C^1 finite element time discretization of 4 th order

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Problem and FE space

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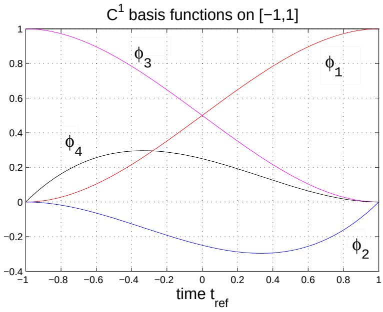
reference transformation : $\omega_n : [-1,1] \to I_n$

$$t = \omega_n(\hat{t}) = t_{n-1/2} + \tfrac{\tau_n}{2}\hat{t}\,, \qquad \hat{u}(\hat{t}) := u(t) \quad \Rightarrow \quad \hat{u}'(\hat{t}) = \tfrac{\tau_n}{2} u'(t)$$

Hermite basis functions



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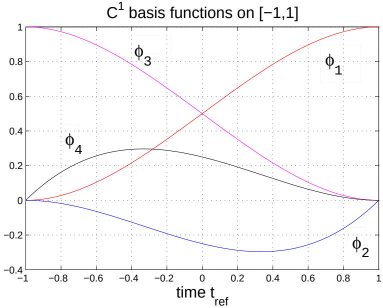

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thus, for $u_\tau\big|_{I_n} \in \mathbb{P}(I_n, \mathbb{R}^d)$, it holds:

$$u_\tau(t) = \underbrace{u_\tau(t_n)}_{U^1}\,\phi_1(t) + \underbrace{\frac{\tau_n}{2}u_\tau'(t_n)}_{U^2}\,\phi_2(t) + \underbrace{u_\tau(t_{n-1})}_{U^3}\,\phi_3(t) + \underbrace{\frac{\tau_n}{2}u_\tau'(t_{n-1})}_{U^4}\,\phi_4(t)$$

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- by the C^1**-continuity** it holds for $n > 1$:

$$\begin{aligned} U_n^3 &= u_\tau(t_{n-1}) = U_{n-1}^1 \\ \frac{2}{\tau_n} U_n^4 &= u_\tau'(t_{n-1}) = \frac{2}{\tau_{n-1}} U_{n-1}^2 \end{aligned}$$

and for $n = 1$:

$$U_1^3 = u_0 \,, \qquad \frac{2}{\tau_n} U_1^4 = u'(t_0) = F(t_0, u_0)$$

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- U_n^2 depends **explicitly** on U_n^1 :

$$\frac{2}{\tau_n} U_n^2 = u_\tau'(t_n) = F(t_n, U_n^1) \qquad \textbf{collocation at } t_n$$

- **conclusion :**
  we need only one more equation to determine U_n^1
- this is the variational test equation :

$$\int_{I_n} u_\tau'(t) \cdot 1 \, dt = \int_{I_n} \pi_n F\big(t, u_\tau(t)\big) \cdot 1 \, dt$$

- where

$$\pi_n F\big(t, u_\tau(t)\big) := \sum_{k=1}^{4} F\big(t_{n,k}, u_\tau(t_{n,k})\big) L_k(t)$$

with

$$t_{n,k} := \omega_n(\hat{t}_k)\,, \qquad \hat{t}_k \in [-1, 1]$$ 4 Gauss-Lobatto-points

and

$\hat{L}_k \in \mathbb{P}_3$ are Lagrange basis functions w.r.t. $\hat{t}_k$

The final scheme

solve for U_n^1 by a quasi-Newton method :

$$U_n^1 - U_n^3 = g_1 U_n^4 + g_4 \underbrace{\frac{\tau_n}{2} F(t_n, U_n^1)}_{=U_n^2}$$

$$+ \tau_n \left\{ g_2 F\Big(t_{n,2}, \underbrace{\sum_{j=1}^{4} c_{j,2} U_n^j}_{\text{with } U_n^1}\Big) + g_3 F\Big(t_{n,3}, \underbrace{\sum_{j=1}^{4} c_{j,3} U_n^j}_{\text{with } U_n^1}\Big) \right\}$$

Theoretical results

Theorem (Sch. '10)

Assume $V := \mathbb{R}^d$ *and*

- $\|F(t,u^1) - F(t,u^2)\|_V \le L\|u^1 - u^2\|_V \quad \forall\, u^1, u^2 \in V$
- $L\tau \le \delta_0$ *sufficiently small* $\qquad (\tau := \max_n \tau_n)$

Then, for our method,

- *a* **unique solution** $u_\tau \in X_\tau$ **exists**
- *it holds the optimal* **error estimate**

$$\|u - u_\tau\|_{C(I_n,V)} \le C \max_{1\le k\le n} \tau_k^4 \|d_t^4 u\|_{C(I_k,V)}$$

where $C = \tilde{C} e^{2Lt_{n-1}}$.

Furthermore, the method is **A-stable**.

Remark: generalization to **arbitrary order** is possible

Simple numerical example

- **problem :** Find $u : [0,1] \to \mathbb{R}^{2\times 2}$ such that

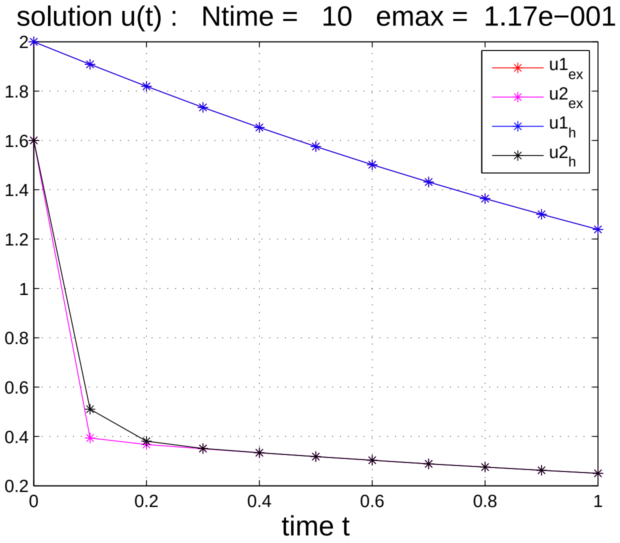
$$\begin{aligned} d_t u(t) &= A u(t) \quad \forall\, t \in [0,1], \\ u(0) &= u_0, \end{aligned}$$

$$A = \begin{pmatrix} -0.5 & 0.1 \\ 10 & -50 \end{pmatrix}, \quad u_0 = \begin{pmatrix} 2 \\ 1.6 \end{pmatrix}$$

- eigenvalues of A : $\lambda_1 \approx -0.48,\ \lambda_2 \approx -50 \ \Rightarrow$ relatively stiff
- we use our method with $\tau_n = \tau \ \forall\, n$
- **error norm :**

$$\|u - u_\tau\|_\infty := \max_{1 \le n \le N} \|u(t_n) - u_\tau(t_n)\|$$

Plot of the components of the solution: $\tau = \frac{1}{10}$



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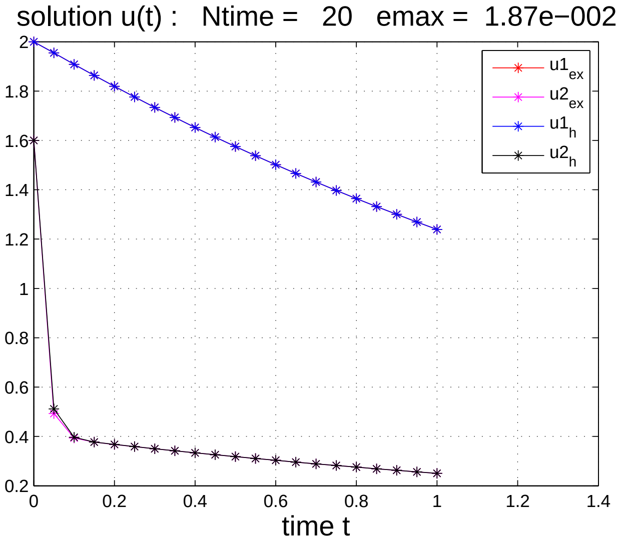

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τ	new method		dG(1)	
	$\|u - u_\tau\|_\infty$	EOC	$\|u - u_\tau\|_\infty$	EOC
1/10	1.170 e-01		5.655 e-02	
1/20	1.875 e-02	2.6413	8.233 e-02	-0.542
1/40	1.592 e-03	3.5583	5.191 e-02	0.666
1/80	9.301 e-05	4.0968	1.829 e-02	1.505
1/160	5.858 e-06	3.9891	5.713 e-03	1.679
1/320	3.645 e-07	4.0063	1.594 e-03	1.842