hp-Adaptive Shell Solver

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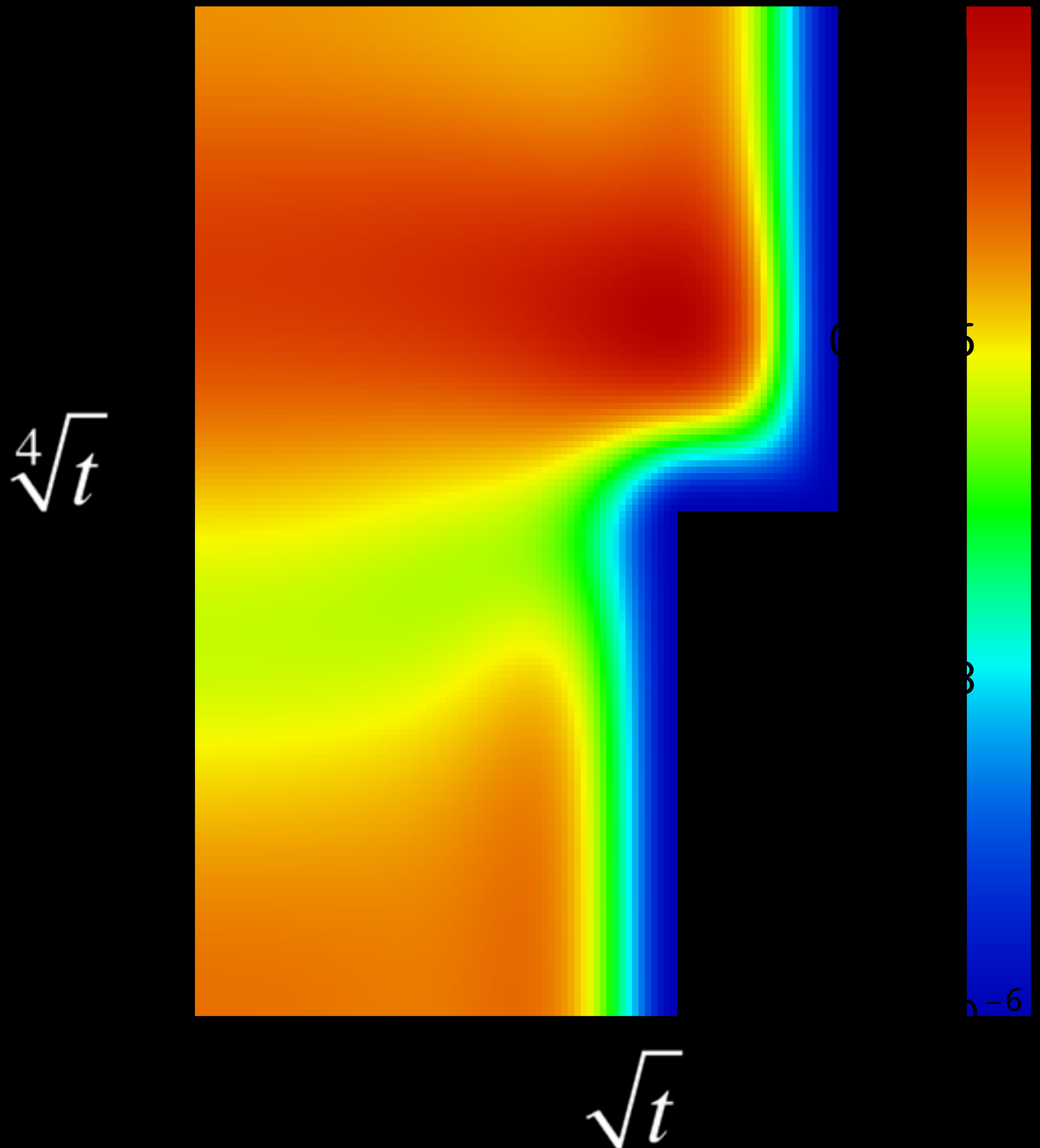
We Are Getting There

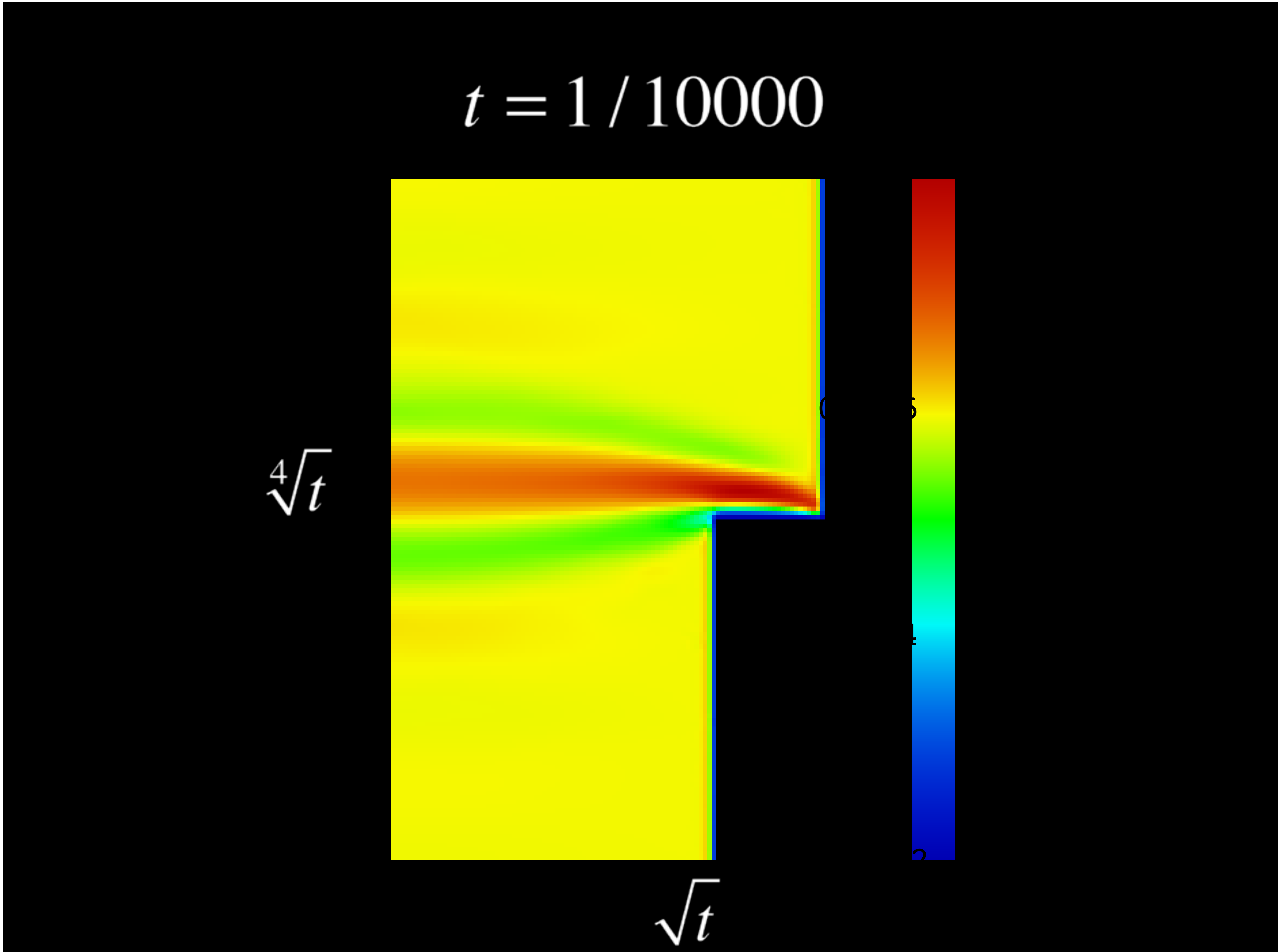
- Shells
- Model Problems
- Locking
  - Standard finite elements used here
- hp-Algorithm
- Remaining Challenges

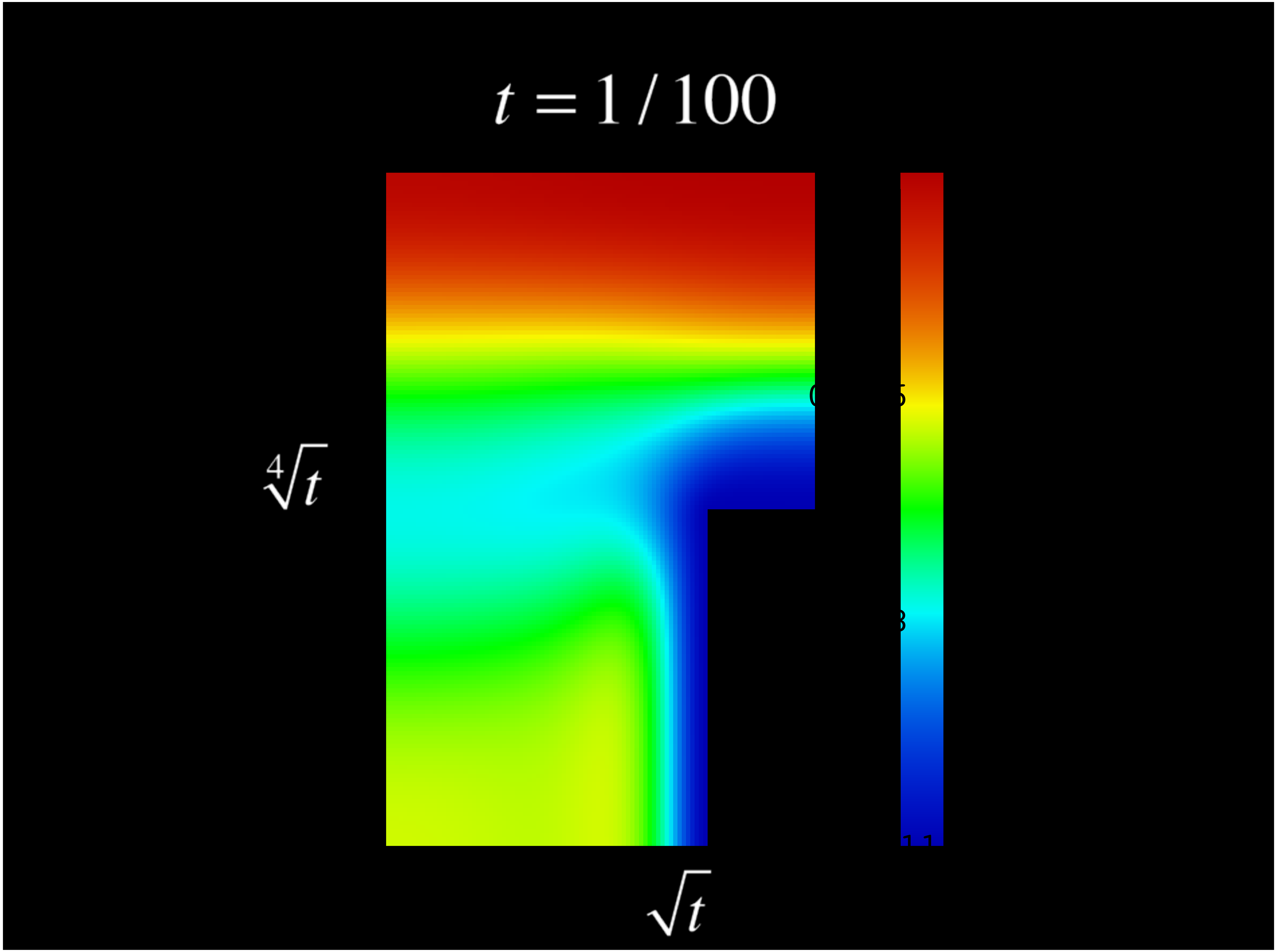
Characteristic Scales

- Every solution can be considered as
  - a linear combination of characteristic features each with its own length scale
  - these may be boundary layers, internal layers or span the whole domain
- Layers are generated by boundaries, point or line loads, or non-continuous changes in curvature

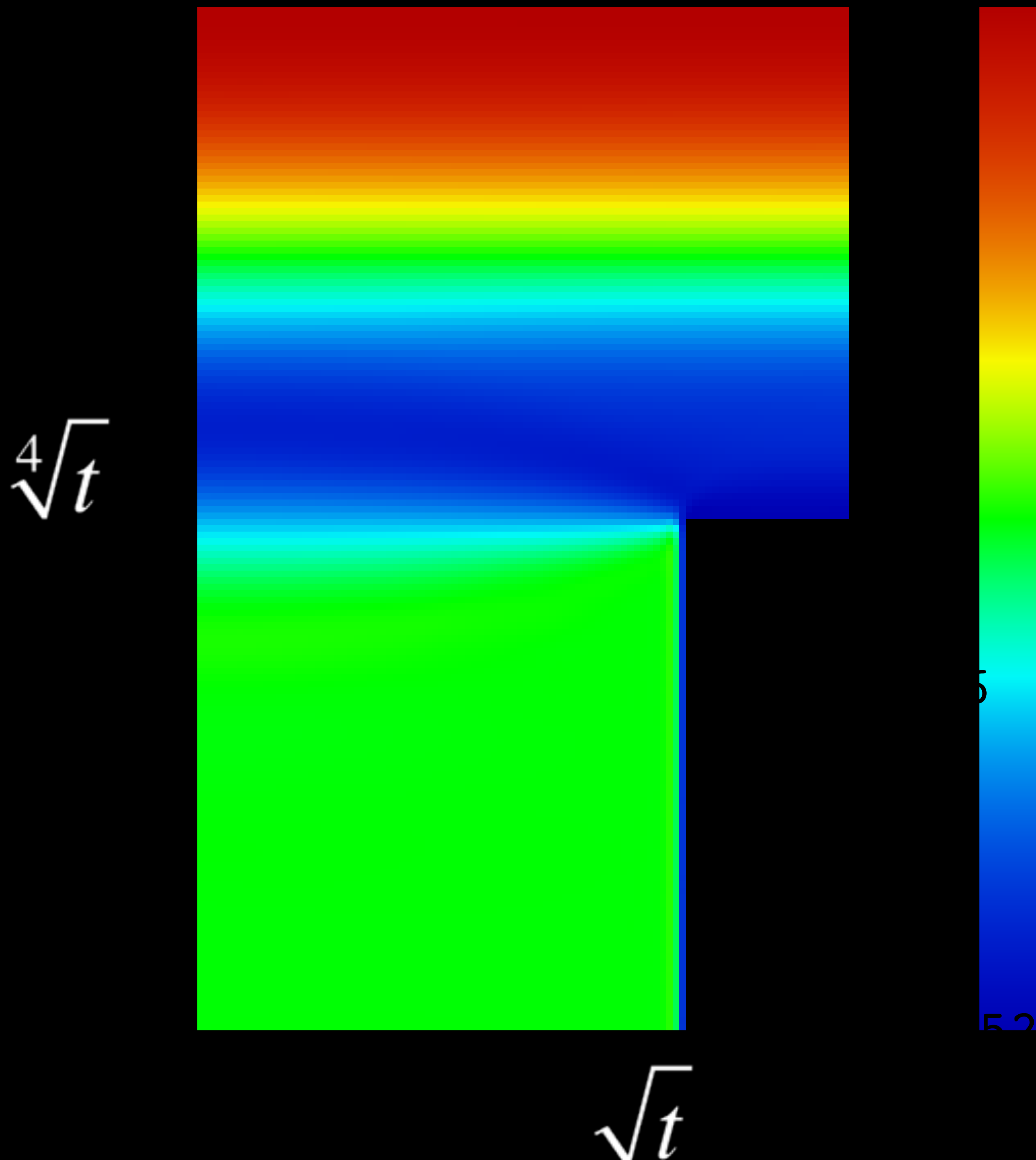
$t = 1/100$




t = 1 / 10000
⁴√t
√t


$t = 1/100$
$\sqrt[4]{t}$
$\sqrt{t}$

$t = 1/10000$



Shell Models

In these models the total energy of the shell is given by

$$\mathcal{F}(\mathbf{u}) = \frac{1}{2}\mathcal{A}(\mathbf{u},\mathbf{u}) - Q(\mathbf{u}) \tag{1}$$

where $\mathcal{A}$ represents the (possibly scaled) deformation energy, Q denotes the load potential and $\mathbf{u} = (u, v, w, \theta, \psi)$ is the vector of three translations and two rotations. The deformation energy is further split into bending, membrane and transverse shear energy:

$$\mathcal{A}(\mathbf{u},\mathbf{u}) = t^3\mathcal{A}_B(\mathbf{u},\mathbf{u}) + t\mathcal{A}_M(\mathbf{u},\mathbf{u}) + t\mathcal{A}_S(\mathbf{u},\mathbf{u}). \tag{2}$$

Here t denotes the thickness of the shell.

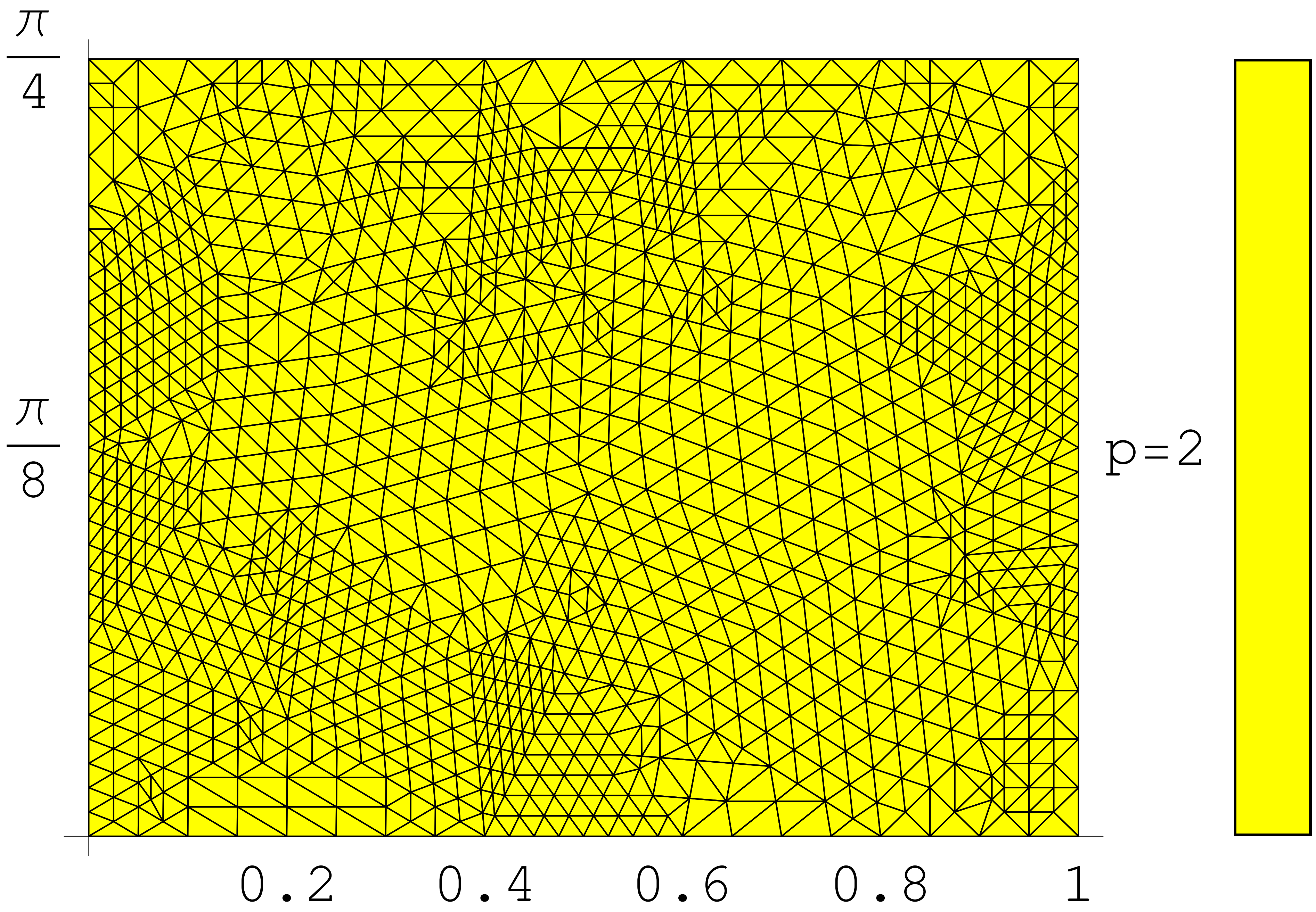
Numerical Locking: Free Cylinder

- Energy distribution
- h-adaptive meshes

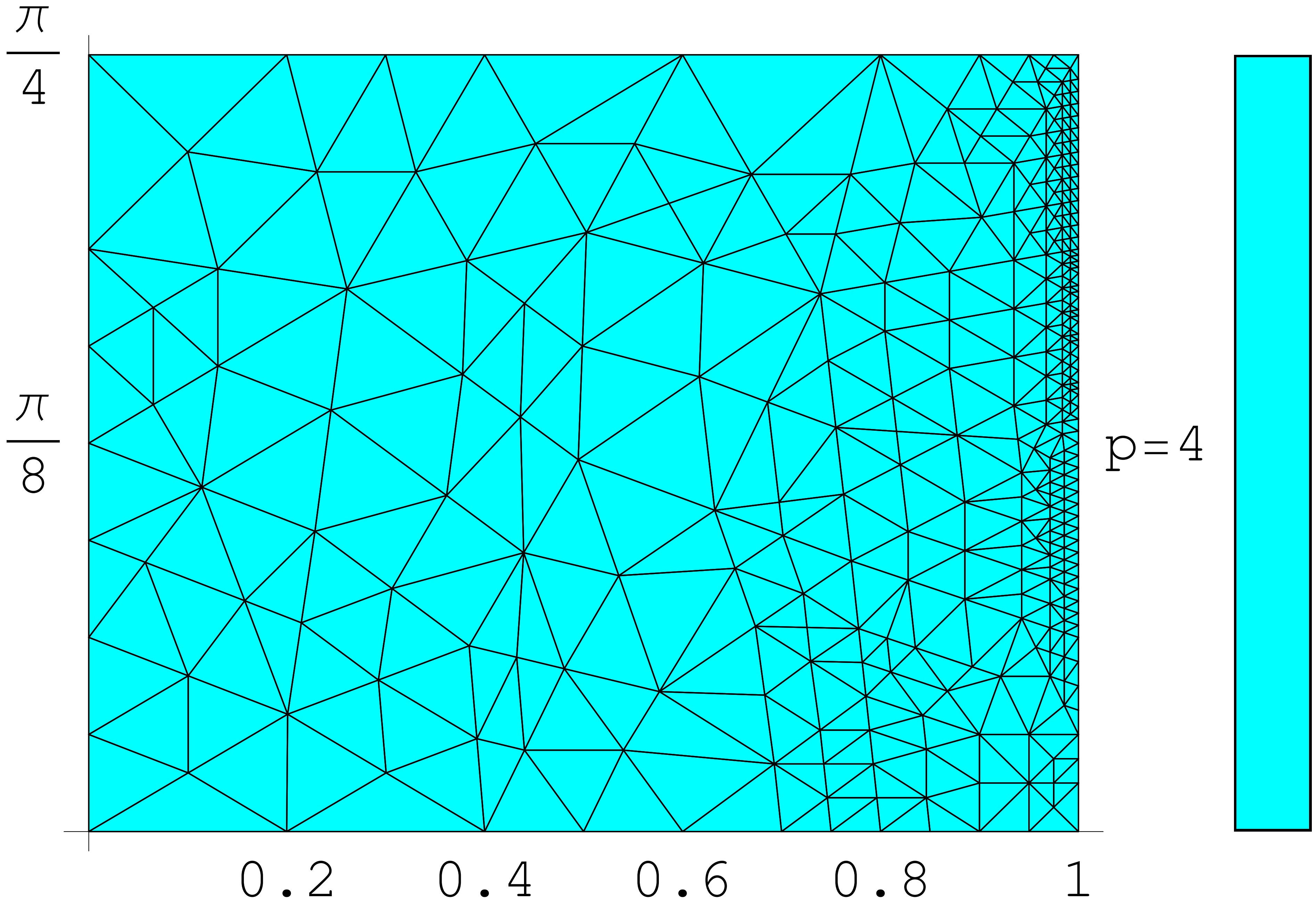
[Rank & al, 06]

p	Bending/E	Membrane/E	Shear/E
1	0.02	40.8	59.1
2	1.36	92.2	6.44
3	94.2	5.5	0.32
4	99.4	0.56	0.03

Numerical Locking



Numerical Locking



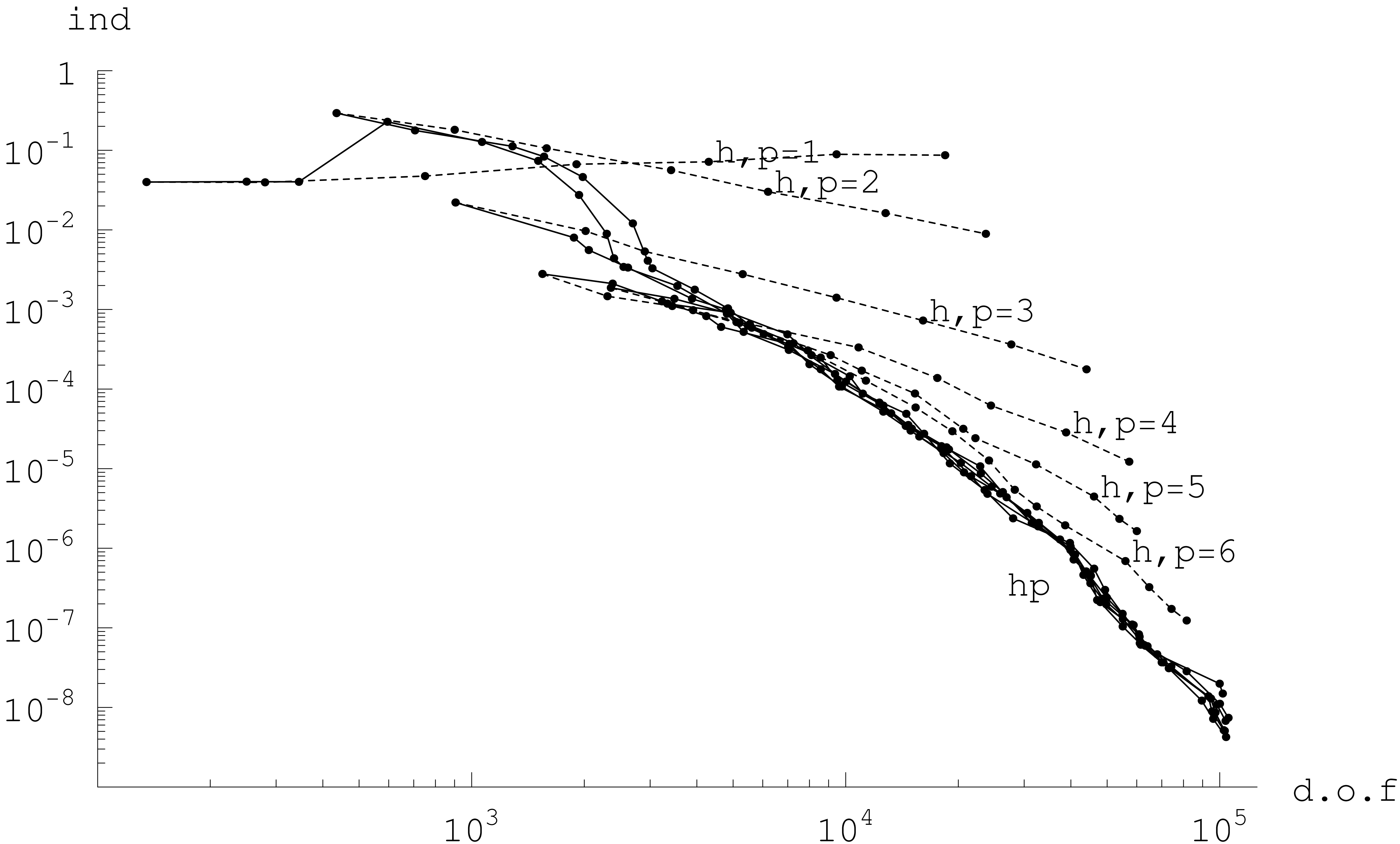
Bending vs Membrane
(or Plan to Throw One Away)

- Any successful scheme must determine the dominating mode before adaptive steps.
- Our solution is to probe first:
  - Solve using a minimal mesh with sufficiently high p.
- No proof.
  Hint: Pitkäranta, Numer Math, 1993

Adaptive Algorithm

- Combination of
  - bubble-mode error indicators
  - Sobolev regularity estimation [Houston & Süli, 03]
- Refine/coarsen the mesh
- Raise/lower the elemental polynomial degree

Indicator



ind
1
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
10^{-8}
h,p=1
h,p=2
h,p=3
h,p=4
h,p=5
h,p=6
hp
10^3
10^4
10^5
d.o.f

Sobolev Regularity

Let us first consider the reference interval $(-1, 1)$ and a function $\hat{u} \in L^2(-1, 1)$ with Legendre series

$$\hat{u}(\xi) = \sum_{i=0}^{\infty} \hat{a}_i \hat{L}_i(\xi), \tag{6}$$

where $\hat{L}_i$ is a Legendre polynomial of degree i. Legendre polynomials are orthogonal so the coefficients $\hat{a}_i$ can be written as

$$\hat{a}_i = \frac{2i+1}{2} \int_{-1}^{1} \hat{u}(\xi) \hat{L}_i(\xi) d\xi. \tag{7}$$

Let us define a sequence $\{l_i\}_{i \geq 2}$ using $\hat{a}_i$:

$$l_i = \frac{\log\left(\frac{2i+1}{2|a_i|^2}\right)}{2 \log i}. \tag{8}$$

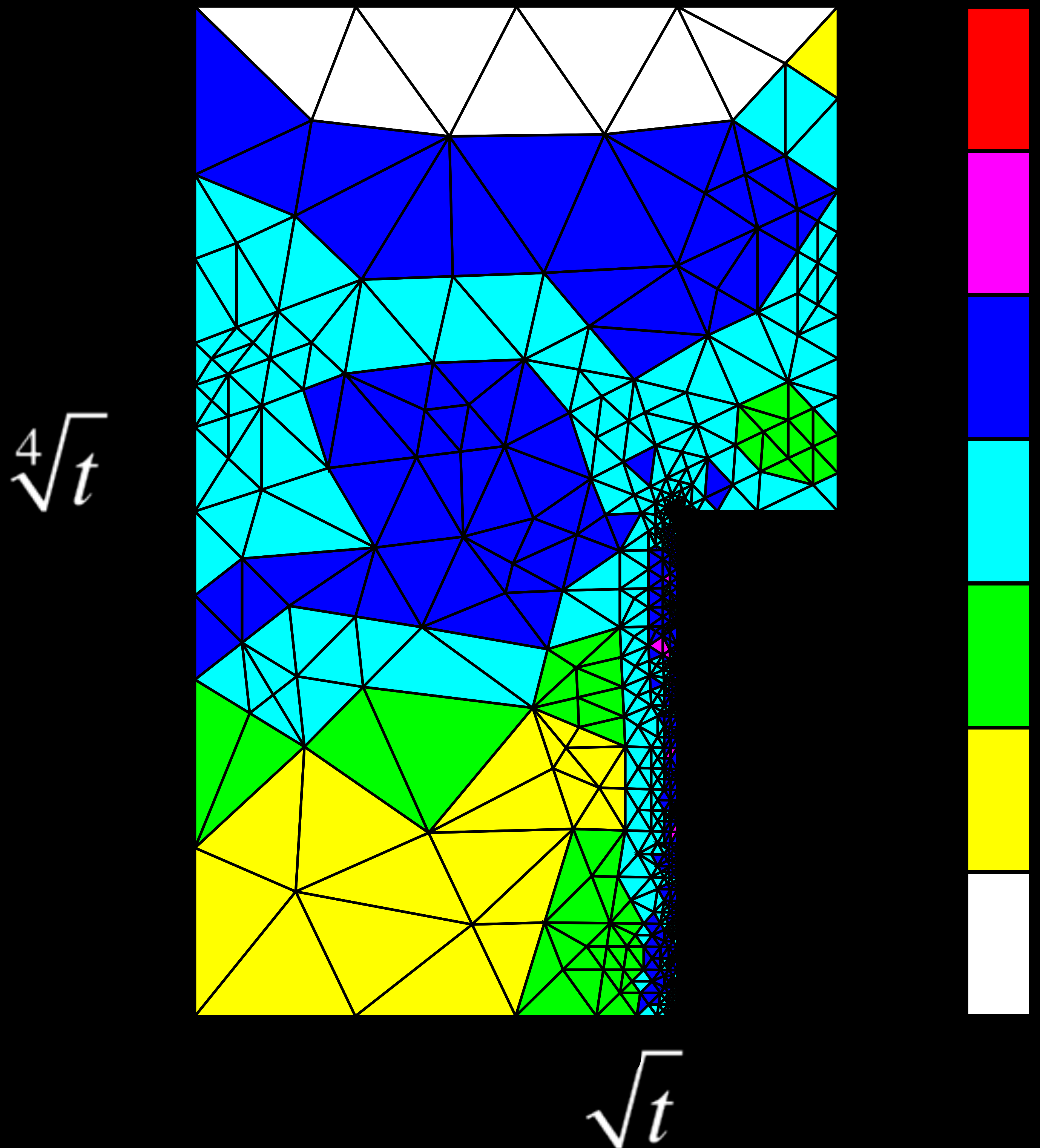
If $l = \lim_{i \to \infty} l_i$ exists and $l > 1/2$, then

$$u \in H_{loc}^{l-1/2-\epsilon}(-1, 1), \qquad 0 < \epsilon < l - 1/2.$$

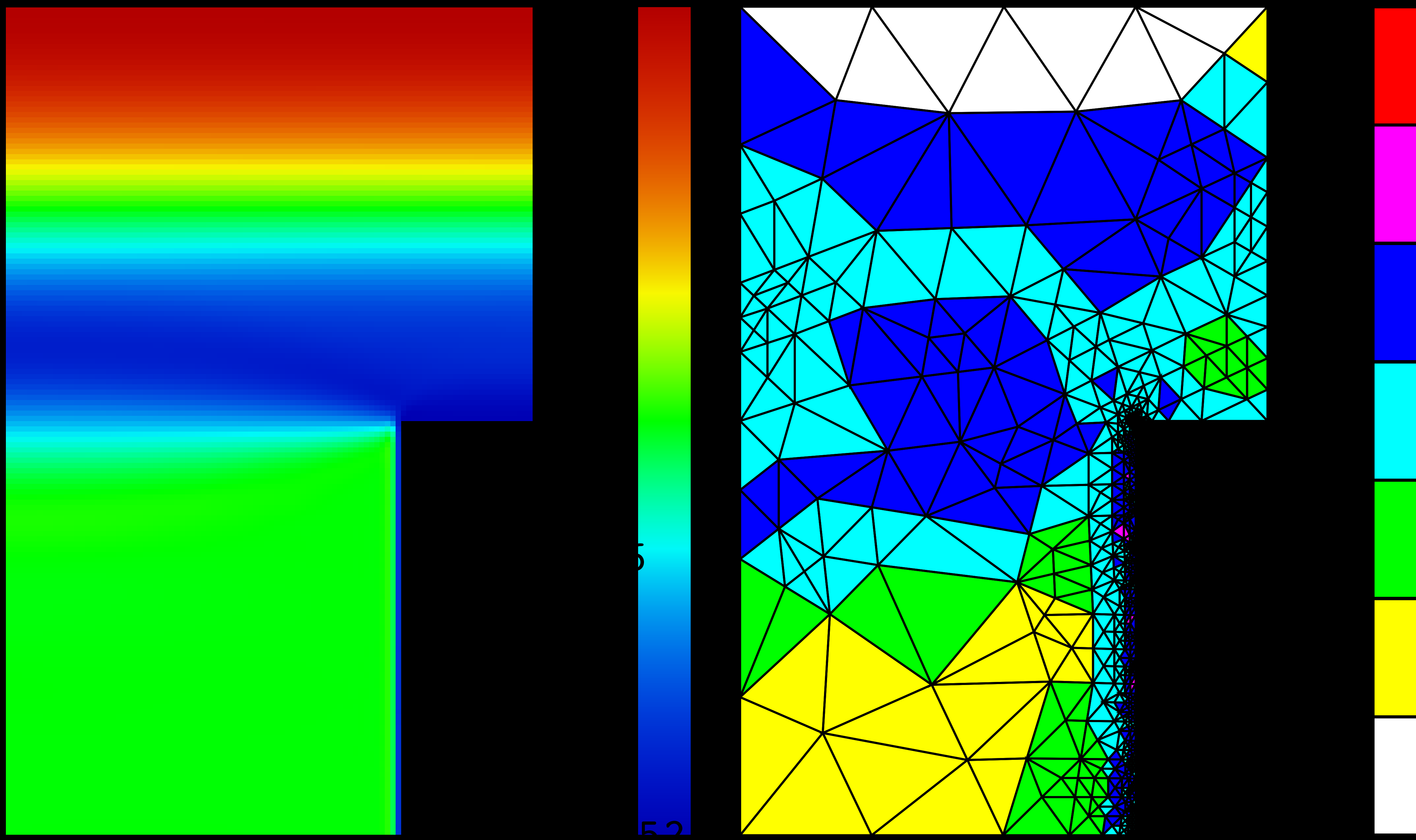
Step by Step

1. Compute the elemental indicators
2. Estimate the highest p for every element
3. Divide the elements in sets:
   3.1. Split, Raise, Lower
4. Check choices made at the previous step:
   4.1. For instance, if the elemental p is higher than that suggested now, lower p and split instead
5. Update elements and solve again

$t = 1/10000$



$t = 1 / 10000$



It Works, but

- There is a wealth of a priori knowledge of the layers.
- What is the most efficient way to use it?
- Boundary layer meshes are difficult to modify.

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Error Indicators

Let us denote the solution space (without bubbles) with $\mathcal{U}_h$ and the additional bubble modes with $\mathcal{U}_h^+$. Let $\underline{u}_h$ be the discrete solution: Find $\underline{u}_h \in \mathcal{U}_h$ such that

$$\mathcal{A}(\underline{u}_h, \underline{v}) = \mathcal{Q}(\underline{v}) \quad \forall \underline{v} \in \mathcal{U}_h.$$

Taking $\underline{u}_h$ as known, add bubbles $\underline{u}_h^+ \in \mathcal{U}_h^+$ to the solution vector. Thus the problem becomes: Find $\underline{u}_h^+ \in \mathcal{U}_h^+$ such that

$$\mathcal{A}(\underline{u}_h + \underline{u}_h^+, \underline{v}) = \mathcal{Q}(\underline{v}) \quad \forall \underline{v} \in \mathcal{U}_h^+. \tag{1}$$

Since every bubble is supported by exactly one element, the problem (1) can be solved element-by-element:

$$\mathcal{A}(\underline{u}_h^+, \underline{v})_e = \mathcal{Q}(\underline{v})_e - \mathcal{A}(\underline{u}_h, \underline{v})_e \quad \forall \underline{v} \in \mathcal{U}_h^+, \tag{2}$$

$e = 1, \ldots, e_{max}$.

Since the solution lies in a subspace of $\mathcal{U}$ we can transform (2) with variational problem so that we end up with

$$\mathcal{A}(\underline{u}_h^+, \underline{v})_e = \mathcal{A}(u - \underline{u}_h, \underline{v})_e \quad \forall \underline{v} \in \mathcal{U}_h^+ \tag{3}$$

The problem (3) can be interpreted so that the error $\underline{u}_{err} = \underline{u} - \underline{u}_h$ is approximated in subspace $\mathcal{U}_h^+ \subset \mathcal{U}$. Error is measured in the energy norm, so the elemental error indicator is

$$\eta_e^+ := |||\underline{u}_h^+|||_{K_e} \tag{4}$$

and corresponding global indicator

$$\eta^+ := \sqrt{\sum_e (\eta_e^+)^2}. \tag{5}$$

Notation

- κ_{ij}, β_{ij} and ρ_i denote the bending, membrane and transverse shear strains, respectively,
- ν is the Poisson number of the material.
- The integrals are calculated over the midsurface Ω of the shell which is parametrized by the (generally curvilinear) principal curvature coordinates α_i.
- The metric of the shell surface is given by the Lamé parameters A_i.
- R_i's are the principal radii of curvature of the shell at the point (α_1, α_2).

Naghdi Shell of Revolution

- Geometry: $f(x), x \in [-L, L]$
- $A_1(x) = \sqrt{1 + (f'(x))^2}$
- $A_2(x) = f(x)$
- $R_1(x) = -\frac{[A_1(x)]^3}{f''(x)}$
- $R_2(x) = A_1(x) A_2(x)$

$$
\begin{aligned}
t^3\mathcal{A}_B(\mathbf{u},\mathbf{u}) &= t^3 \cdot \int_\Omega \{\nu(\kappa_{11}(\mathbf{u}) + \kappa_{22}(\mathbf{u}))^2 \\
&\quad +(1-\nu)\sum_{i,j=1}^{2} \kappa_{ij}(\mathbf{u})^2\} A_1 A_2 \, d\alpha_1 d\alpha_2 \\
t\mathcal{A}_M(\mathbf{u},\mathbf{u}) &= t \cdot 12 \int_\Omega \{\nu(\beta_{11}(\mathbf{u}) + \beta_{22}(\mathbf{u}))^2 \\
&\quad +(1-\nu)\sum_{i,j=1}^{2} \beta_{ij}(\mathbf{u})^2\} A_1 A_2 \, d\alpha_1 d\alpha_2 \\
t\mathcal{A}_S(\mathbf{u},\mathbf{u}) &= t \cdot 6(1-\nu) \int_\Omega \{\rho_1(\mathbf{u})^2 + \rho_2(\mathbf{u})^2\} A_1 A_2 \, d\alpha_1 d\alpha_2.
\end{aligned}
$$

Membrane

$$\beta_{11} = \frac{1}{A_1}\frac{\partial u}{\partial \alpha_1} + \frac{v}{A_1 A_2}\frac{\partial A_1}{\partial \alpha_2} + \frac{w}{R_1}$$

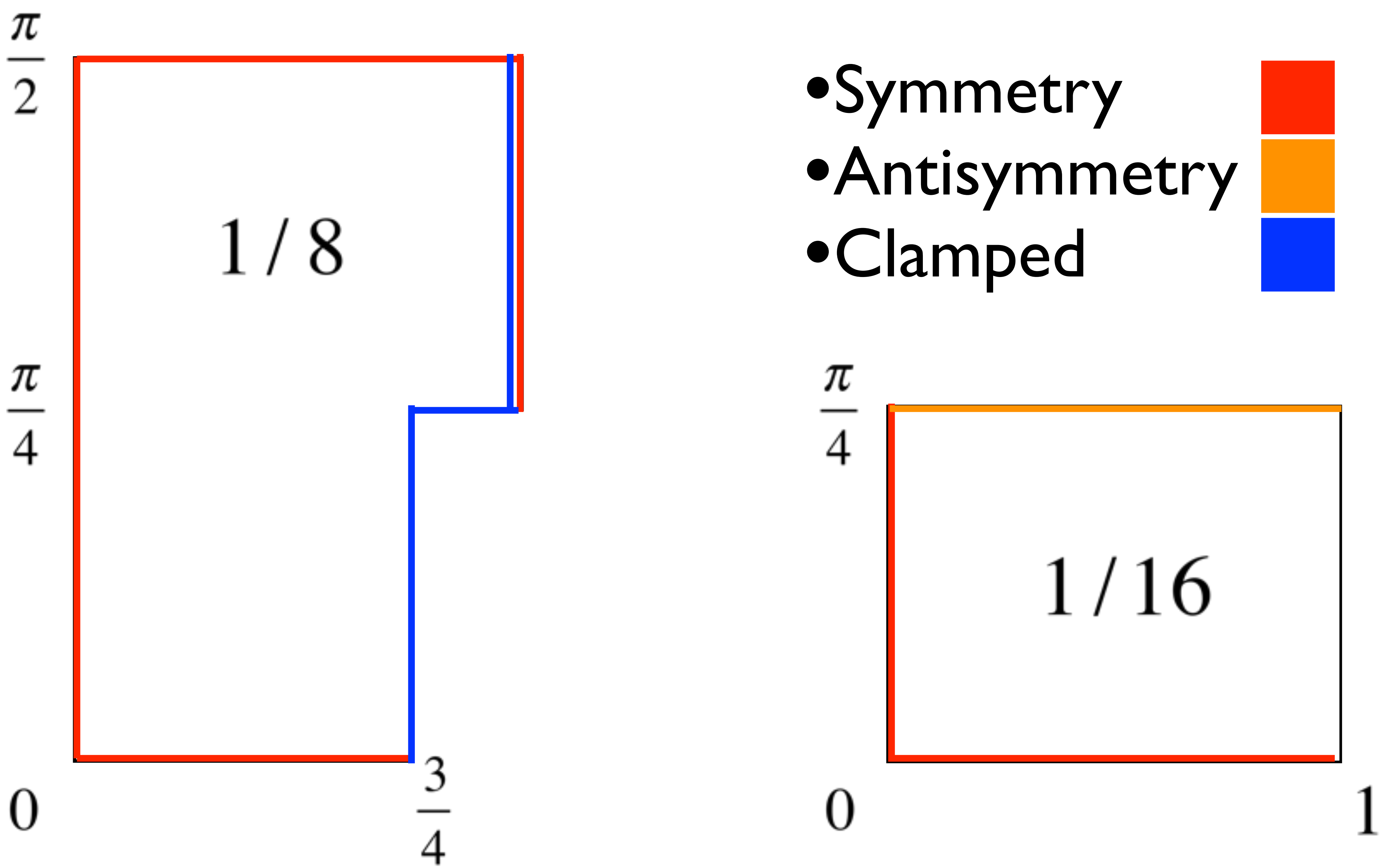
$$\beta_{22} = \frac{1}{A_2}\frac{\partial v}{\partial \alpha_2} + \frac{u}{A_1 A_2}\frac{\partial A_2}{\partial \alpha_1} + \frac{w}{R_2}$$

$$\beta_{12} = \frac{1}{2}(\frac{1}{A_1}\frac{\partial v}{\partial \alpha_1} + \frac{1}{A_2}\frac{\partial u}{\partial \alpha_2} - \frac{u}{A_1 A_2}\frac{\partial A_1}{\partial \alpha_2} - \frac{v}{A_1 A_2}\frac{\partial A_2}{\partial \alpha_1}) = \beta_{21}$$

Shallow

$$\beta_{11} = \frac{\partial u}{\partial \alpha_1} + aw,\ \beta_{22} = \frac{\partial v}{\partial \alpha_2} + bw,\ \beta_{12} = \frac{1}{2}\left(\frac{\partial v}{\partial \alpha_1} + \frac{\partial u}{\partial \alpha_2}\right) + cw = \beta_{21}$$

Two Cylinders



Loading: Constant Pressure | Transverse f(y)=cos(2y)

Membrane-Dominated | Bending-Dominated