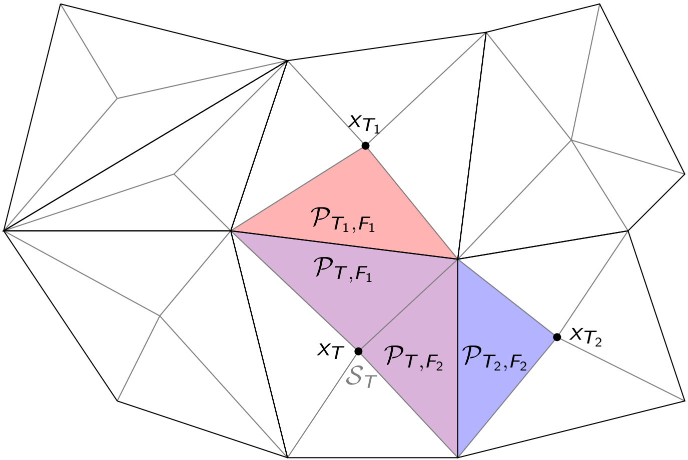
- Local reconstruction handling heterogeneities
- Galerkin method on general meshes in arbitrary dimension based on an incomplete polynomial space
- Easy to implement in existing dG codes (ongoing work on the `Life`/`Feel` platform by C. Prud'homme)
- The procedure virtually extends to every problem for which a dG method has been conceived
- Main drawback: large stencil

# Support of SωIP-ccG basis functions




Minimal (blue) and maximal (blue+red) support for $\varphi_T = \mathfrak{I}_h(\mathbf{e}_T)$.

# Reducing the stencil: The $\mathcal{S}_h$-ccG method I




$\mathcal{S}_h = \{\mathcal{P}_{T,F}\}_{T\in\mathcal{T}_h,\, F\in\mathcal{F}_T}$, $\quad \mathcal{S}_T =$ lateral pyramid faces in $T$

# Reducing the stencil: The $\mathcal{S}_h$-ccG method II

- The L-construction naturally yields a gradient in each face-based pyramid $\mathcal{P}_{T,F}$
- We therefore consider the alternative gradient reconstruction

$$\mathfrak{G}_h : \mathbb{R}^{\mathcal{T}_h} \to [\mathbb{P}^0_d(\mathcal{S}_h)]^d, \qquad \mathbf{v} \mapsto \mathfrak{G}_h(\mathbf{v}),$$

  where, for all $T \in \mathcal{T}_h$ and all $F \in \mathcal{F}_T$

$$\mathfrak{G}_h(\mathbf{v})|_{\mathcal{P}_{T,F}} = G_{\mathcal{P}_{T,F}}$$

- No need for trace reconstructions this time!

# Reducing the stencil: The $\mathcal{S}_h$-ccG method III

- Define the injective operator

$$\mathfrak{I}_h : \mathbb{R}^{\mathcal{T}_h} \to \mathbb{P}^1_d(\mathcal{S}_h), \qquad \mathbb{R}^{\mathcal{T}_h} \ni \mathbf{v} \mapsto v_h = \mathfrak{I}_h(\mathbf{v}) \in \mathbb{P}^1_d(\mathcal{S}_h),$$

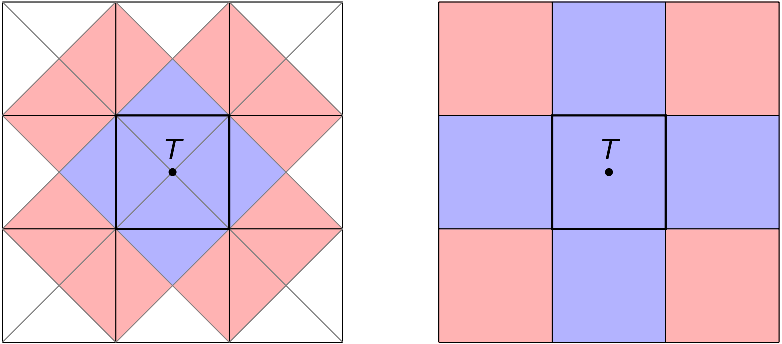
where, for all $T \in \mathcal{T}_h$, all $F \in \mathcal{F}_T$, and all $x \in \mathcal{P}_{T,F}$,

$$v_h|_{\mathcal{P}_{T,F}}(x) = v_T + \mathfrak{G}_h(\mathbf{v})|_{\mathcal{P}_{T,F}} \cdot (x - x_T)$$

- As before, we set

$$\boxed{V_h \stackrel{\text{def}}{=} \mathfrak{I}_h(\mathbb{R}^{\mathcal{T}_h}) \subset \mathbb{P}^1_d(\mathcal{S}_h)}$$

# Reducing the stencil: The $\mathcal{S}_h$-ccG method IV




Minimal (blue) and maximal (blue+red) support for $\varphi_T = \mathfrak{I}_h(\mathbf{e}_T)$
(left panel: $\mathcal{S}_h$-ccG, right panel: S$\omega$IP-ccG)

# A second method with subgrid stabilization

- Based on the bilinear form

$$
\begin{aligned}
a_h(w, v_h) \stackrel{\text{def}}{=} & \int_\Omega \kappa \nabla_h w \cdot \nabla_h v_h \\
& + \sum_{T \in \mathcal{T}_h} \sum_{S \in \mathcal{S}_T} \int_S \frac{\eta k_S}{h_S} [\![ w ]\!] [\![ v_h ]\!] + \sum_{F \in \mathcal{F}_h^b} \int_F \frac{\eta k_F}{h_F} w v_h \\
& - \sum_{T \in \mathcal{T}_h} \sum_{S \in \mathcal{S}_T} \int_S \left( \kappa \{ \nabla_h w \} \cdot \mathrm{n}_S [\![ v_h ]\!] + \kappa \{ \nabla_h v_h \} \cdot \mathrm{n}_S [\![ w ]\!] \right) \\
& - \sum_{F \in \mathcal{F}_h^b} \int_F \left( \kappa \nabla w \cdot \mathrm{n}_F v_h + \kappa \nabla v_h \cdot \mathrm{n}_F w \right) .
\end{aligned}
$$

- Similar theoretical results as for the S$\omega$IP-ccG method

# Numerical examples I

| card($\mathcal{T}_h$) | $\|u - u_h\|_{L^2(\Omega)}$ | order | $\|\|u - u_h\|\|_{\kappa}$ | order | stencil |
|---|---|---|---|---|---|
| | | | S$\omega$IP-ccG | | |
| 3584 | 3.4013e-04 | – | 1.9112e-02 | – | 25.3 |
| 14336 | 8.5152e-05 | 2.00 | 9.5397e-03 | 1.00 | 25.72 |
| 57344 | 2.1298e-05 | 2.00 | 4.7656e-03 | 1.00 | 26.11 |
| 229376 | 5.3253e-06 | **2.00** | 2.3821e-03 | **1.00** | **26.10** |
| | | | $\mathcal{S}_h$-ccG | | |
| 3584 | 9.5453e-04 | – | 2.4299e-02 | – | 15.02 |
| 14336 | 2.4312e-04 | 1.97 | 1.1908e-02 | 1.03 | 15.15 |
| 57344 | 6.1328e-05 | 1.99 | 5.8797e-03 | 1.02 | 15.22 |
| 229376 | 1.5420e-05 | **2.00** | 2.9466e-03 | **1.00** | **15.25** |

Homogeneous isotropic test case

# Numerical examples II

| card($\mathcal{T}_h$) | $\|u - u_h\|_{L^2(\Omega)}$ | order | $\|\|\|u - u_h\|\|\|_\kappa$ | order | stencil |
|---|---|---|---|---|---|
| | | S$\omega$IP-ccG | | | |
| 3584 | 5.2227e-04 | – | 2.8501e-02 | – | 25.26 |
| 14336 | 1.2926e-04 | 2.01 | 1.4194e-02 | 1.01 | 25.73 |
| 57344 | 3.2545e-05 | 1.99 | 7.1126e-03 | 1.00 | 26.11 |
| 229376 | 7.9803e-06 | **2.03** | 3.5293e-03 | **1.01** | **26.10** |
| | | $\mathcal{S}_h$-ccG | | | |
| 3584 | 1.3027e-03 | – | 4.5746e-02 | – | 15.03 |
| 14336 | 3.1039e-04 | 2.07 | 2.2608e-02 | 1.02 | 15.16 |
| 57344 | 8.6435e-05 | 1.84 | 1.1581e-02 | 0.97 | 15.22 |
| 229376 | 1.9971e-05 | **2.11** | 5.6241e-03 | **1.04** | **15.25** |

Heterogeneous test case (heterogeneity ratio $= 1000$)

# Numerical examples III

| card($\mathcal{T}_h$) | $\|u-u_h\|_{L^2(\Omega)}$ | order | $\|\|u-u_h\|\|_\kappa$ | order | stencil |
|---|---|---|---|---|---|
| | | S$\omega$IP-ccG | | | |
| 806 | 4.0093e-03 | – | 4.8250e-02 | – | 25.33 |
| 3162 | 1.7803e-03 | 1.17 | 2.7231e-02 | 0.83 | 26.35 |
| 12632 | 4.9217e-04 | 1.85 | 1.3674e-02 | 0.99 | 26.85 |
| 50548 | 1.3945e-04 | **1.82** | 6.9776e-03 | **0.97** | **27.12** |
| | | $\mathcal{S}_h$-ccG | | | |
| 806 | 6.2131e-03 | – | 9.0381e-02 | – | 14.69 |
| 3162 | 1.7417e-03 | 1.83 | 4.5942e-02 | 0.98 | 15.25 |
| 12632 | 5.7562e-04 | 1.60 | 2.2897e-02 | 1.00 | 15.38 |
| 50548 | 1.4492e-04 | **1.99** | 1.1341e-02 | **1.01** | **15.52** |

Anisotropic test case (anisotropy ratio $=1000$)

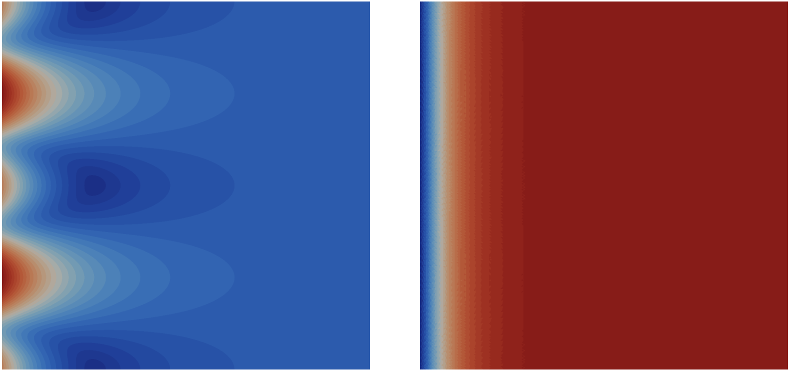
# Application to the incompressible Navier-Stokes problem I

| card($\mathcal{T}_h$) | $\|u - u_h\|_{[L^2(\Omega)]^d}$ | ord | $\|p - p_h\|_{L^2(\Omega)}$ | ord |
|---|---|---|---|---|
| 224 | 1.5288e-01 | – | 2.5693e-01 | – |
| 896 | 4.1691e-02 | 1.87 | 1.0847e-01 | 1.24 |
| 3584 | 1.1115e-02 | 1.91 | 4.0251e-02 | 1.43 |
| 14336 | 2.9261e-03 | 1.93 | 1.7487e-02 | 1.20 |
| 57344 | 7.6622e-04 | **1.93** | 8.7005e-03 | **1.01** |

| card($\mathcal{T}_h$) | $\|\|\|(u - u_h, p - p_h)\|\|\|_{\mathrm{sto}}$ | ord |
|---|---|---|
| 224 | 4.5730e-01 | – |
| 896 | 2.1185e-01 | 1.11 |
| 3584 | 1.0319e-01 | 1.04 |
| 14336 | 5.1495e-02 | 1.00 |
| 57344 | 2.6540e-02 | **0.96** |

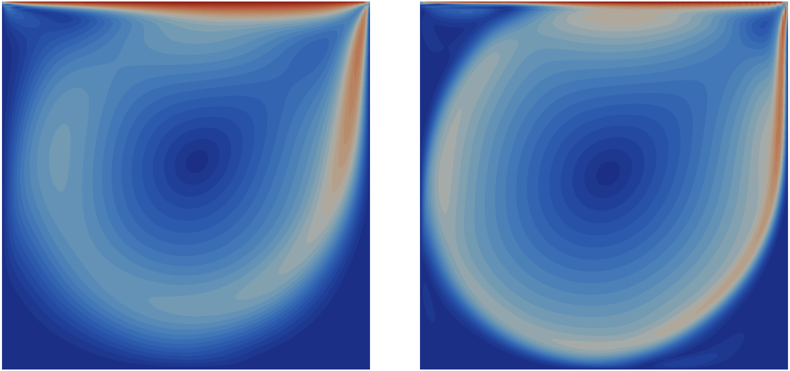
Kovasznay's problem, IP-ccG method inspired by [DP & Ern, 2010]

# Application to the incompressible Navier-Stokes problem II



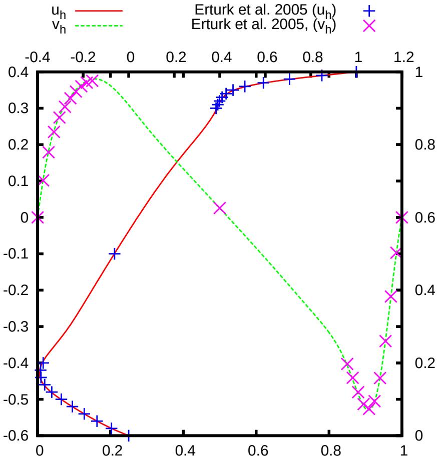
Kovasznay's problem, velocity magnitude and pressure

# Application to the incompressible Navier-Stokes problem III



Lid-driven cavity problem, Re=1000,7500, velocity magnitude

# Application to the incompressible Navier-Stokes problem IV




Lid-driven cavity problem, Re=1000, comparison with [Erturk, Corke, Gökçöl, 2005]