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Numerical methods for option pricing in Feller Lévy models

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Pricing problem

- Numerical approximation of (weak!) solution $u(x,t) \in \mathrm{Domain}(A_X)$ in $G \subseteq \mathbb{R}^q$ of fwd Kolmogoroff PIDE

$$u_t + (A_X(x;D)u)(x) = f \in (0,T] \times G, \quad u|_{t=0} = u_0.$$

- $$(A_X(x;D)u)(t,x) := \\ c(x)u(t,x) + \gamma(x)^\top \nabla_x u(t,x) + \frac{1}{2}\sigma(x)\sigma(x)^\top D^2 u(t,x) \\ + \\ \int_{y\in\mathbb{R}^d} \left(u(x+y) - u(x) - y \cdot \nabla_x u(x) \frac{1}{1+\|y\|^2} \right) N(x,dy),$$

Applications:

- Markovian projection of Semimartingales
- Modelling of bounded (jump) processes

Feller-processes I

Definition

Assume $q = 1$. Let X be a strong $\mathbb{R}$-valued Markov process and let

$$(T_t g)(x) = \mathbb{E}[g(X_t)|X_0 = x].$$

X is called Feller iff

1. $T_t : C_0(\mathbb{R}) \to C_0(\mathbb{R})$,
2. $\lim_{t\to 0^+} \|u - T_t u\|_{L^\infty(\mathbb{R})} = 0$ for all $u \in C_0(\mathbb{R})$.

Theorem

Feller generators admit the positive maximum principle, i.e., if $u \in D(A_X)$ and $\sup_{x\in\mathbb{R}} u(x) = u(x_0) > 0$, then $(A_X u)(x_0) \leq 0$.

Feller-processes II

Theorem

Let A_X be the generator of a Feller-process with $C_0^\infty(\mathbb{R}) \subset D(A_X)$, then $A|_{C_0^\infty(\mathbb{R})}$ is a pseudodifferential operator (PDO):

$$\begin{aligned}(A_X u)(x) &= -a(x,D)u(x) \\ &= -(2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}} a(x,\xi)\hat{u}(\xi)e^{ix\xi}\, d\xi, u \in C_0^\infty(\mathbb{R}),\end{aligned}$$

with symbol $a(x,\xi) : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ which is measurable and locally bounded in (x,ξ) and which admits the Lévy-Khintchine representation.

Feller-processes III

$$
\begin{aligned}
a(x,\xi) &= c(x) - i\gamma(x)\xi + \frac{1}{2}(\sigma(x))^2\xi^2 \\
&\quad + \int_{\mathbb{R}} \left(1 - e^{iy\xi} + \frac{iy\xi}{1+y^2}\right) N(x,dy),
\end{aligned}
$$

where $\sup_{x\in\mathbb{R}} \int_{\mathbb{R}} \min(1,y^2) N(x,dy) < \infty$.
Examples:

1. Brownian motion (local vol) $a(x,\xi) = \frac{1}{2}\sigma(x)^2\xi^2$.
2. Lévy process
   $a(x,\xi) = c - i\gamma\xi + \frac{1}{2}(\sigma)^2\xi^2 + \int_{\mathbb{R}} \left(1 - e^{iy\xi} + \frac{iy\xi}{1+y^2}\right) \nu(dy)$.

Feller-processes IV

Definition

(Symbol class $S^{m(x)}_{\rho,\delta}$)
Let $0 \leq \delta \leq \rho \leq 1$ and let $m(x) \in C^\infty(\mathbb{R})$. A symbol $a(x,\xi)$ belongs to $S^{m(x)}_{\rho,\delta}(\mathbb{R})$ iff

1. $a(x,\xi) \in C^\infty(\mathbb{R} \times \mathbb{R})$,
2. $m(x) = s + \widetilde{m}(x)$ with $\widetilde{m}(x) \in S(\mathbb{R})$, $s \in \mathbb{R}$,
3. for $\alpha, \beta \in \mathbb{N}_0$ there are constants $c_{\alpha,\beta}$ such that

$$\forall x, \xi \in \mathbb{R} : \quad \left| D^\beta_x D^\alpha_\xi a(x,\xi) \right| \leq c_{\alpha,\beta} \langle \xi \rangle^{m(x) - \rho\alpha + \delta\beta},$$

where $\langle \xi \rangle := (1+\xi^2)^{\frac{1}{2}}$, $\xi \in \mathbb{R}$.

The corresponding set of PDOs is denoted by $\Psi^{m(x)}_{\rho,\delta}(\mathbb{R})$.

Feller-processes V

Theorem

(Komatsu, Strook, Jacod 1976, Hoh 1998)
For every symbol $a(x,\xi) \in S^{m(x)}_{\rho,\delta}$ *there exists a unique Feller process* X *with generator* A_X*.*

Domain of A_X? Answer: Sobolev spaces of variable order.

Definition

The PDO $\Lambda^{m(x)}$ with symbol $a(x,\xi) = \langle\xi\rangle^{m(x)} \in S^{m(x)}_{1,\delta}$, $\delta \in (0,1)$, is called *(variable order) Riesz potential*.

Corollary

$(\Lambda^{m(x)})^\top \in \Psi^{m(x)}_{1,\delta}$ *and* $(\Lambda^{m(x)})^\top(\Lambda^{m(x)}) \in \Psi^{2m(x)}_{1,\delta}$.

Alternative characterization of PDOs

A PDO in distributional sense can be written as:

$$\begin{aligned} Au(x) &= \int_{\mathbb{R}} K_A(x,y)u(y)\,dy, \\ K_A(x,y) &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{i(x-y)\xi} a_A(x,\xi,y)\,d\xi, \end{aligned}$$

where $K_A(x,y)$ is an oscillatory integral, i.e.,

$$\begin{aligned} K_A(x,y) &= \lim_{\epsilon\to 0}\frac{1}{2\pi}\int_{\mathbb{R}} e^{i(x-y)\xi} a^{\epsilon}_A(x,\xi,y)\,d\xi, \\ a^{\epsilon}_A(x,\xi,y) &= a_A(x,\xi,y)\mu(\epsilon y,\epsilon\xi), \quad \mu \in C^{\infty}_0(\mathbb{R}\times\mathbb{R}),\ \mu(0,0)=1. \end{aligned}$$

Kikuchi & Negoro 1997, Bass 2002:

$$a_{(\Lambda^{m(x)})^{\top}(\Lambda^{m(x)})}(x,\xi,y) = \langle\xi\rangle^{m(x)+m(y)}.$$

Sobolev spaces of variable order

In what follows we always assume $m(x) \in (0,1)$.

Definition

The Sobolev space of variable order is
$H^{m(x)}(\mathbb{R}) := \{u \in L_2(\mathbb{R}) | \, \|u\|_{H^{m(x)}(\mathbb{R})} < \infty\}$ where

$$\|u\|^2_{H^{m(x)}(\mathbb{R})} := \left\|\Lambda^{m(x)} u\right\|^2_{L_2(\mathbb{R})} + \|u\|^2_{L_2(\mathbb{R})} \, .$$

On a bounded domain I we define the space

$$\widetilde{H}^{m(x)}(I) \quad = \quad \{u|_I \big| u \in H^{m(x)}(\mathbb{R}), \quad u|_{\mathbb{R}\setminus\overline{I}} = 0\}.$$

The norm on $\widetilde{H}^{m(x)}(I)$ is given as

$$\|u\|_{\widetilde{H}^{m(x)}(I)} = \|\widetilde{u}\|_{H^{m(x)}(\mathbb{R})} \, ,$$

Wavelets I

Aim: Prove a norm equivalence on $\widetilde{H}^{m(x)}(I)$ and obtain a preconditioner for the wavelet matrix of $N(x, dz)$. We require the following properties of the wavelets:

1. Biorthogonality, i.e., $\psi_{l,k}, \widetilde{\psi}_{l',k'}$ satisfy $\langle \psi_{l,k}, \widetilde{\psi}_{l',k'} \rangle = \delta_{l,l'} \delta_{k,k'}$.
2. Local support:
$$\text{diam supp}\psi_{l,k} \leq C2^{-l}, \quad \text{diam supp}\widetilde{\psi}_{l,k} \leq C2^{-l}.$$
3. Conformity:
$$\mathcal{W}^l \subset \widetilde{H}^1(I), \widetilde{\mathcal{W}}^l \subset \widetilde{H}^\delta(I) \quad \text{for some } \delta > 0,\ l \geq -1.$$
4. Density: $\bigoplus_{l=-1}^{\infty} \mathcal{W}^l$,$\bigoplus_{l=-1}^{\infty} \widetilde{\mathcal{W}}^l$ dense in $L_2(I)$.
5. Vanishing moments for primal and dual wavelets.

Estimates for extended symbols

Theorem

For any $\delta \in (0,1)$ the Schwartz kernel $K_{(\Lambda^{m(x)})^\top(\Lambda^{m(x)})}$ satisfies the Caldéron-Zygmund type estimate

$$\left| D_x^\alpha D_y^\beta K_{(\Lambda^{m(x)})^\top(\Lambda^{m(x)})}(x,y) \right| \leq C_{\alpha,\beta,\delta} \, |x-y|^{-(1+m(x)+m(y)+(1-\delta)(\alpha+\beta))}$$

where $x \neq y$ and $|x-y|$ is small. For large values of $|x-y|$ the kernel decays faster than $|x-y|^{-N}$, for any $N \in \mathbb{N}$.

Proof:

- Littlewood Paley decomposition of unity
- Decomposition of the symbol

Norm Equivalences I

We consider the infinite matrix ($\lambda = (l,k)$, $\lambda' = (l,k')$):

$$\mathbf{M} := \left(\langle \Lambda^{m(x)} \psi_{\lambda'}, \Lambda^{m(x)} \psi_\lambda \rangle \right)_{\lambda,\lambda' \in \mathcal{I}} = \left(\langle (\Lambda^{m(x)})^\top \Lambda^{m(x)} \psi_{\lambda'}, \psi_\lambda \rangle \right)_{\lambda,\lambda' \in \mathcal{I}} .$$

The following variables will be useful:

$$\begin{aligned} \overline{m}_\lambda &= \sup\{m(x) : x \in \Omega_\lambda\}, \quad \underline{m}_\lambda = \inf\{m(x) : x \in \Omega_\lambda\}, \\ \Omega_\lambda &= \bigcup_{l'>l} \{\text{supp}\psi_{\lambda'} : \text{supp}\psi_\lambda \cap \text{supp}\psi_{\lambda'} \neq \emptyset\} . \end{aligned}$$

Norm Equivalences II

Theorem

Let $\mathbf{D}^{-m(x)} := (2^{-l\overline{m}_\lambda}\delta_{\lambda,\lambda'})_{\lambda,\lambda'}$ *and*

$$\mathbf{A} := \mathbf{D}^{-m(x)}\mathbf{M}\mathbf{D}^{-m(x)}.$$

Then $\mathbf{A}$ *is compressible i.e. there exists* $s > 0$ *s.t.*

$$\left|A_{\lambda,\lambda'}\right| \lesssim 2^{-|l-l'|(s+\frac{1}{2})}(1 + \mathit{dist}(\mathit{supp}\psi'_\lambda, \mathit{supp}\psi_\lambda)^{-1-2(d-\overline{m})(1-\delta)}.$$

As compressible matrices have a bounded spectral norm and $\mathbf{D}^{-m(x)}\mathbf{D}^{m(x)}$ also has a bounded spectral norm, we obtain the norm equivalence:

$$\|u\|_{\widetilde{H}^{m(x)}(I)} \sim u^\top\mathbf{D}^{2m(x)}u.$$

Implementation of the PIDE

- Implementation of FFT methods for the PDO not feasible, due to nonstationarity of X
- Alternative: solve PIDE in "x-space" $\mathbb{R}^q$
- Weak solutions: FEM
- Compression of jump measure: Wavelets

Assumptions

Let $N(x, dz) = k(x, z)dz$. Assume that the jump density $k(x, z)$ satisfies: there exist constants $\beta^- > 0$ and $\beta^+ > 1$, $0 \leq \delta \leq \rho \leq 1$ independent of x s.t.

1.
$$k(x, z) \leq C \begin{cases} e^{-\beta^-|z|}, & z < -1, \\ e^{-\beta^+ z}, & z > 1. \end{cases}$$

2.
$$\frac{1}{|z|^{2m(x)}} \sim k(x, z), \quad 0 < |z| \leq 1.$$

3.
$$\left| D_x^\beta D_z^\alpha k(x, z) \right| \leq c\alpha!\beta! \, |z|^{-1-2m(x)-\alpha\rho-\beta\delta} \quad \forall \alpha, \beta \in \mathbb{N}_0, z \neq 0.$$

Derivation of the PIDE I

Let X be a pure jump process without drift. Then

$$a(x,\xi) = \int_{\mathbb{R}} (1 - e^{iz\xi} + iz\xi) k(x,z) dz.$$

We can derive for all $u(x) \in S(\mathbb{R})$:

$$\begin{aligned} (A_X u)(x) &= -\frac{1}{2\pi} \int_{\mathbb{R}} e^{ix\xi} a(x,\xi) \hat{u}(\xi) \, d\xi \\ &= \int_{\mathbb{R}} (u(x+z) - u(x) - z\partial_x u(x)) k(x,z) \, dz. \end{aligned}$$

For sufficiently smooth u this can be written as:

$$(A_X u)(x) = \int_{\mathbb{R}} u''(x+z) k^{(-2)}(x,z) \, dz,$$

where $k^{(-i)}$ is the i-th antiderivative w.r.t. z.

Derivation of the PIDE II

The bilinear form for a test function $v \in C_0^\infty(\mathbb{R})$ reads:

$$
\begin{aligned}
b(u,v) &= \int_{\mathbb{R}} (A_X u)(x) v(x)\, dx \\
&= -\int_{\mathbb{R}} \int_{\mathbb{R}} u'(x+z) v'(x) k^{(-2)}(x,z)\, dz\, dx \\
&\quad -\int_{\mathbb{R}} \int_{\mathbb{R}} u'(x+z) v(x) k_x^{(-2)}(x,z)\, dz\, dx.
\end{aligned}
$$

Numerical quadrature

Density $k(x,z)$ of $N(x,dz)$ satisfies conditions of Chernov, von Peterdorff & Schwab 2009

- Composite Gauss quadrature used to deal with singularity at $x = y$.
- Idea: geometric quadrature node refinement towards singularity of $k(x,z)$.
- Exponential convergence of the tensorized (composite) Gauss quadrature

Extension to multidimensional framework

- Symbol classes can analogously be defined on $\mathbb{R}^q$
- $H^{\mathbf{m}(x)}$, for $\mathbf{m}(x) = (m_1(x_1), \ldots, m_q(x_q))$ can be characterized as

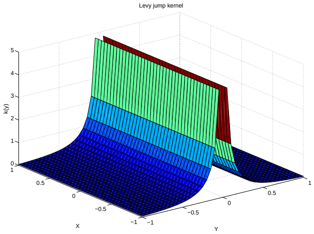
$$H^{\mathbf{m}(x)} = \bigcap_{j=1}^{q} H_j^{m_j(x_j)}. \quad (1)$$

- Construction of the jump measure using a Lévy copula function
- Discretization using tensor product Wavelet basis $\Rightarrow$ Norm equivalences.

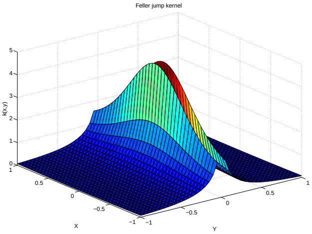
Model problem I

We consider CGMY-type processes:

$$
\begin{aligned}
k(x,z) &= C \begin{cases} e^{-\beta^- z} z^{-1-\alpha(x)}, & z > 0 \\ e^{-\beta^+ |z|} |z|^{-1-\alpha(x)}, & z < 0, \end{cases} \\
\alpha(x) &= k e^{-x^2} + 0.5.
\end{aligned}
$$



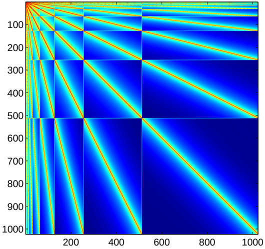
(a) $\alpha(x) = 1.75$



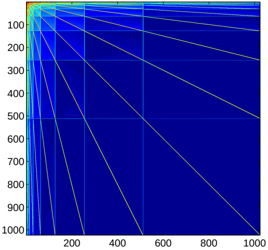
(b) $\alpha(x) = 1.25 e^{-x^2} + 0.5$

Stiffness matrices

Stiffness matrices for Example I with $Y(x) = 1.25e^{-x^2} + 0.5$:



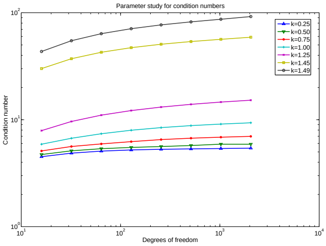
(a) $k^{(-2)}(x, z)$



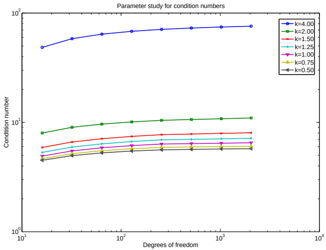
(b) $k_x^{(-2)}(x, z)$

Figure: Stiffness matrices

Preconditioning



(a) Model problem I (smooth α)



(b) Model problem II (Lipschitz continuous α)

Figure: Condition numbers for different levels and choices of k.

Option prices

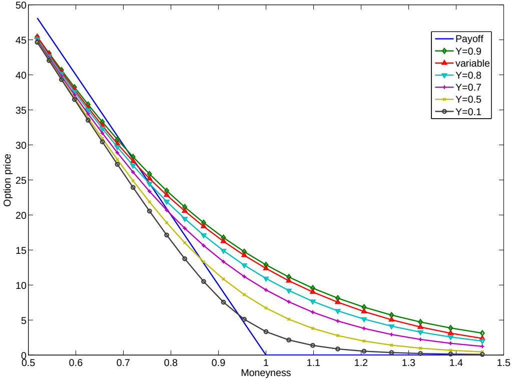


Figure: Option prices for several models for a European put option with $T = 1$ and $K = 100$.

Outlook

- Quadratic Hedging
- Analysis of model risk via hierarchical models, i.e. BS $\subseteq$ Local Volatility $\subseteq$ Feller-Lévy.
- Preconditioning methods for $\alpha(x) \approx 2$.
- (Piecewise) smooth time dependent coefficients.
- Fast Calibration (P. Carr 2009).

References

- R. Schneider, O.R., Ch. Schwab, Wavelet solution of variable order pseudodifferential equations, Calcolo'09.
- O.R. and Ch. Schwab, Numerical analysis of additive, Lévy and Feller processes with applications to option pricing, SAM-Report 06-2010.