Tailored discrete concepts for pde constrained optimization

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Topic of this talk

Optimal control of pdes with pointwise constraints:

$$(P) \qquad \min_{u \in U_{\text{ad}}, y \in Y_{\text{ad}}} J(y,u) \text{ s.t. } PDE(y) = B(u)$$

Discrete counterpart:

$$(P_h) \qquad \min_{u_h \in U_{\text{ad}}^h, y_h \in Y_{\text{ad}}^h} J_h(y_h,u_h) \text{ s.t. } PDE_h(y_h) = B_h(u_h)$$

Questions:

- **Appropriate choice of U_{ad}^h and Ansatz for u_h?**
- **Appropriate choice of Y_{ad}^h and Ansatz for y_h?**

Aim: Capture as much structure as possible of (P) on the discrete level.

Model problem, general setting

$$(P) \qquad \min_{u \in U_{ad}, y \in Y_{ad}} J(y,u) = \frac{1}{2}\int_D |y-z|^2 + \frac{\alpha}{2}\|u\|_U^2 \quad \text{subject to } y = \mathcal{G}(Bu).$$

Here, U denotes a Hilbert space, $U_{ad} \subseteq U$ closed and convex, B linear bounded control operator, which maps abstract controls to feasible rhs. Furthermore, $Y_{ad} \subseteq Y$ with state space Y and $y = \mathcal{G}(Bu)$ iff

$$Ay = Bu \text{ in } D, \text{ plus b.c. (plus i.c.)}$$

- elliptic case: $D = \Omega$ and $Ay := -\sum_{i,j=1}^d \partial_{x_j}\left(a_{ij} y_{x_i}\right) + \sum_{i=1}^d b_i y_{x_i} + cy$ uniformly elliptic operator, $Y = H^1(\Omega)$
- parabolic case: $D = (0,T] \times \Omega$ and $Ay := y_t - \sum_{i,j=1}^d \partial_{x_j}\left(a_{ij} y_{x_i}\right) + \sum_{i=1}^d b_i y_{x_i} + cy$ with strongly elliptic leading part, $Y = W(0,T)$.

Typical choices:

$$U_{ad} = \{a \leq u \leq b\} \text{ and } Y_{ad} = \{y \leq c\}, \text{ or } \{|\nabla y| \leq \delta\}.$$

Existence and uniqueness of solutions, derivatives, $Y_{ad} = \{y \le c\}$

- **For every u we have a unique $y(u) = \mathcal{G}(Bu)$. So we may minimize the reduced functional**

  $$J(u) := J(\mathcal{G}(Bu), u)$$

  instead.

- **Problem (P) admits a unique solution u with corresponding state $y = \mathcal{G}(Bu)$.**

- **$J'(u) = \alpha u + RB^* p$ with p denoting the adjoint state determined by**

  $$A^* p = y(u) - z,$$

  and $R : U^* \longrightarrow U$ denoting the Riesz isomorphism.

Optimality conditions, $Y_{ad} = \{y \leq c\}$(Casas 86,93)

Let $u \in U_{ad}$ denote the unique optimal control with associated state $y = \mathcal{G}(Bu)$ and let the

$$\text{Slater condition: } \exists \tilde{u} \in U_{ad} \text{ such that } \mathcal{G}(B\tilde{u}) < b \text{ in } \bar{D}$$

be satisfied. Then there exist $\mu \in \mathcal{M}(\bar{D})$ and some p such that there holds

$$\begin{gathered} A^* p = y - z + \mu \\ u = P_{U_{ad}} \left(-\frac{1}{\alpha} RB^* p \right), \\ \mu \geq 0, \;\; y \leq c \text{ in } D \text{ and } \int_{\bar{D}} (c - y) d\mu = 0, \end{gathered}$$

where

- elliptic case: $p \in W^{1,s}(\Omega)$ for all $s < d/(d-1)$.
- parabolic case: $p \in L^s(W^{1,\sigma})$ for all $s, \sigma \in [1, 2)$ with $2/s + d/\sigma > d + 1$.

Only control constraints, $Y_{ad} \equiv Y$

Optimality conditions:

$$
\begin{aligned}
Ay &= Bu \\
A^* p &= y - z \\
u &= P_{U_{ad}} \left(-\frac{1}{\alpha} R B^* p \right),
\end{aligned}
$$

- **The adjoint p in general is smoother than the state y.**
- **The optimal control and the adjoint state p are coupled via an algebraic relation $\Rightarrow$ a discretization of p induces a discretization of u.**
- **Reduction to p, say delivers**

$$AA^* p - B P_{U_{ad}} \left(-\frac{1}{\alpha} R B^* p \right) = Az,$$

which in the parabolic case is a bvp for p in space-time.

Only state constraints, $U_{ad} \equiv U$

Optimality conditions:

$$
\begin{aligned}
Ay &= Bu \\
A^*p &= y - z + \mu \\
u &= -\frac{1}{\alpha} RB^*p, \\
\mu \geq 0,\ y \leq c \text{ in } D &\text{ and } \int_{\bar{D}} (c - y)\, d\mu = 0,
\end{aligned}
$$

- **The state y in general is smoother than the adjoint p.**
- **The optimal control and the adjoint state p are coupled via an algebraic relation $\Rightarrow$ a discretization of p induces a discretization of u.**
- **A discretization of y ideally should deliver feasible discrete states.**

Discretization – a variational concept (H., COAP 2005)

Discrete optimal control problem:

$$\min_{u \in U_{ad}} J_h(u) := \frac{1}{2}\int_D |y_h - z|^2 + \frac{\alpha}{2}\|u\|_U^2$$
$$\text{subject to} \quad y_h = \mathcal{G}_h(Bu) \text{ and } y_h \le I_h c.$$

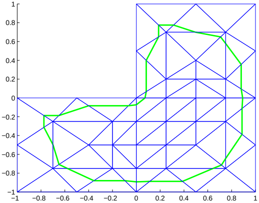
Here, $y_h(u) = \mathcal{G}_h(Bu)$ denotes the

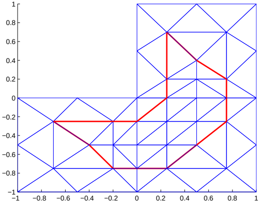
- p.l. and continuous fe approximation to $y(u)$ (elliptic case),
- $dg(0)$ in time and p.l. and continuous fe in space approximation to $y(u)$ (parabolic case), i.e.

$$a(y_h, v_h) = \langle Bu, v_h \rangle \text{ for all } v_h \in X_h.$$

We do not discretize the control!

Variational versus conventional discretization





Variational discretization for time-dependent problems

Movie time-dependent problems

Discrete optimality conditions

Let $u_h \in U_{ad}$ denote the unique variational–discrete optimal control, $y_h = \mathcal{G}(Bu_h)$. There exist $\mu \in \mathbb{R}^k$ and $p_h \in X_h$ such that with

- $\mu_h = \sum_{j=1}^{nv} \mu_j \delta_{x_j}$ (elliptic case, x_j fe nodes, $k = nv$),
- $\mu_h = \sum_{i=1}^{m} \sum_{j=1}^{nv} \mu_{ij} \delta_{x_j} \circ \frac{1}{|I_i|} \int_{I_i} \bullet dt$ (parabolic case, x_j fe nodes, I_i dg intervals, $k = nv + m$),

we have

$$\begin{aligned} a(v_h, p_h) = \int_D (y_h - z) v_h + \int_{\bar{D}} v_h d\mu_h \qquad & \forall v_h \in X_h, \\ \langle B^* p_h + \alpha u_h, v - u_h \rangle_{U^*,U} \geq 0 \qquad & \forall v \in U_{ad}, \\ \mu_j \geq 0,\ y_h \leq I_h c, \text{ and } \int_{\bar{D}} (I_h c - y_h) d\mu_h = 0. & \end{aligned}$$

Here, δ_x denotes the Dirac measure concentrated at x and I_h is the usual Lagrange interpolation operator.

Algorithms for variational discretization

Define

$$G_h(u) = P_{U_{\rm ad}}\left(-\frac{1}{\alpha}RB^* p_h(y_h(u))\right).$$

The optimality condition reads $G_h(u) = 0$ and motivates the fix–point iteration

- **u given, do until convergence**

$$u^+ = P_{U_{\rm ad}}\left(-\frac{1}{\alpha}RB^* p_h(y_h(u))\right), \quad u = u^+.$$

Two questions immediately arise.

1. **Is this algorithm numerically implementable?**
2. **Does this algorithm converge?**

Answers:
1. Yes, whenever for given u it is possible to numerically evaluate the expression

$$P_{U_{\rm ad}}\left(-\frac{1}{\alpha}RB^* p_h(y_h(u))\right)$$

in the $i - th$ iteration, with an numerical overhead which is *independent* of the iteration counter of the algorithm.

Algorithms for variational discretization, cont.

2. Yes, if $\alpha > \|RB^* S_h^* S_h B\|_{\mathcal{L}(U)}$, since $P_{U_{\mathrm{ad}}}$ is non–expansive.

Condition too restrictive $\longrightarrow$ semi–smooth Newton method applied to $G_h(u) = 0$:

- u given, solve until convergence

$$G_h'(u)u^+ = -G_h(u) + G_h'(u)u, \quad u = u^+.$$

1. This algorithm is implementable whenever the fix–point iteration is, since

$$-G_h(u) + G_h'(u)u =$$
$$= -P_{U_{\mathrm{ad}}}\left(-\frac{1}{\alpha}RB^* p_h(u)\right) - \frac{1}{\alpha}P'_{U_{\mathrm{ad}}}\left(-\frac{1}{\alpha}RB^* p_h(u)\right)RB^* S_h^* S_h Bu.$$

2. For every $\alpha > 0$ this algorithm is locally fast convergent (H. (COAP 2005), Vierling).

Results

Let $u_h \in U_{ad}$ denote the variational–discrete optimal solution with corresponding state $y_h \in X_h$ and $\mu_h \in \mathcal{M}(\bar{D})$. Then for h small enough

$$\|y_h\|,\ \|u_h\|_U,\ \|\mu_h\|_{\mathcal{M}(\bar{D})} \leq C.$$

For the proof a discrete counterpart to the Slater condition is needed, which is deduced from uniform convergence of the discrete states associated to the Slater point $B\tilde{u}$.

Results, cont.

Let u denote the solution of the continuous problem and u_h the variational discrete optimal control. Then

$$\begin{aligned}\alpha\|u-u_h\|^2+\|y-y_h\|^2 &\le \\ \le C(\|\mu\|_{\mathcal{M}(\bar{D})},\|\mu_h\|_{\mathcal{M}(\bar{D})})&\left\{\|y-y_h(u)\|_\infty+\|y^h(u_h)-y_h\|_\infty\right\}+\\ &+C(\|u\|,\|u_h\|)\left\{\|y-y_h(u)\|+\|y^h(u_h)-y_h\|\right\}.\end{aligned}$$

Here, $y_h(u)=\mathcal{G}_h(Bu), y^h(u_h)=\mathcal{G}(Bu_h)$.

We need uniform estimates for discrete approximations.

Error estimates, parabolic case

Deckelnick, H. (JCM 2010)

Controls $u \in L^2(0,T)^m$, and $f_i \in H^1(\Omega)$ given actuations.

$$Bu := \sum_{i=1}^{m} u_i(t) f_i(x), \quad y_0 \in H^2(\Omega).$$

Then $y = \mathcal{G}(Bu) \in \{v \in L^\infty(H^2), v_t \in L^2(H^1)\}$ and we have with $y_h = \mathcal{G}_h(Bu)$ and time stepping $\delta t \sim h^2$

$$\|y - y_h\|_\infty \le C \begin{cases} h\sqrt{|\log h|}, & (d = 2) \\ \sqrt{h}, & (d = 3) \end{cases}$$

This is not an *off-the-shelf* result! It yields

$$\alpha \|u - u_h\|^2 + \|y - y_h\|^2 \le C \begin{cases} h\sqrt{|\log h|}, & (d = 2) \\ \sqrt{h}, & (d = 3). \end{cases}$$

Error estimates, elliptic case

Deckelnick, H. (SINUM 2007, ENUMATH 2007)

- $Bu \in L^2(\Omega)$:

$$\|u-u_h\|_U,\ \|y-y_h\|_{H^1} = \begin{cases} O(h^{\frac{1}{2}}), & \text{if } d=2, \\ O(h^{\frac{1}{4}}), & \text{if } d=3, \end{cases}$$

- $Bu \in W^{1,s}(\Omega)$:

$$\|u-u_h\|_U,\ \|y-y_h\|_{H^1} \le Ch^{\frac{3}{2}-\frac{d}{2s}}\sqrt{|\log h|}.$$

- $Bu \in L^\infty(\Omega)$:

$$\|u-u_h\|_U,\ \|y-y_h\|_{H^1} \le Ch|\log h|.$$

- $U = L^2(\Omega), U_{ad} = \{u \le d\}$, u_h p.c.:

$$\|u-u_h\|_U,\ \|y-y_h\|_{H^1} \le C\begin{cases} h|\log h|, & \text{if } d=2, \\ \sqrt{h}, & \text{if } d=3. \end{cases}$$

Similar results obtained by C. Meyer for discrete controls.

Numerical experiment 1

$\Omega := B_1(0)$, $\alpha > 0$,

$$z(x) := 4 + \frac{1}{\pi} - \frac{1}{4\pi}|x|^2 + \frac{1}{2\pi}\log|x|,\; b(x) := |x|^2 + 4,$$

and $u_0(x) := 4 + \frac{1}{4\alpha\pi}|x|^2 - \frac{1}{2\alpha\pi}\log|x|$.

$$J(u) := \frac{1}{2}\int_\Omega |y - z|^2 + \frac{\alpha}{2}\int_\Omega |u - u_0|^2,$$

where $y = \mathcal{G}(u)$.
Unique solution $u \equiv 4$ with corresponding state $y \equiv 4$ and multipliers

$$p(x) = \frac{1}{4\pi}|x|^2 - \frac{1}{2\pi}\log|x| \quad \text{and} \quad \mu = \delta_0.$$

Experimental order of convergence

RL	$\|\|u - u_h\|\|$	$\|\|y - y_h\|\|$
1	0.788985	0.536461
2	0.759556	1.147861
3	0.919917	1.389378
4	0.966078	1.518381
5	0.986686	1.598421

Thank you very much for coming, and for your attention

Relaxing constraints – Lavrentiev (H., Meyer COAP 2008)

Lavrentiev Regularization: relax $y \le c$ to $\lambda u + y \le c$ $(\lambda > 0)$. Numerical analysis yields

- $Bu^\lambda \in L^2(\Omega)$ **uniformly:**

$$\|u - u_h^\lambda\| \sim \|u - u^\lambda\| + \|u^\lambda - u_h^\lambda\| \sim \sqrt{\lambda} + h^{1-d/4},$$

- $Bu^\lambda \in W^{1,s}(\Omega)$ **uniformly for all** $s \in (1, \frac{d}{d-1})$:

$$\|u - u_h^\lambda\| \sim \|u - u^\lambda\| + \|u^\lambda - u_h^\lambda\| \sim \sqrt{\lambda} + h^{2-d/2-\epsilon},$$

- $Bu^\lambda \in L^\infty(\Omega), Bu_h^\lambda \in L^\infty(\Omega)$ **uniformly:**

$$\|u - u_h^\lambda\| \sim \|u - u^\lambda\| + \|u^\lambda - u_h^\lambda\| \sim \sqrt{\lambda} + h|\log h|.$$

Relaxing constraints – penalization (Hintermüller, H., SINUM 2009)

Relax $y \le c$ with $\frac{\gamma}{2}\int_\Omega |(y-c)^+|^2 dx$ in cost functional.

- $Bu^\gamma \in L^2(\Omega)$ **uniformly:**

$$\|u - u_h^\gamma\| \sim \|u - u^\gamma\| + \|u^\gamma - u_h^\gamma\| \sim$$

$$\sim \left(h^{1-d/p} + \frac{1}{\sqrt{\gamma}} h^{-d/2}\right)^{1/2} + h^{1-d/4},$$

- $Bu^\gamma \in W^{1,s}(\Omega)$ **for all** $s \in (1, \frac{d}{d-1})$ **uniformly:**

$$\|u - u_h^\gamma\| \sim \|u - u^\gamma\| + \|u^\gamma - u_h^\gamma\| \sim$$

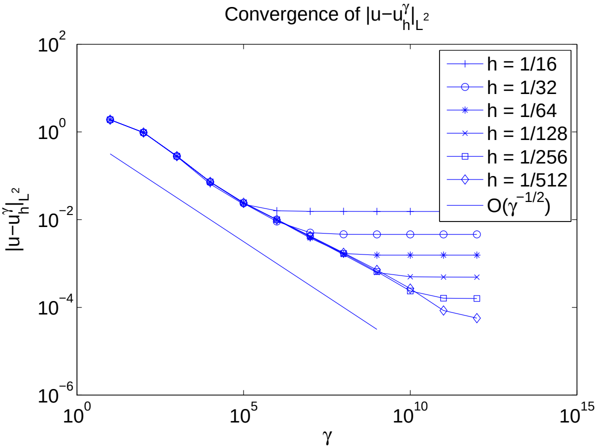
$$\sim \left(h^{1-d/p} + \frac{1}{\sqrt{\gamma}} h^{-d/2}\right)^{1/2} + h^{2-d/2-\epsilon},$$

- $Bu^\gamma \in L^\infty(\Omega), Bu_h^\gamma \in L^\infty(\Omega)$ **uniformly:**

$$\|u - u_h^\gamma\| \sim \|u - u^\gamma\| + \|u^\gamma - u_h^\gamma\| \sim$$

$$\sim \left(h^{1-d/p} + \frac{1}{\sqrt{\gamma}} h^{-d/2}\right)^{1/2} + h|\log h|.$$

Relaxing constraints – penalization, numerical results



Relaxing constraints – barriers (H., Schiela, COAP 2009)

Barriers: relax $y \le c$ by adding $-\mu \int_\Omega \log(c-y)dx$ to cost functional ($\mu > 0$). Numerical analysis yields

- **$Bu^\mu \in L^2(\Omega)$ uniformly:**

$$\|u - u_h^\mu\| \sim \|u - u^\mu\| + \|u^\mu - u_h^\mu\| \sim \sqrt{\mu} + h^{1-d/4},$$

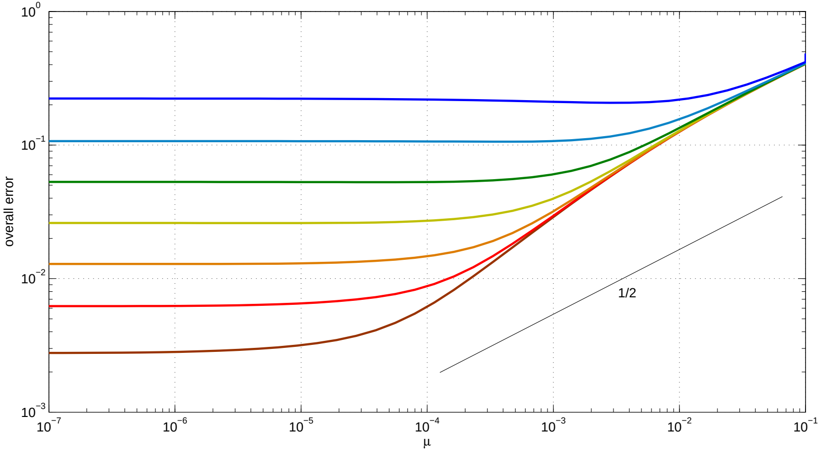
- **$Bu^\mu \in W^{1,s}(\Omega)$ for all $s \in (1, \frac{d}{d-1})$ uniformly:**

$$\|u - u_h^\mu\| \sim \|u - u^\mu\| + \|u^\mu - u_h^\mu\| \sim \sqrt{\mu} + h^{2-d/2-\epsilon},$$

- **$Bu^\mu \in L^\infty(\Omega), Bu_h^\mu \in L^\infty(\Omega)$ uniformly:**

$$\|u - u_h^\mu\| \sim \|u - u^\mu\| + \|u^\mu - u_h^\mu\| \sim \sqrt{\mu} + h|\log h|.$$

Relaxing constraints – barriers, numerical results



Consequence: Grid size h and parameters (λ, γ, μ) should be coupled;

Lavrentiev: $\sqrt{\lambda} \sim h^{2-d/2}$,

Barriers: $\sqrt{\mu} \sim h^{2-d/2}$,

Penalization ($p = \infty$): $\frac{1}{\sqrt{\gamma}} \sim h^{1+d/2}$ **(optimal ?).**

Constraints on the gradient

Consider

$$\min_{u \in U_{\mathrm{ad}}} J(u) = \frac{1}{2}\int_\Omega |y - z|^2 + \frac{\alpha}{r}\int_\Omega |u|^r \left(+ \frac{\alpha}{2}\int_\Omega |u|^2 \right)$$

where $y = \mathcal{G}(u)$, **i.e. solves the pde, and** $\nabla y \in Y_{\mathrm{ad}}$.

Here

$$Y_{\mathrm{ad}} = \{z \in C^0(\bar{\Omega})^d \mid |z(x)| \leq \delta, x \in \bar{\Omega}\},$$

and

$$\begin{array}{ll} r = 2: & U_{\mathrm{ad}} = \{u \in L^2(\Omega) \mid a \leq u \leq b \text{ a.e. in } \Omega\}(a, b \in L^\infty), \\ r > d: & U_{\mathrm{ad}} = L^r(\Omega). \end{array}$$

Then $U_{\mathrm{ad}} \subset L^r(\Omega)$ **for** $r > d \Rightarrow \nabla y \in C^0(\bar{\Omega})^d$.

Slater condition:

$$\exists \hat{u} \in U_{\mathrm{ad}} \, |\nabla \hat{y}(x)| < \delta, \; x \in \bar{\Omega}, \text{ where } \hat{y} \text{ solves the pde with } u = \hat{u}.$$

Optimality conditions (Casas & Fernandez)

An element $u \in U_{\mathrm{ad}}$ is a solution if and only if there exist $\vec{\mu} \in \mathcal{M}(\bar{\Omega})^d$ and $p \in L^t(\Omega)$ $(t < \frac{d}{d-1})$ such that

$$\begin{array}{rll} \int_\Omega p\mathcal{A}z - \int_\Omega (y-z)z & = \int_{\bar{\Omega}} \nabla z \cdot d\vec{\mu} & \forall z \in W^{2,t'}(\Omega) \cap W_0^{1,t'}(\Omega) \\ \int_{\bar{\Omega}} (z - \nabla y) \cdot d\vec{\mu} & \leq 0 & \forall z \in Y_{\mathrm{ad}}, \\ \int_\Omega (p + \alpha u)(\tilde{u} - u) & \geq 0 & \forall \tilde{u} \in U_{\mathrm{ad}} \text{ for } r = 2, \text{ or} \\ p + \alpha((u+)|u|^{r-2}u) & = 0 & \text{in } \Omega \text{ for } r > d. \end{array}$$

Structure of multiplier: $\vec{\mu} = \frac{1}{\delta} \nabla y\, \mu$, where $\mu \in \mathcal{M}(\bar{\Omega}) \geq 0$ is concentrated on $\{x \in \bar{\Omega} \,|\, |\nabla y(x)| = \delta\}$.

FE discretization, conventional

Piecewise linear, continuous Ansatz for the state $y_h = \mathcal{G}_h(u) \in X_h$.

The discrete control problem reads

$$\min_{u \in U_{\text{ad}}} J_h(u) := \frac{1}{2}\int_\Omega |y_h - z|^2 + \frac{\alpha}{r}\int_\Omega |u|^r \left(+ \frac{\alpha}{2}\int_\Omega |u|^2 \right)$$
$$\text{subject to} \quad y_h = \mathcal{G}_h(u) \text{ and } \left(\frac{1}{|T|}\int_T \nabla y_h \right)_{T \in \mathcal{T}_h} \in Y_{\text{ad}}^h,$$

where

$$Y_{\text{ad}}^h := \{ \mathrm{c_h} : \bar{\Omega} \longrightarrow \mathbb{R}^d \mid \mathrm{c_h}_{|T} \text{ is constant and } |\mathrm{c_h}_{|T}| \le \delta,\ T \in \mathcal{T}_h \}.$$

FE discretization, conventional, optimality conditions

The variational discrete problem has a unique solution $u_h \in U_{\mathrm{ad}}$. There exist $\mu_T \in \mathbb{R}^d$, $T \in \mathcal{T}_{h,X}$ and $p_h \in X_h$ such that with $y_h = \mathcal{G}_h(u_h)$ we have

$$a(v_h, p_h) = \int_\Omega (y_h - z)v_h + \sum_{T\in\mathcal{T}_{h,X}} |T|\,\nabla v_{h|T} \cdot \mu_T \qquad \forall v_h \in X_h,$$

$$\sum_{T\in\mathcal{T}_{h,X}} |T|\big(c_{h_T} - \nabla y_{h|T}\big) \cdot \mu_T \le 0 \quad \forall c_h \in C_h,$$

$$p_h + \alpha((u_h+)|u_h|^{r-2}u_h) = 0 \quad \text{in } \Omega.$$

Structure of the multiplier: $\vec{\mu}_T = \mu_T \, \frac{1}{\delta} \nabla y_{h_T}$, where $\mu_T \in \mathbb{R}$. Furthermore, $\mu_T \ge 0$ and $\mu_T > 0$ only if $|\nabla y_{h_T}| = \delta$.

Results

Deckelnick, Günther, H. (Oberwolfach Report 2008): Let $u_h \in U_{ad}$ be the variational discrete optimal solution with corresponding state $y_h \in X_h$ and adjoint variables $p_h \in X_h$, $\vec{\mu}_T (T \in \mathcal{T}_h)$.

Then for h small enough

- $\|y_h\|$, $\|u_h\|_{L^r}$, $\|p_h\|_{L^{\frac{r}{r-1}}}$, $\sum_{T \in \mathcal{T}_{h,X}} |T|\,|\mu_T| \le C$,
- $\|y - y_h\| \le C h^{\frac{1}{2}(1-\frac{d}{r})}$, $\|u - u_h\|_{L^r} \le C h^{\frac{1}{r}(1-\frac{d}{r})}$, **and** $\|u - u_h\|_{L^2} \le C h^{\frac{1}{2}(1-\frac{d}{r})}$.

These results are also valid for a piecewise constant Ansatz of the control.

FE discretization, Raviart Thomas

Mixed fe approximation of the state with lowest order Raviart–Thomas element, i.e.

$$(y_h, \mathbf{v_h}) = \mathcal{G}_h(u) \in Y_h \times \mathbf{V_h}$$

denotes the solution of

$$\begin{array}{rcll} \int_\Omega A^{-1}\mathbf{v_h} \cdot \mathbf{w_h} + \int_\Omega y_h \,\mathrm{div}\mathbf{w_h} & = & 0 & \forall \mathbf{w_h} \in \mathbf{V_h} \\ \int_\Omega z_h \,\mathrm{div}\mathbf{v_h} - \int_\Omega a_0 y_h\, z_h + \int_\Omega u\, z_h & = & 0 & \forall z_h \in Y_h. \end{array}$$

FE discretization, cont.

The discrete control problem reads

$$\min_{u \in U_{\text{ad}}} J_h(u) := \frac{1}{2}\int_\Omega |y_h - z|^2 + \frac{\alpha}{2}\int_\Omega |u|^2$$
$$\text{subject to } \ (y_h, \mathrm{v_h}) = \mathcal{G}_h(u) \text{ and } \left(\frac{1}{|T|}\int_T A^{-1}\mathrm{v_h}\right)_{T \in \mathcal{T}_h} \in Y^h_{\text{ad}},$$

where

$$Y^h_{\text{ad}} := \{\mathrm{c_h} : \bar{\Omega} \longrightarrow \mathbb{R}^d \mid \mathrm{c_h}_{|T} \text{ is constant and } |\mathrm{c_h}_{|T}| \le \delta,\ T \in \mathcal{T}_h\}.$$

FE discretization, optimality conditions

The discrete problem has a unique solution $u_h \in U_{\mathrm{ad}}$. Furthermore, there are $\vec{\mu}_T \in \mathbb{R}^d$ and $(p_h, \chi_h) \in Y_h \times \mathrm{V_h}$ such that with $(y_h, \mathrm{v_h}) = \mathcal{G}_h(u_h)$ we have

$$\begin{aligned}
\int_\Omega A^{-1}\chi_h \cdot \mathrm{w_h} + \int_\Omega p_h \,\mathrm{div}\mathrm{w_h} + \sum_{T\in\mathcal{T}_h} \vec{\mu}_T \cdot ⨍_T A^{-1}\mathrm{w_h} &= 0 \forall \mathrm{w_h} \in \mathrm{V_h} \\
\int_\Omega z_h \,\mathrm{div}\chi_h - \int_\Omega a_0 p_h\, z_h + \int_\Omega (y_h - z)\, z_h &= 0 \forall z_h \in Y_h. \\
\int_\Omega (p_h + \alpha u_h)(\tilde{u} - u_h) &\geq 0 \forall \tilde{u} \in U_{\mathrm{ad}} \\
\sum_{T\in\mathcal{T}_h} \vec{\mu}_T \cdot \left(\mathrm{c}_{\mathrm{h}|T} - ⨍_T A^{-1}\mathrm{v_h}\right) &\leq 0 \,\forall \mathrm{c_h} \in Y^h_{\mathrm{ad}}.
\end{aligned}$$

Structure of the multiplier: $\vec{\mu}_T = \mu_T \, \frac{1}{\delta} ⨍_T A^{-1}\mathrm{v_h}$, where $\mu_T \in \mathbb{R}$. Furthermore, $\mu_T \geq 0$ and $\mu_T > 0$ only if $\left|⨍_T A^{-1}\mathrm{v_h}\right| = \delta$.

Results

Deckelnick, Günther, H. (Numer. Math 2008): Let $u_h \in U_{\mathrm{ad}}$ be the optimal solution of the discrete problem with corresponding state $(y_h, \mathrm{v_h}) \in Y_h \times \mathrm{V_h}$ and adjoint variables $(p_h, \chi_\mathrm{h}) \in Y_h \times \mathrm{V_h}$, $\vec{\mu}_T$, $T \in \mathcal{T}_h$.

Then for h small enough

- $\|y_h\|$, $\sum_{T\in\mathcal{T}_h} |\vec{\mu}_T| \le C$, **and**
- $\|u - u_h\| + \|y - y_h\| \le C h^{\frac{1}{2}} |\log h|^{\frac{1}{2}}$.

Constraints on the gradient, example

We take $\Omega = B_2(0)$ and consider

$$\min J(u) = \frac{1}{2}\|y - z\|^2_{L^2(\Omega)} + \frac{1}{2}\|u\|^2_{L^2(\Omega)}$$

with pointwise bounds on the constraints, i.e. $\{a \leq u \leq b\}$, where $a, b \in L^\infty(\Omega)$, and pointwise bounds on the gradient, i.e. $|\nabla y(x)| \leq \delta := 1/2$. State and control satisfy

$$-\Delta y = f + u \text{ in } \Omega, \quad y = 0 \text{ on } \partial\Omega.$$

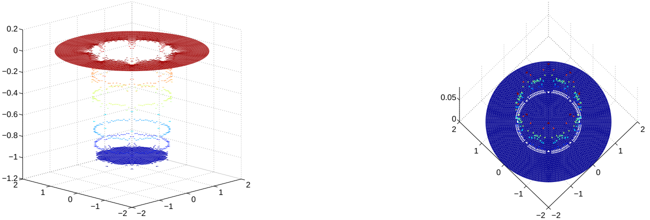
Data:

$$z(x) := \begin{cases} \frac{1}{4} + \frac{1}{2}\ln 2 - \frac{1}{4}|x|^2 & , 0 \leq |x| \leq 1 \\ \frac{1}{2}\ln 2 - \frac{1}{2}\ln|x| & , 1 < |x| \leq 2 \end{cases} \qquad f(x) := \begin{cases} 2 & , 0 \leq |x| \leq 1 \\ 0 & , 1 < |x| \leq 2 \end{cases}$$

Solution:

$$y(x) \equiv z(x) \text{ and } u(x) = \begin{cases} -1 & , 0 \leq |x| \leq 1 \\ 0 & , 1 < |x| \leq 2 \end{cases}$$

Numerical experiment, piecewise constant control Ansatz

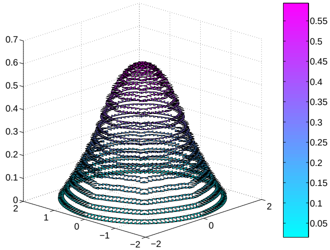


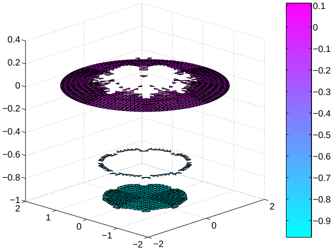
Experimental order of convergence

RL	$\|\|u - u_h\|\|_{L^4}$	$\|\|u - u_h\|\|$	$\|\|y - y_h\|\|$
1	0.76678	0.72339	1.90217
2	0.33044	0.64248	1.25741
3	0.27542	0.54054	1.23233
4	0.28570	0.53442	1.16576

Results show the predicted behaviour, since $r = \infty$.

Numerical solution, mixed finite elements





Experimental order of convergence

RL	$\|\|u - u_h\|\|$	$\|\|y - y_h\|\|$	$\|\|y^P - y_h^P\|\|$
1	0.98576	1.06726	1.08949
2	0.51814	1.02547	1.09918
3	0.50034	1.01442	1.08141

Superscript P denotes post–processed piecewise linear state. It attains the same order of convergence but yields significantly smaller approximation error.

Thank you very much for your attention

Goal oriented adaptivity

Let $a_{ij} = \delta_{ij}, b_i = 0$, and $c = 1$. Further let

$$U_{ad} = \{c \le u \le d\}, \quad Y_{ad} = \{a \le y \le b\},$$

and, for example,

$$J(y,u) = \frac{1}{2}\int_\Omega |y-z|^2 + \frac{\alpha}{2}\|u-u_0\|_U^2,$$

$$J_h(y_h,u_h) = \frac{1}{2}\int_\Omega |y_h-z|^2 + \frac{\alpha}{2}\|u_h-u_0\|_U^2.$$

Aim: Extend DWR method of Becker, Kapp, Rannacher,...) to construct ideal meshes w.r.t. the error $J(y_h,u_h) - J(y,u)$.

Goal oriented adaptivity, error representation

Let

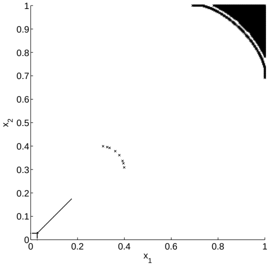
$$\begin{aligned}\rho^p(\cdot) &:= \quad J_y(y_h,u_h)(\cdot) - a(\cdot,p_h) + \langle \mu_h,\cdot\rangle,\\ \rho^u(\cdot) &:= \quad J_u(y_h,u_h)(\cdot) - (\cdot,p_h) \text{ and}\\ \rho^y(\cdot) &:= \quad -a(y_h,\cdot) + (u_h,\cdot).\end{aligned}$$

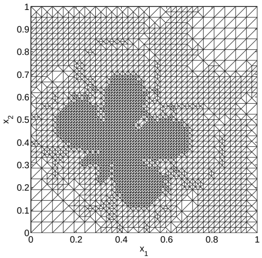
Then for $b = I_h b$ (Günther, H. (J. Numer. Math. 2008))

$$\begin{aligned}J(y,u) - J(y_h,u_h) = \frac{1}{2}\rho^p(y - i_h y) + \frac{1}{2}\rho^y(p - i_h p) + \\ + \frac{1}{2}\{\langle \mu + \mu_h, y_h - y\rangle + \langle \lambda + \lambda_h, u_h - u\rangle\}\end{aligned}$$

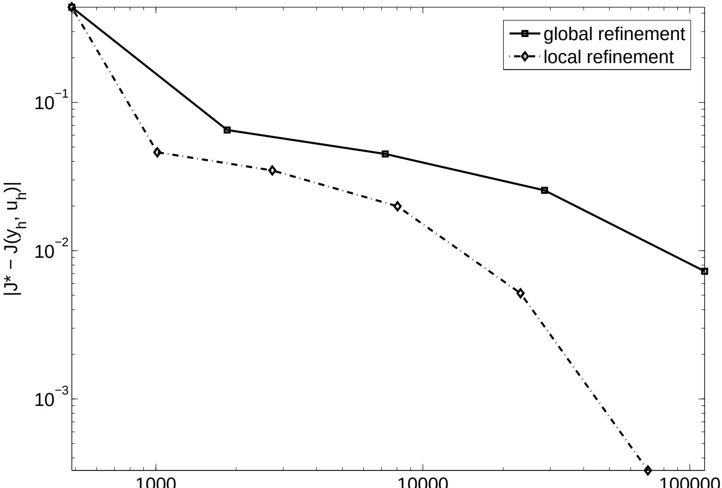
- $\rho^u(\cdot)$ does not appear in this representation.
- No differences of the multipliers $\mu, \mu_h, \lambda, \lambda_h$ appear in this representation.
- Constraints on gradient: $\rho^p(\cdot) = J_y(y_h,u_h)(\cdot) - a(\cdot,p_h) - \langle \mathrm{div}\vec{\mu}_h,\cdot\rangle$ and $\langle \mu + \mu_h, y_h - y\rangle \longrightarrow \langle -\mathrm{div}(\vec{\mu} + \vec{\mu}_h), y_h - y\rangle$

Goal oriented adaptivity, multiplier support and mesh





Goal oriented adaptivity, error



Goal oriented adaptivity, efficiency

m	h	h_{min}	$\lvert J^* - J(y_h, u_h)\rvert$	I_{eff}
484	0.0673	0.0476	0.43855	2.0
1013	0.0673	0.0238	0.04606	0.5
2730	0.0673	0.0119	0.03477	1.2
8038	0.0673	0.0060	0.01992	1.9
23216	0.0673	0.0030	0.00516	1.4
69645	0.0673	0.0015	0.00033	0.3

Work in progress, collaboration with O. Benedix and B. Vexler

Thank you very much for your attention