FEM on Hierarchial Meshes

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Motivation

- Many preconditioners for hierarchial meshes : BPX, Hierarchial basis, Multigrid
- How many DOF's can be done on a single workstation ?
- Has to fit into Matlab - framwork

Outline

- Few words on FEM : assembly - stencils
- Hierarchial meshes : implicit
- System matrix as an operator
- As much as possible with Matrix operations : someone else optimizes
- Combine with multigrid : 100M DOFs with 8GB Desktop, 20M DOFs with 2GB Laptop

The problem

- Poisson problem with zero Dirichlet BC.

$$\boxed{\begin{aligned} -\Delta u &= f \quad \text{in} \quad \Omega \\ u &= 0 \quad \text{in} \quad \partial\Omega \end{aligned}}$$

- Find $u \in H_0^1$ such that

$$\boxed{(\nabla u, \nabla v) = (f, v) \quad \forall \quad v \in H_0^1}$$

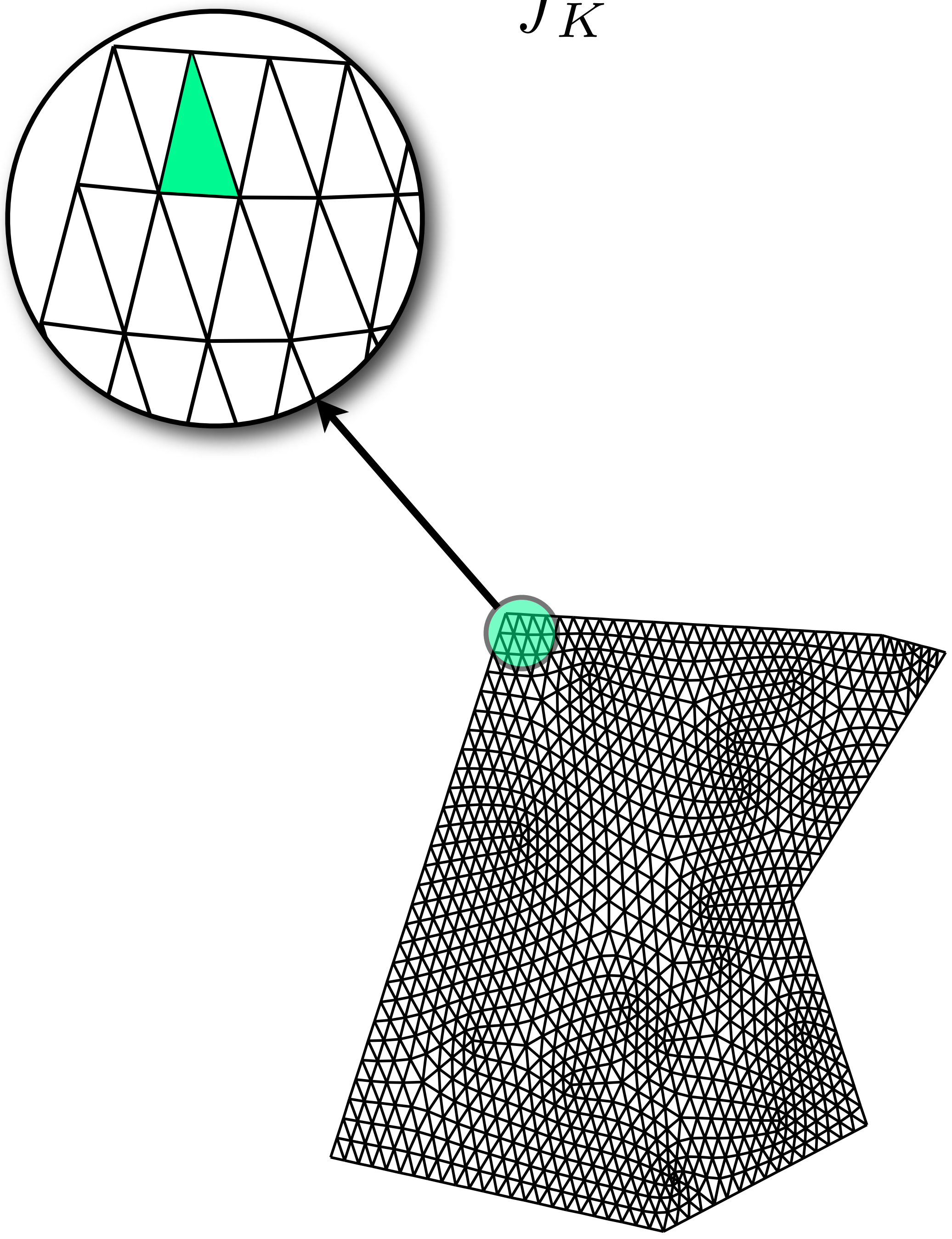
Assembly

```
for i=1:Nelements
   for p=1:Nbasis
        for k=1:Nbasis
          % Integrate numerically
         end
   end
end
```

p-fem : numerical integration
Today : Many basisfunctions

Integration

$$S_{i,j} = \int_K \nabla \phi_i \cdot \nabla \phi_j \, dx$$



Integration

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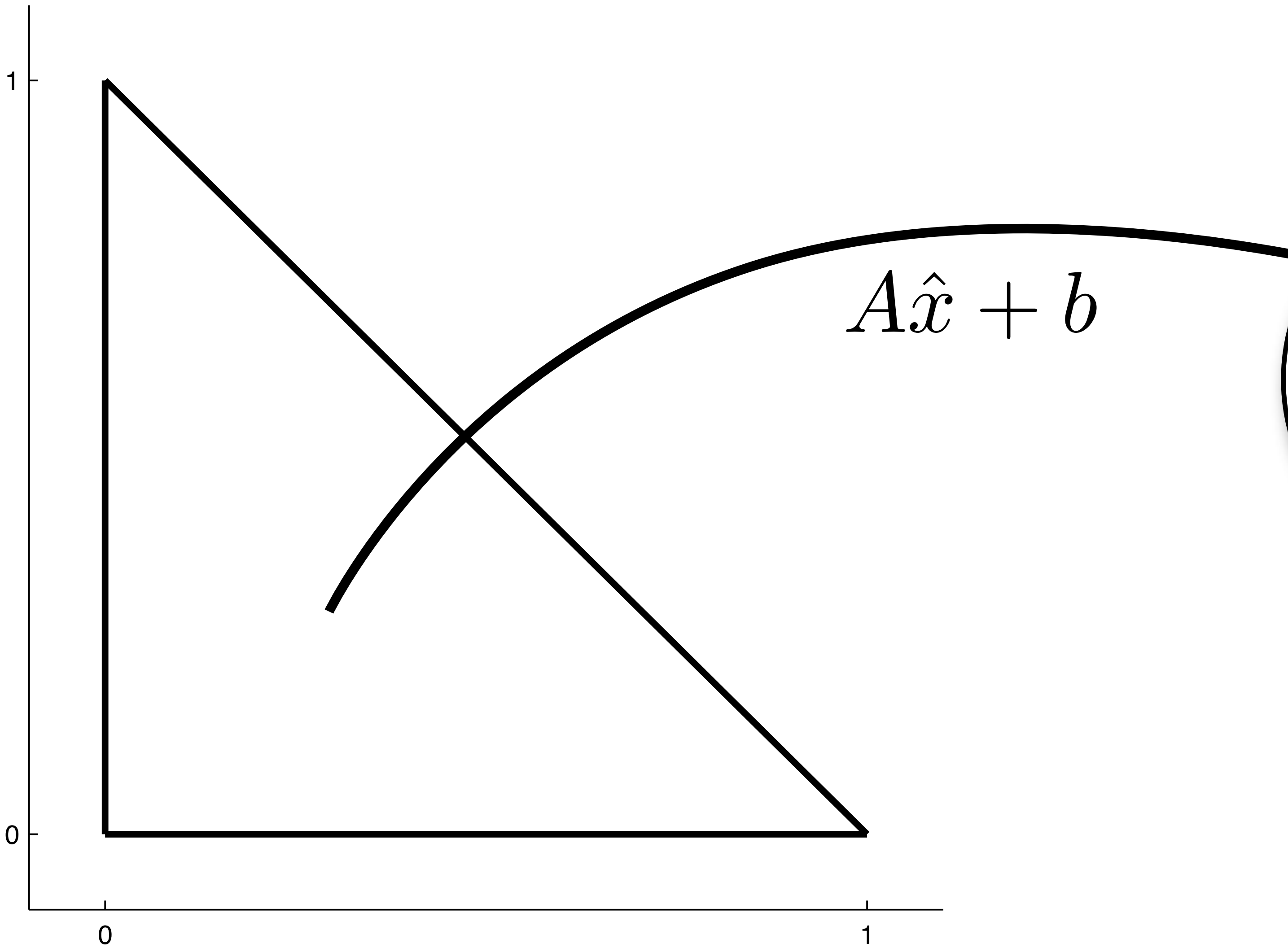

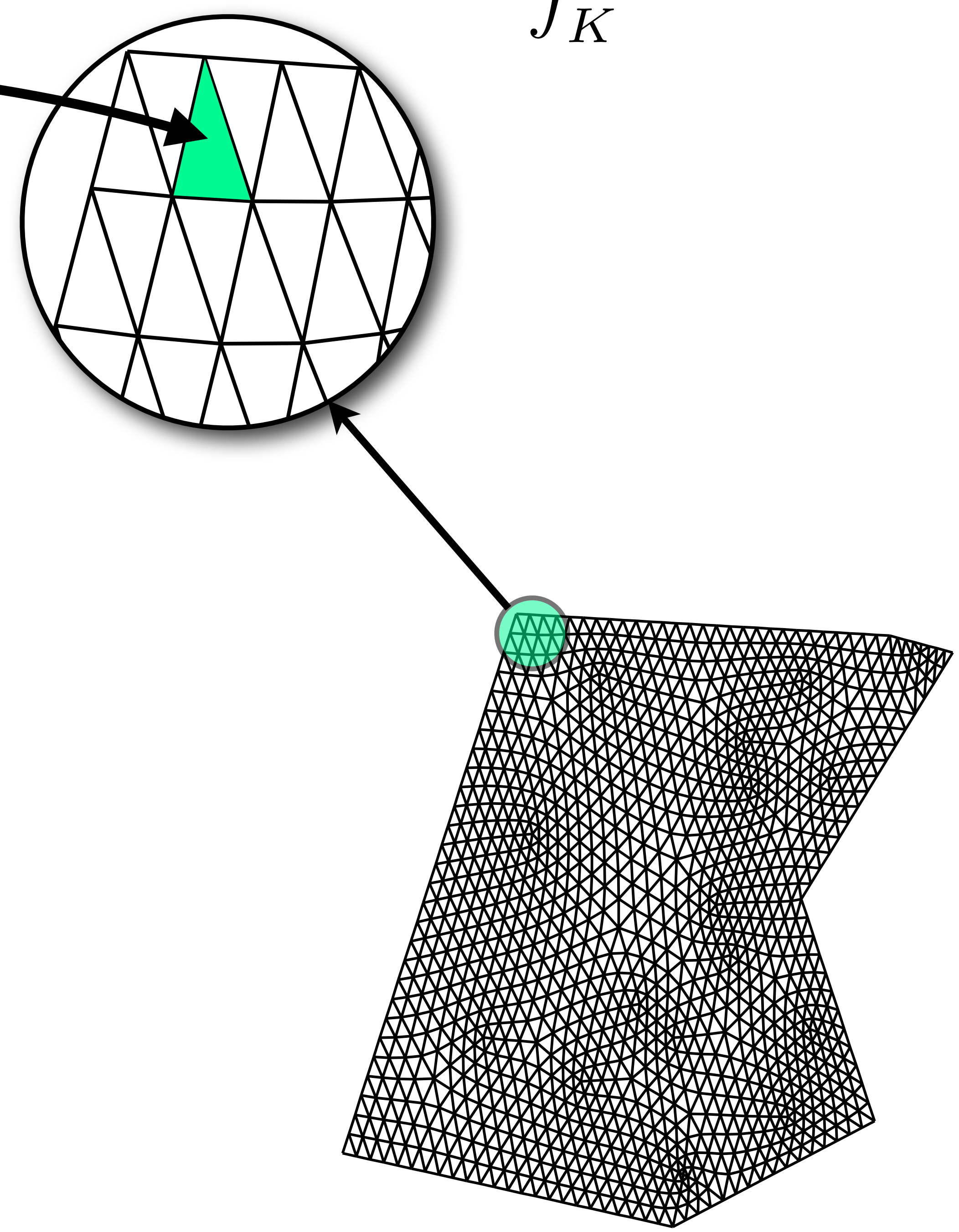

$$\int_{\hat{K}} (A^{-T} \nabla \hat{\phi}_i) \cdot (A^{-T} \nabla \hat{\phi}_j) |\det A| \; d\hat{x}$$

$$\hat{\phi}_i(\hat{x}) = \phi_i(A\hat{x} + b)$$

Stencils - Eliminate inner loops !

$$\int_{\hat{K}} (A^{-T}\nabla\hat{\phi}_i) \cdot (A^{-T}\nabla\hat{\phi}_j) |\det A| \, d\hat{x}$$

$$= \int_{\hat{K}} \nabla\hat{\phi}_i^T A^{-1} A^{-T} \nabla\hat{\phi}_j |\det A| \, d\hat{x}$$

$$= \int_{\hat{K}} \nabla\hat{\phi}_i^T B \nabla\hat{\phi}_j \, d\hat{x}$$

$$A^{-1}A^{-T}|\det A| = B = \begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{1,2} & b_{2,2} \end{bmatrix}$$

Stencils

$$\boxed{\int_{\hat{K}} \nabla \hat{\phi}_i^T B \nabla \hat{\phi}_j \, d\hat{x} = \sum_{k,l} c_{k,l} K_{k,l}}$$

$$c_{1,1} = b_{1,1} + b_{1,2}$$

$$K_{1,1} = \int_{\hat{K}} \partial_{\hat{x}_1} \hat{\phi}_i (\partial_{\hat{x}_1} \hat{\phi}_j + \partial_{\hat{x}_2} \hat{\phi}_j) \, d\hat{x}$$

- Integration : only on reference element
- Piecewise constant materials
- Can be done for Piola transformations

...Assembly done...

$$Sx = b$$

- Direct solvers : requires matrix in the memory
- Iterative solvers : requires only operator S(x)

Storage Requirements

- Solution vector
- Mesh : 3 x triangle index, nodal coords (greater than 5 x solution vector)
- Matrix : index vectors, nonzero entries (about 7 x solution vector)

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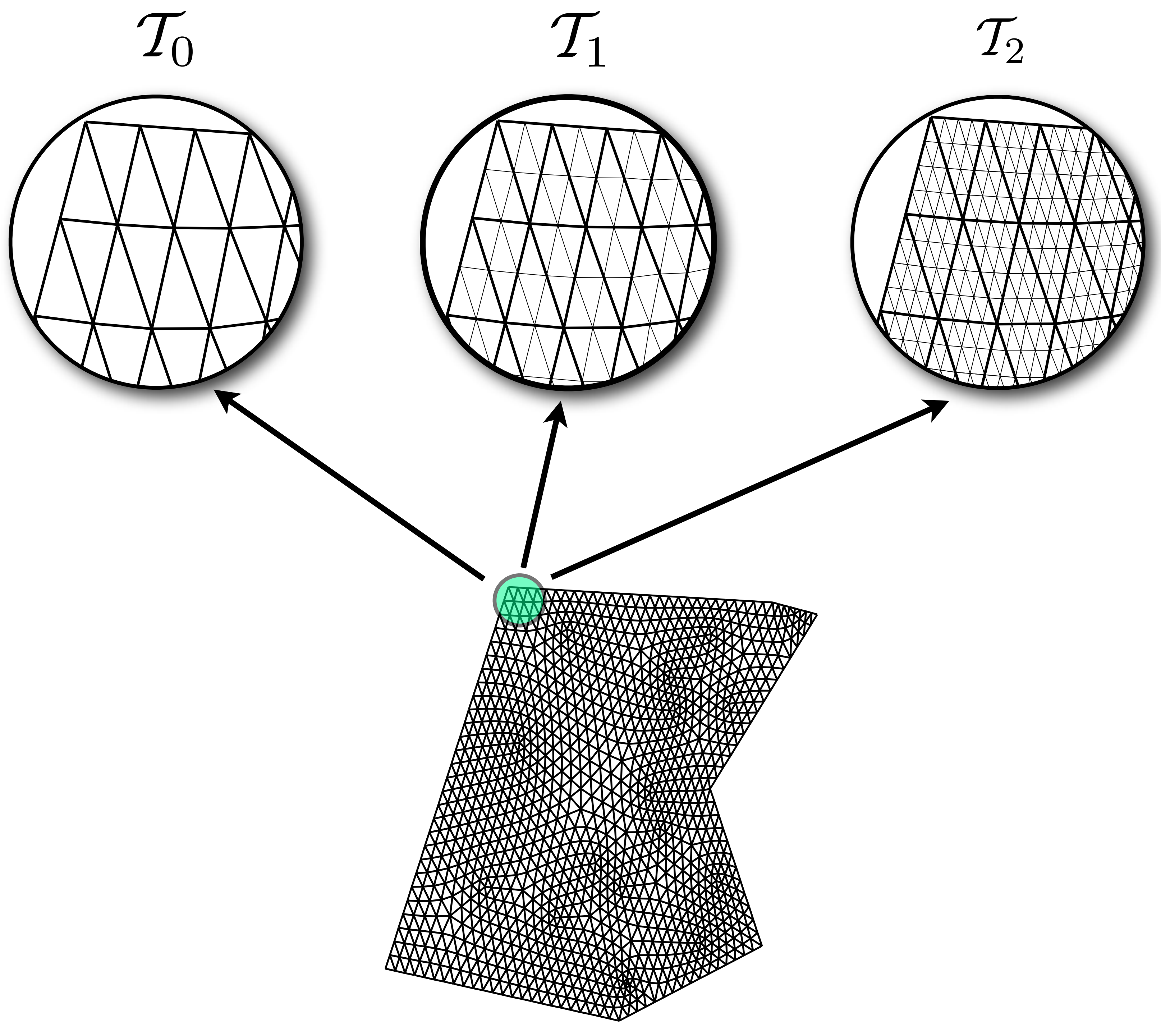
Storage Requirements

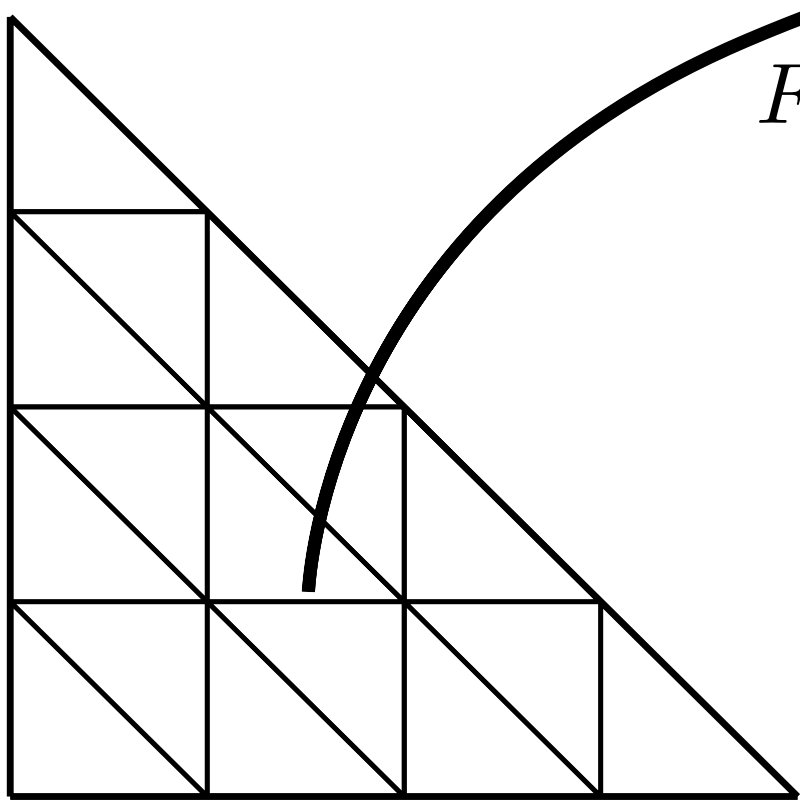
- Solution vector : CAN BE 6 times larger !
- ~~Mesh : 3 x triangle index, nodal coords (greater than 5 x solution vector)~~
- ~~Matrix : index vectors, nonzero entries (about 7 x solution vector)~~

Hierarchial mesh

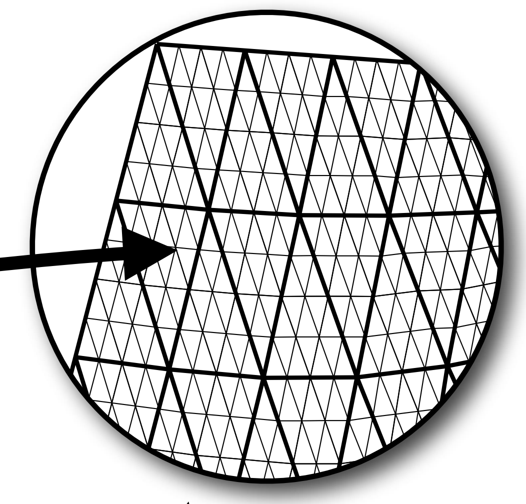


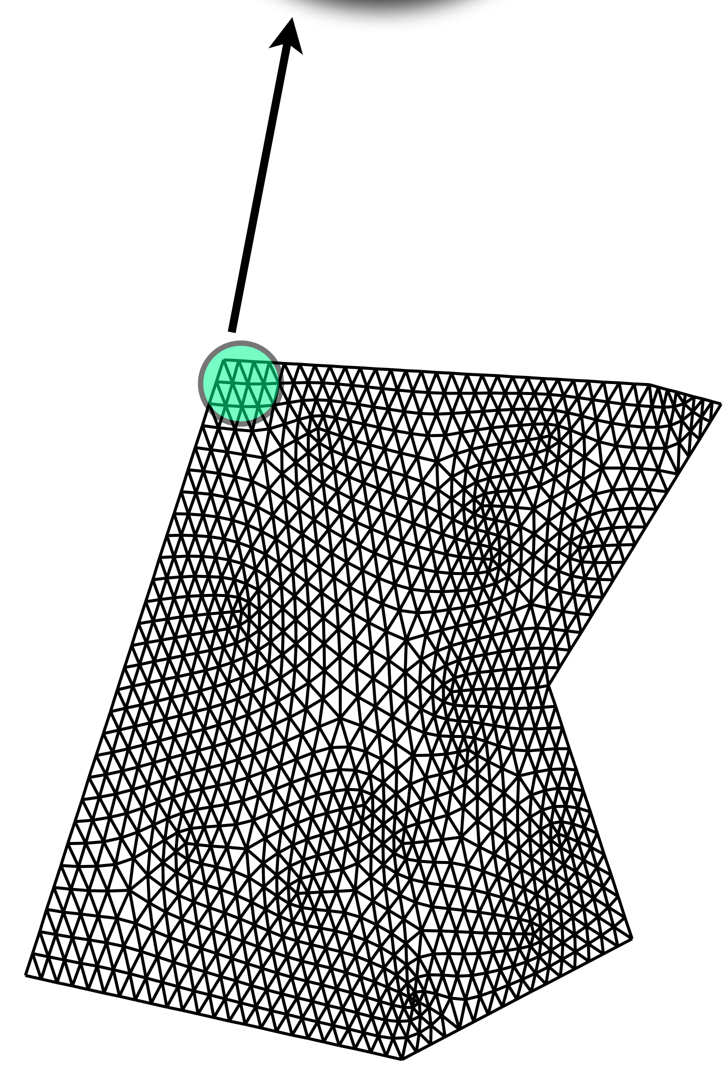
“Implicit mesh”

$$F_K(\hat{\mathcal{T}}_2) = \mathcal{T}_2 \cap K, K \in \mathcal{T}_0$$



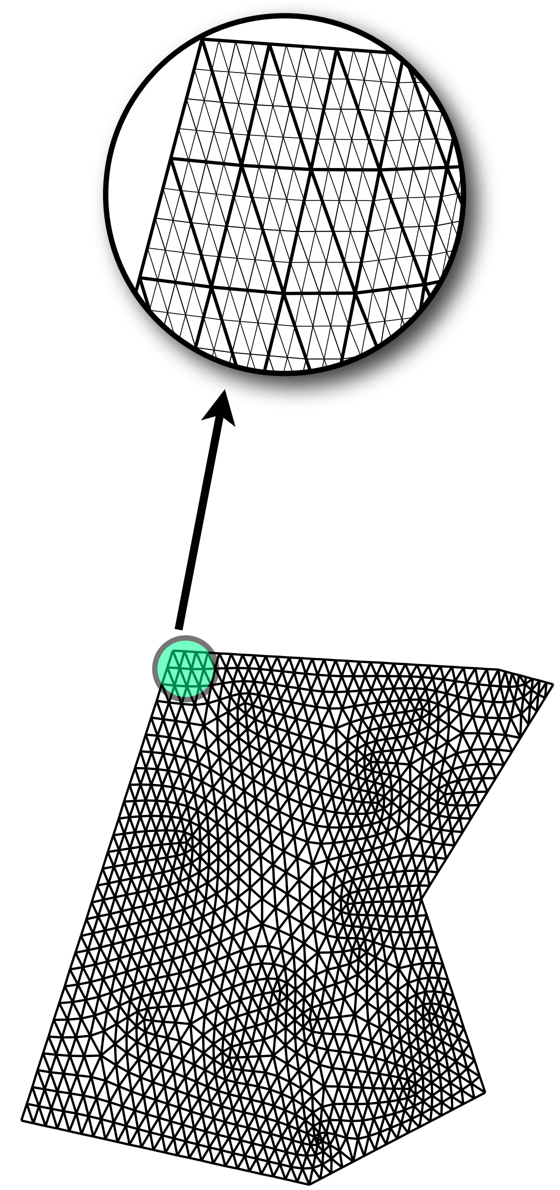
$$F_K(\hat{x}) = A_K\hat{x} + b_K$$





Assembly on "Implicit mesh"

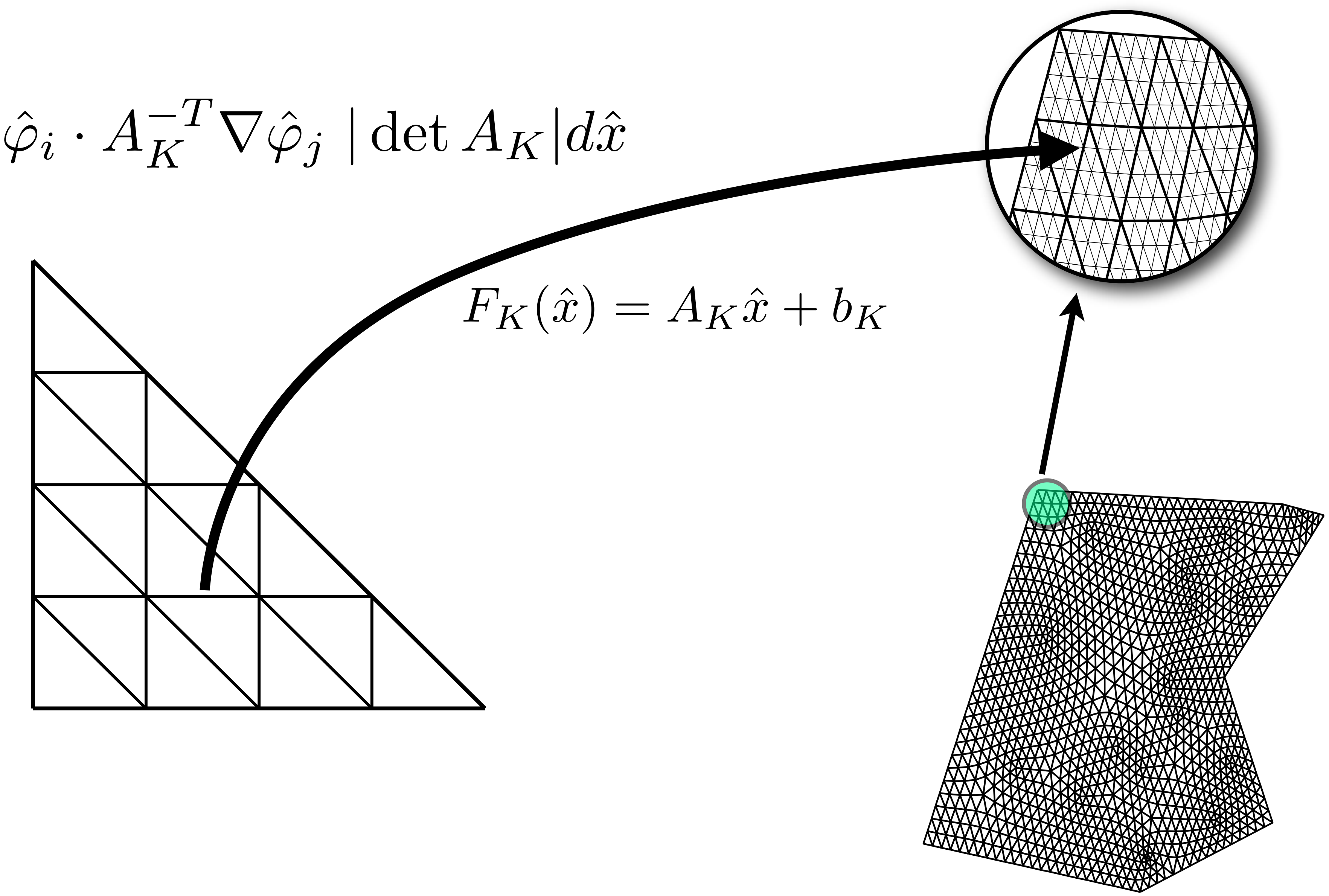
$$\int_{\mathcal{T}_2 \cap K} \nabla \varphi_i \cdot \nabla \varphi_j \, dx$$



Assembly on "Implicit mesh"

$$\int_{\mathcal{T}_2 \cap K} \nabla \varphi_i \cdot \nabla \varphi_j \, dx$$

$$\int_{\hat{\mathcal{T}}_2} A_K^{-T} \nabla \hat{\varphi}_i \cdot A_K^{-T} \nabla \hat{\varphi}_j \, |\det A_K| d\hat{x}$$



Loop over elements

Elements in the coarse mesh

```
for i=1:Nelements
  for p=1:Nbasis
      for k=1:Nbasis
         % Integrate numerically
        end
  end
end
```

Now : Several basisfunctions supported on the refined reference element

Stencils on "Implicit mesh"

On "Standard" mesh

$$\int_K \nabla \varphi_i^T \nabla \varphi_j \, dx = \sum_{i,j} C_{i,j} K_{i,j}$$

$$C_{1,1} = B_{1,1} + B_{1,2}$$

$$K_{1,1} = \int_{\hat{K}_2} \partial_{\hat{x}_1} \hat{\varphi}_i (\partial_{\hat{x}_1} \hat{\varphi}_j + \partial_{\hat{x}_2} \hat{\varphi}_j) \, d\hat{x}$$

On Hierarchial mesh

$$\int_{\mathcal{T}_2 \cap K} \nabla \varphi_i^T \nabla \varphi_j \, dx = \sum C_{i,j} K_{i,j}$$

$$K_{1,1} = \int_{\hat{\mathcal{T}}_2} \partial_{\hat{x}_1} \hat{\varphi}_i (\partial_{\hat{x}_1} \hat{\varphi}_j + \partial_{\hat{x}_2} \hat{\varphi}_j) \, d\hat{x}$$

Stencils on “Implicit mesh”

- Compute FE matrix on refined reference element = Stencil
- Same stencil can be used for each coarse mesh element.

FE - Operator

- In Iterative methods only operation

$$x \to Sx$$

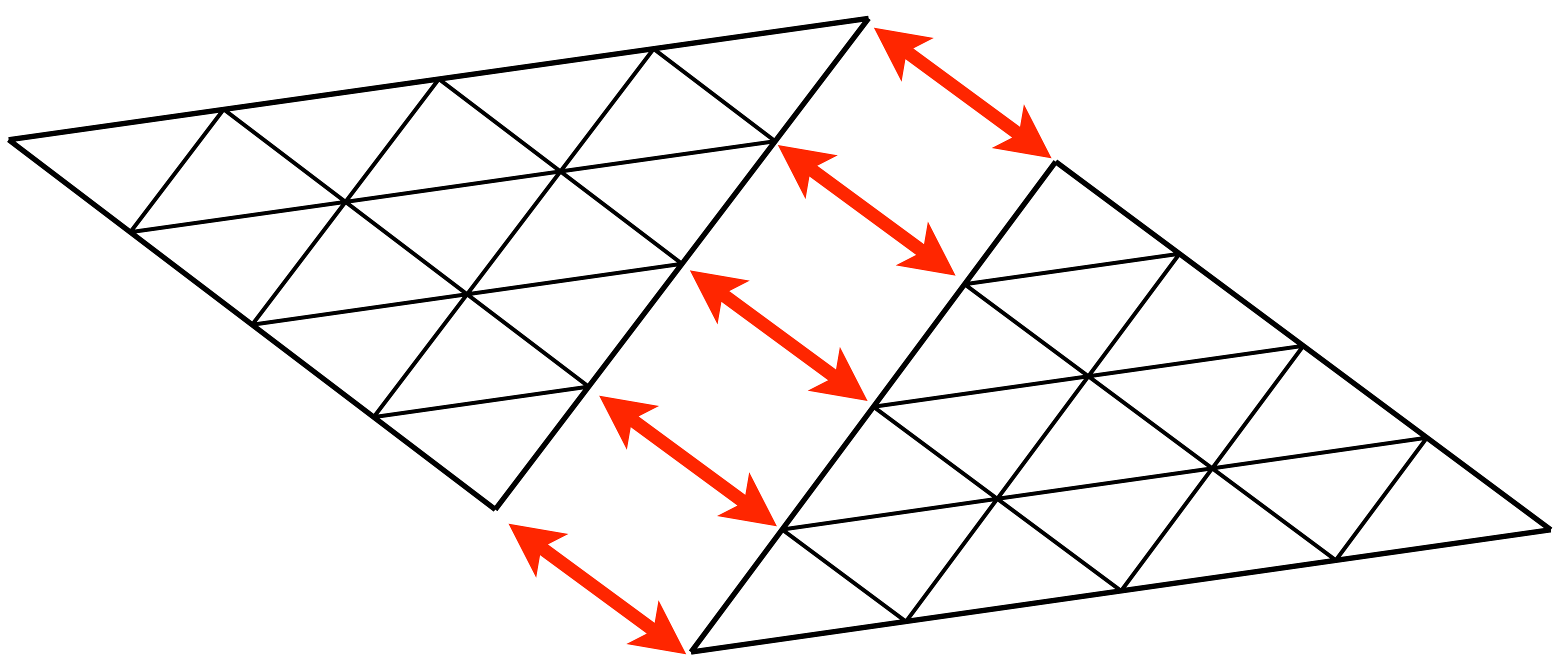
Is needed

$$Sx = \sum_{K \in \mathcal{T}_0} S_{\mathcal{T}_2 \cap K} x_{\mathcal{T}_2 \cap K}$$

$$S_{\mathcal{T}_2 \cap K} x_{\mathcal{T}_2 \cap K} = \sum C_{i,j} (K_{i,j} x_{\mathcal{T}_2 \cap K})$$

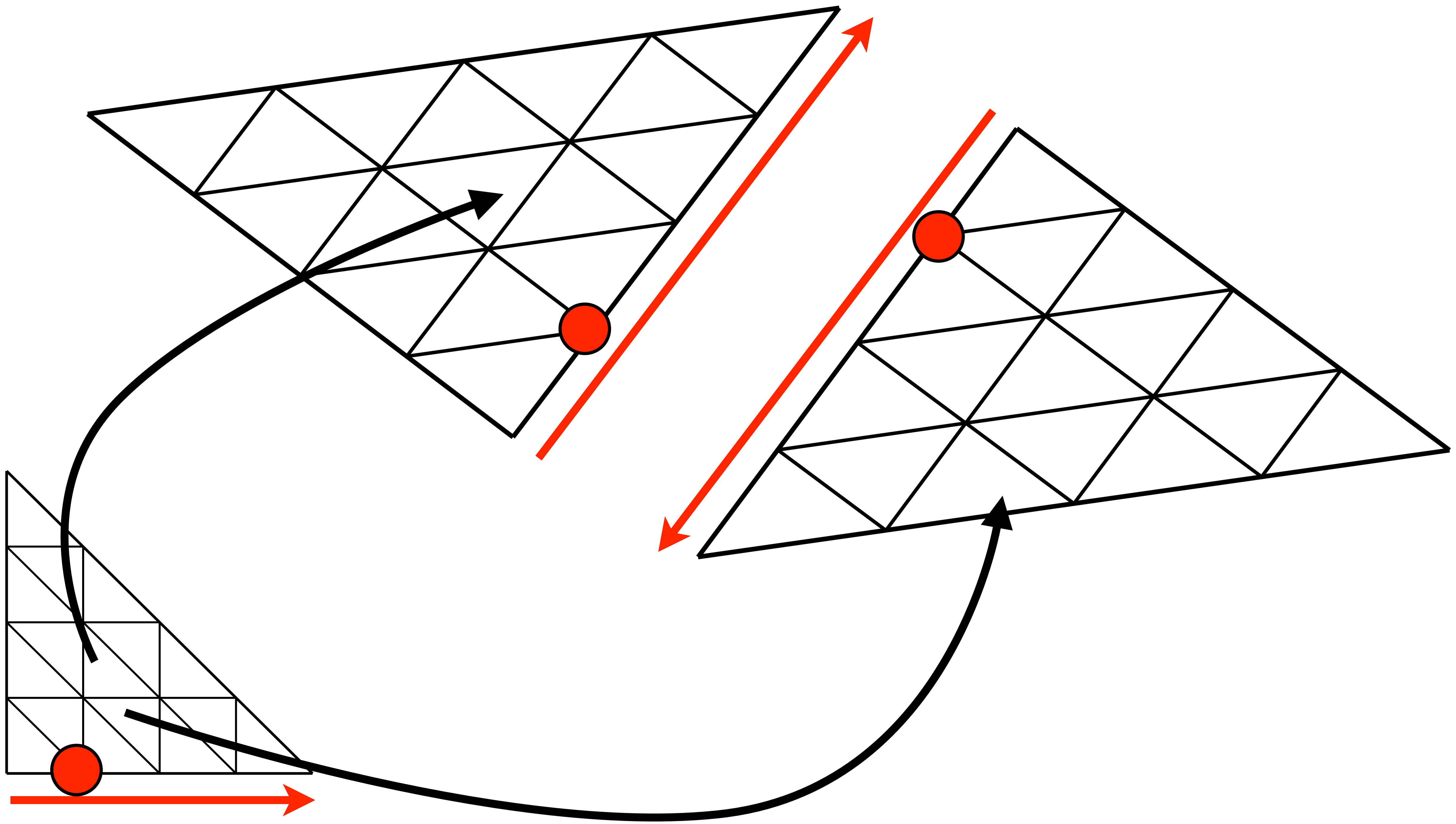
Summing up !

$$S_{\mathcal{T}_2 \cap K} = \int_{\mathcal{T}_2 \cap K} \nabla \varphi_i \cdot \nabla \varphi_j \, dx \qquad S = \sum_{K \in \mathcal{T}_0} S_{\mathcal{T}_2 \cap K}$$



Sum only over nodes on the edges of the coarse mesh

Orientation



FE - Operator

- In practice : $X \in R^{\text{nodes in } \hat{\mathcal{T}}_2 \times \text{elemens in } \mathcal{T}_0}$

$X(:, K) = x_{\mathcal{T}_2 \cap K}$

1. Compute $\tilde{X} = \sum C_{i,j}(K_{i,j} X)$

2. Sum over edge / node DOF's

$$SX, SX(:, K) = Sx_{\mathcal{T} \cap K}$$

Storage requirement ?

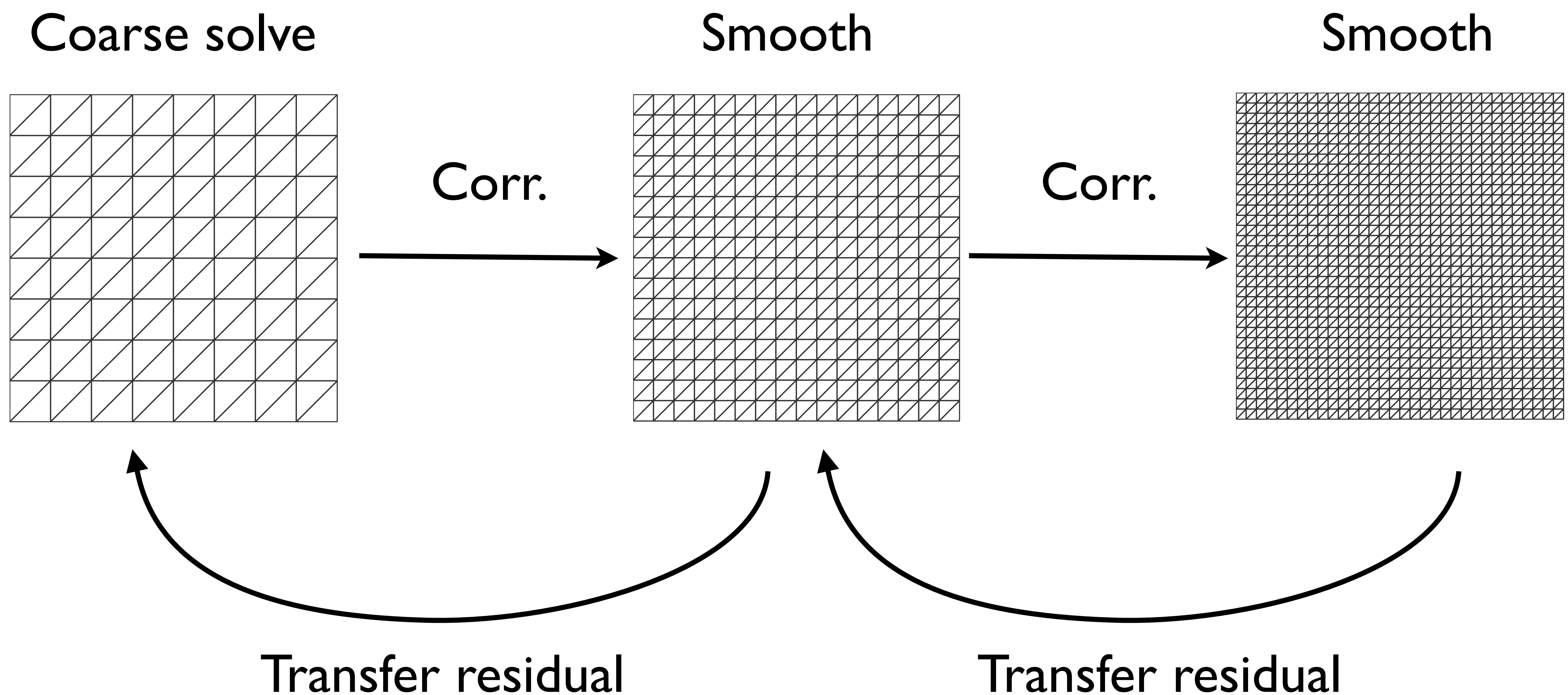
4x2^(Refinement levels)

$$\tilde{X} = \sum C_{i,j}(K_{i,j}X)$$

4xElements in $\mathcal{T}_0$

Number of total DOF'S

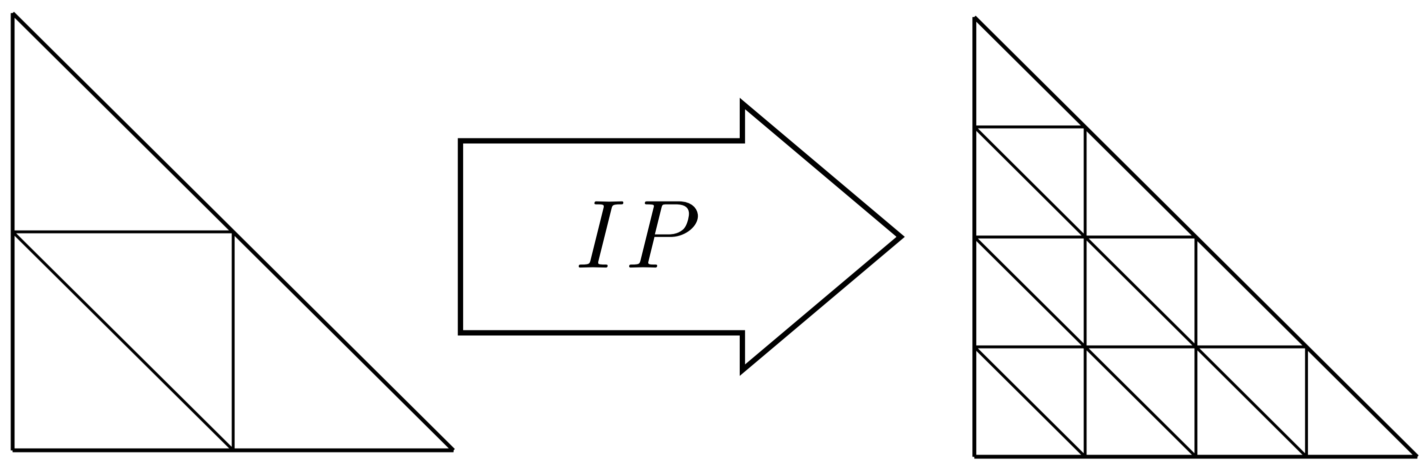
Multigrid algorithm



Interpolation

- Interpolation between levels can be done locally : interpolation operator required only on reference element !

$$\boxed{X_{\mathcal{T}_{i+1}} = IPX_{\mathcal{T}_i}}$$



Smoother

- Solves the high frequency components

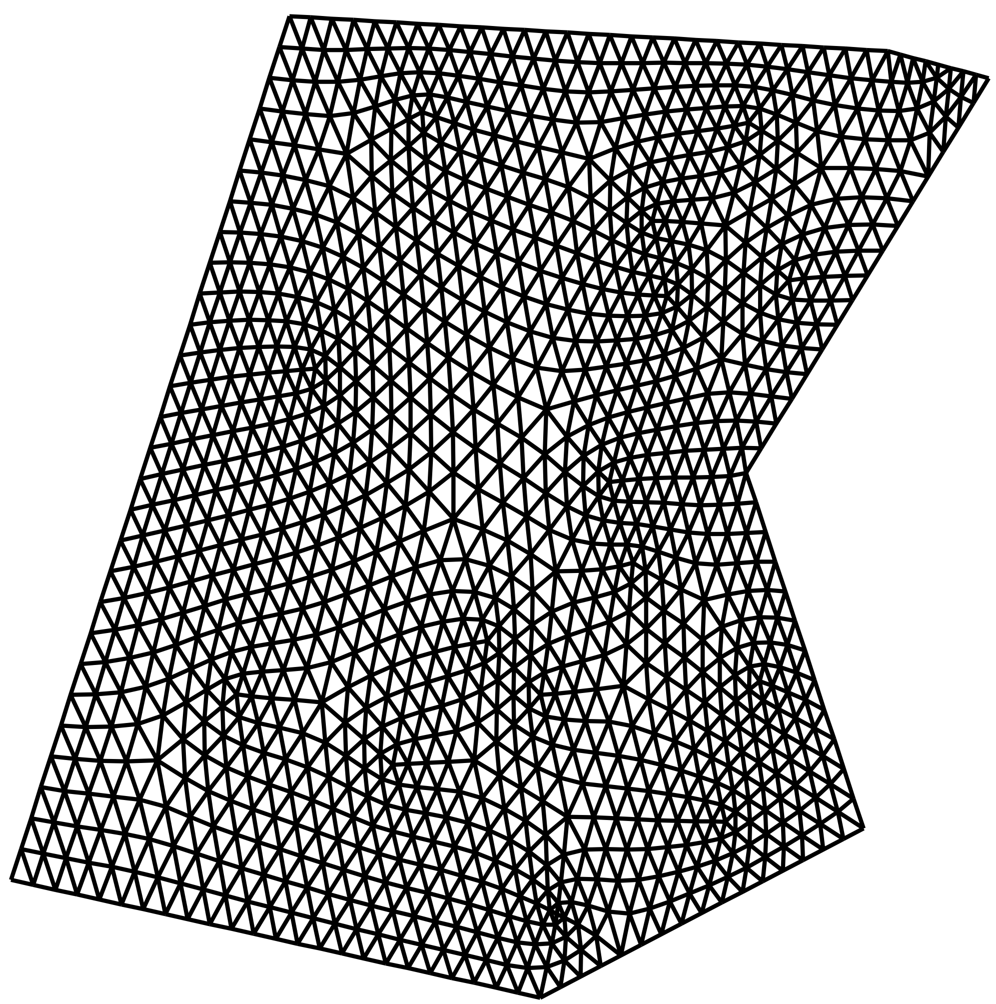
$$x = x + \frac{1}{\Lambda}(b - Ax)$$

- Does not solve low frequency components

- In mesh refinement, new high freq. components are added !

Some Results

- On laptop with 2GB of memory
- A pure Matlab implementation



Ref. Level	Dof's	V-cycle (s)
4	71697	0.432
6	1140801	1.5931
8	18227457	16.4238

Conclusions