A Finite Element Method for Nonvariational Elliptic Problems

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joint work with Omar Lakkis

May 20, 2010

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Outline

Model problem in nonvariational form

Model problem

Given $f \in \mathsf{L}_2(\Omega)$, find $u \in \mathsf{H}^2(\Omega) \cap \mathsf{H}^1_0(\Omega)$ such that

$$\left\langle \mathbf{A}:\mathrm{D}^2 u \,\middle|\, \phi \right\rangle = \langle f, \phi \rangle \quad \forall\, \phi \in \mathsf{H}^1_0(\Omega),$$

$\mathbf{X}:\mathbf{Y} := \operatorname{trace}(\mathbf{X}^\mathsf{T}\mathbf{Y})$ is the Frobenius inner product of two matrices.

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Don't want to rewrite in divergence form!

$$\left\langle \mathbf{A}{:}\mathrm{D}^2 u \,\middle|\, \phi \right\rangle = \langle \operatorname{div}(\mathbf{A}\nabla u) \mid \phi \rangle - \langle \operatorname{div}(\mathbf{A}) \nabla u, \phi \rangle .$$

What should we do?

Main idea

- Define appropriately the Hessian of a function who's not twice differentiable, i.e., as a distribution.

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- Construct a finite element approximation of this distribution. What is meant by the Hessian of a finite element function? [Aguilera and Morin, 2008]
- Discretise the strong form of the PDE directly.

Hessians as distributions

generalised Hessian

Given $v \in \mathrm{H}^1(\Omega)$ it's generalised Hessian is given by

$$\left\langle \mathrm{D}^2 v \,\middle|\, \phi \right\rangle = -\left\langle \nabla v \otimes \nabla \phi \right\rangle + \left\langle \nabla u \otimes \mathbf{n}\ \phi \right\rangle_{\partial\Omega} \quad \forall\, \phi \in \mathrm{H}^1(\Omega),$$

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finite element space notation

$$\mathbb{V} := \left\{ \Phi \in \mathrm{H}^1(\Omega) : \ \Phi|_K \in \mathbb{P}^p \,\forall\, K \in \mathscr{T} \right\},$$
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finite element Hessian

For each $V \in \mathring{\mathbb{V}}$ there exists a unique $\mathbf{H}[V] \in \mathbb{V}^{d\times d}$ such that

$$\left\langle \mathbf{H}[V], \Phi \right\rangle = \left\langle \mathrm{D}^2 V \,\middle|\, \Phi \right\rangle \qquad \forall\, \Phi \in \mathbb{V}.$$

Nonvariational finite element method

Substitute the finite element Hessian directly into the model problem.
We seek $U \in \mathring{\mathbb{V}}$ such that

$$\left\langle \mathbf{A}:\mathbf{H}[U], \mathring{\Phi} \right\rangle = \left\langle f, \mathring{\Phi} \right\rangle \quad \forall\, \mathring{\Phi} \in \mathring{\mathbb{V}}.$$

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Discretisation

$$\begin{aligned}\left\langle f, \mathring{\boldsymbol{\Phi}} \right\rangle &= \sum_{\alpha=1}^{d} \sum_{\beta=1}^{d} \left\langle \mathbf{A}^{\alpha,\beta} \mathbf{H}_{\alpha,\beta}[U], \mathring{\boldsymbol{\Phi}} \right\rangle \\ &= \sum_{\alpha=1}^{d} \sum_{\beta=1}^{d} \left\langle \mathring{\boldsymbol{\Phi}}, \mathbf{A}^{\alpha,\beta} \boldsymbol{\Phi}^{\mathsf{T}} \right\rangle \mathbf{h}_{\alpha,\beta}.\end{aligned}$$

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$$\left\langle \boldsymbol{\Phi}, \boldsymbol{\Phi}^{\mathsf{T}} \right\rangle \mathbf{h}_{\alpha,\beta} = \left\langle \boldsymbol{\Phi}, \mathbf{H}_{\alpha,\beta}[U] \right\rangle$$
$$= \left(- \left\langle \partial_\beta \boldsymbol{\Phi}, \partial_\alpha \mathring{\boldsymbol{\Phi}}^{\mathsf{T}} \right\rangle + \left\langle \boldsymbol{\Phi} \mathbf{n}_\beta, \partial_\alpha \mathring{\boldsymbol{\Phi}}^{\mathsf{T}} \right\rangle_{\partial\Omega} \right) \mathbf{u}.$$

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Linear system

$U = \mathring{\boldsymbol{\Phi}}^{\mathsf{T}}\mathbf{u}$, where $\mathbf{u} \in \mathbb{R}^{\mathring{N}}$ is the solution to the following linear system

$$\mathbf{D}\mathbf{u} := \sum_{\alpha=1}^{d}\sum_{\beta=1}^{d} \mathbf{B}^{\alpha,\beta}\mathbf{M}^{-1}\mathbf{C}_{\alpha,\beta}\mathbf{u} = \mathbf{f}.$$

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Components of the linear system

$$\begin{aligned}
\mathbf{B}^{\alpha,\beta} &:= \left\langle \mathring{\boldsymbol{\Phi}}, \mathbf{A}^{\alpha,\beta}\boldsymbol{\Phi}^{\mathsf{T}} \right\rangle \in \mathbb{R}^{\mathring{N}\times N}, \\
\mathbf{M} &:= \langle \boldsymbol{\Phi}, \boldsymbol{\Phi}^{\mathsf{T}} \rangle \in \mathbb{R}^{N\times N}, \\
\mathbf{C}_{\alpha,\beta} &:= -\left\langle \partial_\beta \boldsymbol{\Phi}, \partial_\alpha \mathring{\boldsymbol{\Phi}}^{\mathsf{T}} \right\rangle + \left\langle \boldsymbol{\Phi}\mathbf{n}_\beta, \partial_\alpha \mathring{\boldsymbol{\Phi}}^{\mathsf{T}} \right\rangle_{\partial\Omega} \in \mathbb{R}^{N\times \mathring{N}}, \\
\mathbf{f} &:= \left\langle f, \mathring{\boldsymbol{\Phi}} \right\rangle \in \mathbb{R}^{\mathring{N}}.
\end{aligned}$$

The system is hard to solve

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Remarks

- The matrix $\mathbf{D}$ is not sparse

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- Notice **D** is the sum of Schur complements

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Remarks

- The matrix $\mathbf{D}$ is not sparse
- Mass lumping only works for $\mathbb{P}^1$ elements AND in this case only gives a reasonable solution U for very simple $\mathbf{A}$
- Notice $\mathbf{D}$ is the sum of Schur complements
- We can create a block matrix to exploit this

Block system

$$\mathbf{E} = \begin{bmatrix} \mathbf{M} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & -\mathbf{C}_{1,1} \\ \mathbf{0} & \mathbf{M} & \cdots & \mathbf{0} & \mathbf{0} & -\mathbf{C}_{1,2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{M} & \mathbf{0} & -\mathbf{C}_{d,d-1} \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{M} & -\mathbf{C}_{d,d} \\ \mathbf{B}^{1,1} & \mathbf{B}^{1,2} & \dots & \mathbf{B}^{d,d-1} & \mathbf{B}^{d,d} & \mathbf{0} \end{bmatrix},$$

$$\mathbf{v} = (\mathbf{h}_{1,1}, \mathbf{h}_{1,2}, \dots, \mathbf{h}_{d,d-1}, \mathbf{h}_{d,d}, \mathbf{u})^{\mathsf{T}},$$

$$\mathbf{h} = (\mathbf{0}, \mathbf{0} \dots, \mathbf{0}, \mathbf{0}, \mathbf{f})^{\mathsf{T}}.$$

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Equivalence of systems

Then solving the system $\mathbf{Du} = \mathbf{f}$ is equivalent to solving

$$\mathbf{Ev} = \mathbf{h}.$$

for $\mathbf{u}$.

structure of the block matrix

the discretisation presented nothing but:

Find $U \in \mathring{\mathbb{V}}$ such that
$$\begin{cases} \langle \mathbf{H}[U], \Phi \rangle = -\langle \nabla U \otimes \nabla \Phi \rangle + \langle \nabla U \otimes \mathbf{n}\ \Phi \rangle_{\partial\Omega} \\ \qquad\qquad\qquad\qquad \forall\, \Phi \in \mathbb{V} \\ \left\langle \mathbf{A}{:}\mathbf{H}[U], \mathring{\Phi} \right\rangle = \left\langle f, \mathring{\Phi} \right\rangle \quad \forall\, \mathring{\Phi} \in \mathring{\mathbb{V}}. \end{cases}$$

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$$\begin{bmatrix} \mathbf{M} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & -\mathbf{C}_{1,1} \\ \mathbf{0} & \mathbf{M} & \cdots & \mathbf{0} & \mathbf{0} & -\mathbf{C}_{1,2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{M} & \mathbf{0} & -\mathbf{C}_{d,d-1} \\ \mathbf{0} & \mathbf{0} & \ldots & \mathbf{0} & \mathbf{M} & -\mathbf{C}_{d,d} \\ \mathbf{B}^{1,1} & \mathbf{B}^{1,2} & \ldots & \mathbf{B}^{d,d-1} & \mathbf{B}^{d,d} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{h}_{1,1} \\ \mathbf{h}_{1,2} \\ \vdots \\ \mathbf{h}_{d,d-1} \\ \mathbf{h}_{d,d} \\ \mathbf{u} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{f} \end{bmatrix}.$$

A Linear PDE in NDform

Operator choice - heavily oscillating

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & a(\mathbf{x}) \end{bmatrix}$$

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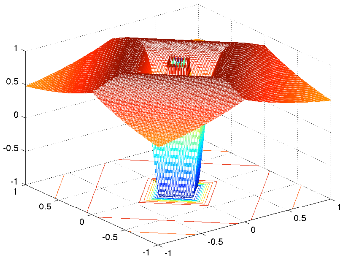
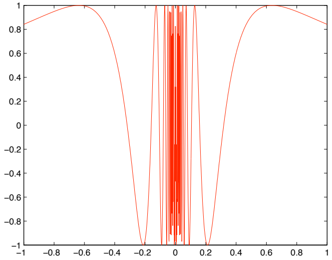
Operator choice - heavily oscillating

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & a(\mathbf{x}) \end{bmatrix}$$

$$a(\mathbf{x}) = \sin\left(\frac{1}{|x_1| + |x_2| + 10^{-15}}\right)$$





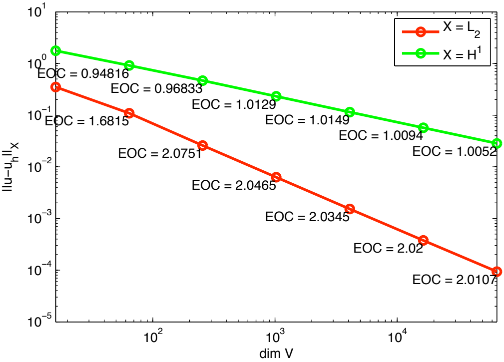


Figure: Choosing f appropriately such that $u(\mathbf{x}) = \exp\left(-10\,|\mathbf{x}|^2\right)$.

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$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & a(\mathbf{x}) \end{bmatrix}$$

$$a(\mathbf{x}) := \left(\arctan\left(5000(|\mathbf{x}|^2 - 1)\right) + 2\right).$$

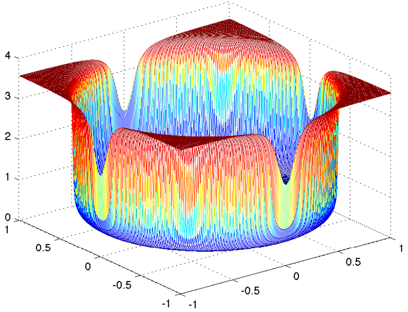
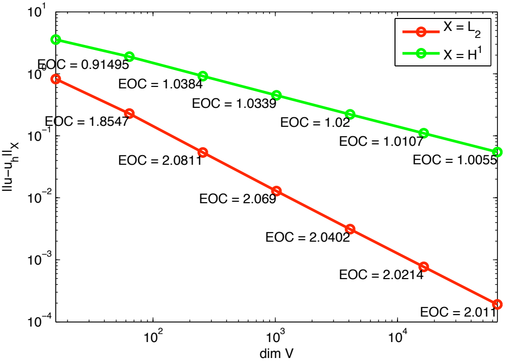
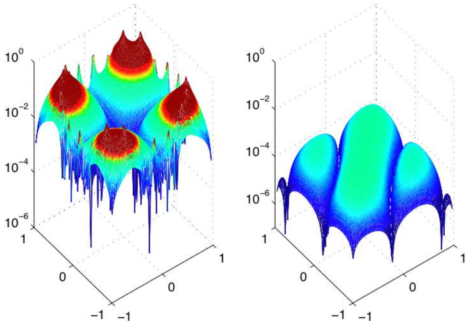


Figure: Choosing f appropriately such that $u(\mathbf{x}) = \sin(\pi x_1)\sin(\pi x_2)$.



The same Linear PDE in NDform

Figure: On the left we present the maximum error of the standard FE-solution. Notice the oscillations apparant on the unit circle. On the right we show the maximum error of the NDFE-solution



Fully nonlinear PDEs

Model problem

Given $f \in \mathrm{L}_2(\Omega)$, find $u \in \mathrm{H}^2(\Omega) \cap \mathrm{H}^1_0(\Omega)$ such that

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Newton's method

Given u^0 for each $n \in \mathbb{N}$ find u^{n+1} such that

$$\left\langle \mathscr{N}'[u^n] \mid u^{n+1} - u^n \right\rangle = -\mathscr{N}[u^n]$$

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$$\begin{aligned} \langle \mathcal{N}'[u] \mid v \rangle &= \lim_{\epsilon \to 0} \frac{\mathcal{N}[u + \epsilon v] - \mathcal{N}[u]}{\epsilon} \\ &= \lim_{\epsilon \to 0} \frac{\mathscr{F}(\mathrm{D}^2 u + \epsilon \mathrm{D}^2 v) - \mathscr{F}(\mathrm{D}^2 u)}{\epsilon} \\ &= \lim_{\epsilon \to 0} \mathscr{F}'(\mathrm{D}^2 u) : \mathrm{D}^2 v + \mathrm{O}(\epsilon). \end{aligned}$$

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Discretisation

VERY roughly

Given $U^0 = \Lambda u^0$ find U^{n+1} such that

$$\mathscr{F}'(\mathbf{H}[U^n]):\mathbf{H}[U^{n+1} - U^n] = f - \mathscr{F}(\mathbf{H}[U^n])$$

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$\mathbf{H}[U^n]$ is given in the solution of the previous iterate!

$$\begin{bmatrix} \mathbf{M} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & -\mathbf{C}_{1,1} \\ \mathbf{0} & \mathbf{M} & \cdots & \mathbf{0} & \mathbf{0} & -\mathbf{C}_{1,2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{M} & \mathbf{0} & -\mathbf{C}_{d,d-1} \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{M} & -\mathbf{C}_{d,d} \\ \mathbf{B}^{1,1}_{n-1} & \mathbf{B}^{1,2}_{n-1} & \dots & \mathbf{B}^{d,d-1}_{n-1} & \mathbf{B}^{d,d}_{n-1} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{h}^n_{1,1} \\ \mathbf{h}^n_{1,2} \\ \vdots \\ \mathbf{h}^n_{d,d-1} \\ \mathbf{h}^n_{d,d} \\ \mathbf{u}^n \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{f} \end{bmatrix}.$$

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$$\mathscr{F}'(\mathbf{H}[U^n]){:}\mathbf{H}[U^{n+1} - U^n] = f - \mathscr{F}(\mathbf{H}[U^n])$$

$\mathbf{H}[U^n]$ is given in the solution of the previous iterate!

$$\begin{bmatrix} \mathbf{M} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & -\mathbf{C}_{1,1} \\ \mathbf{0} & \mathbf{M} & \cdots & \mathbf{0} & \mathbf{0} & -\mathbf{C}_{1,2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{M} & \mathbf{0} & -\mathbf{C}_{d,d-1} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{M} & -\mathbf{C}_{d,d} \\ \mathbf{B}^{1,1}_{n-1} & \mathbf{B}^{1,2}_{n-1} & \cdots & \mathbf{B}^{d,d-1}_{n-1} & \mathbf{B}^{d,d}_{n-1} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{h}^n_{1,1} \\ \mathbf{h}^n_{1,2} \\ \vdots \\ \mathbf{h}^n_{d,d-1} \\ \mathbf{h}^n_{d,d} \\ \mathbf{u}^n \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{f} \end{bmatrix}.$$

Discretisation

VERY roughly

Given $U^0 = \Lambda u^0$ find U^{n+1} such that

$$\mathscr{F}'(\mathbf{H}[U^n]){:}\mathbf{H}[U^{n+1} - U^n] = f - \mathscr{F}(\mathbf{H}[U^n])$$

$\mathbf{H}[U^n]$ is given in the solution of the previous iterate!

$$\begin{bmatrix} \mathbf{M} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & -\mathbf{C}_{1,1} \\ \mathbf{0} & \mathbf{M} & \cdots & \mathbf{0} & \mathbf{0} & -\mathbf{C}_{1,2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{M} & \mathbf{0} & -\mathbf{C}_{d,d-1} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{M} & -\mathbf{C}_{d,d} \\ \mathbf{B}^{1,1}_{n-1} & \mathbf{B}^{1,2}_{n-1} & \cdots & \mathbf{B}^{d,d-1}_{n-1} & \mathbf{B}^{d,d}_{n-1} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{h}^n_{1,1} \\ \mathbf{h}^n_{1,2} \\ \vdots \\ \mathbf{h}^n_{d,d-1} \\ \mathbf{h}^n_{d,d} \\ \mathbf{u}^n \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{f} \end{bmatrix}.$$

- Saves us postprocessing another one! [Vallet et al., 2007] [Ovall, 2007]

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