Reaction diffusion systems on evolving domains

with applications to biological pattern formation

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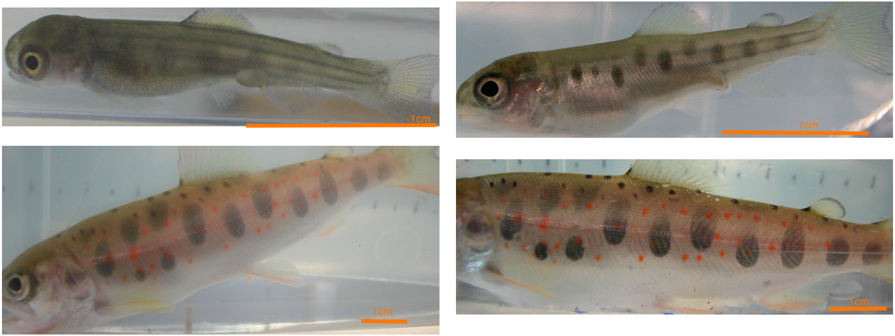
Figure: Transient patterns exhibited during early development



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- Lots of models on evolving domains
- Theoretical results lacking

model problem

$$\partial_t \boldsymbol{u} - \boldsymbol{D}\Delta \boldsymbol{u} + \nabla \cdot (\boldsymbol{a} : \boldsymbol{u}) = \boldsymbol{f}(\boldsymbol{u}) \quad \boldsymbol{x} \in \Omega_t \text{ and } t \in (0, T],$$

$$\frac{\partial \boldsymbol{u}}{\partial \nu} = 0 \quad \boldsymbol{x} \in \partial\Omega_t, t > 0,$$

$$\boldsymbol{u}(\boldsymbol{x}, 0) = \boldsymbol{u}_0(\boldsymbol{x}) \quad \boldsymbol{x} \in \Omega_0, t = 0.$$

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flow assumption

$$\boldsymbol{a} = \partial_t \boldsymbol{x}(t)$$

Gierer-Meinhardt

$$f_1(\boldsymbol{U}) = \gamma\left(a - bu_1 + \frac{u_1^2}{u_2(1 + ku_1^2)}\right)$$

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Thomas

$$f_1(\boldsymbol{U}) = \gamma \left(a - u_1 - g(u_1, u_2)\right),$$
$$f_2(\boldsymbol{U}) = \gamma \left(b - \alpha u_2 - g(u_1, u_2)\right)$$

Bounded spatially linear isotropic evolution.

$$\partial_t \boldsymbol{x}(t) = \rho(t)\boldsymbol{x}(0), \quad \rho(t) \in C^1(\mathbb{R}^+; \mathbb{R}^+)$$

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Rescaled problem

$$\partial_s \boldsymbol{v}(\boldsymbol{x}, s) - \boldsymbol{D}\Delta\boldsymbol{v} + \boldsymbol{v} n\rho\partial_t\rho = \rho^2 \boldsymbol{f}(\boldsymbol{v}) \quad (\boldsymbol{x}, s) \in \Omega_0 \times (0, s],$$

$$\frac{\partial \boldsymbol{v}}{\partial \nu} = 0 \quad \boldsymbol{x} \in \partial\Omega_0, s > 0,$$

$$\boldsymbol{v}(\boldsymbol{x}, 0) = \boldsymbol{v}_0(\boldsymbol{x}) \quad \boldsymbol{x} \in \Omega_0, s = 0,$$

where $\boldsymbol{v}(\boldsymbol{x}(0), s(t)) = \boldsymbol{u}(\rho(t)\boldsymbol{x}(0), t)$ and $s(t) := \int_0^t \frac{dr}{\rho(r)^2}$.

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- e.g., autocatalytic models (Schnakenberg, Brussellator, Gray Scott).

Reference configuration

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- $\mathcal{A} : \hat{\Omega} \times [0, T] \to \Omega_t$ a one to one mapping, such that, for

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Notation

$$g(\boldsymbol{\mathcal{A}}(\xi, t)) = \hat{g}(\xi)$$

$$\boldsymbol{J}_{ij} = \partial_{\xi_i} \boldsymbol{\mathcal{A}}_j \text{ and } J = \det(\boldsymbol{J})$$

$$\boldsymbol{K}_{ij} = \partial_{\boldsymbol{\mathcal{A}}_i} \xi_j \text{ and } \boldsymbol{B} = J\boldsymbol{K}\boldsymbol{K}^T$$

Reference frame

$$\left\langle \frac{d}{dt}(J\hat{\boldsymbol{u}}), \hat{\chi} \right\rangle_{\hat{\Omega}} + \langle \boldsymbol{D}(\boldsymbol{B} : \nabla \hat{\boldsymbol{u}}), \nabla \hat{\chi} \rangle_{\hat{\Omega}} = \langle J\boldsymbol{f}(\hat{\boldsymbol{u}}), \hat{\chi} \rangle_{\hat{\Omega}}$$

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Evolving frame

$$\frac{d}{dt}\langle \boldsymbol{u}, \chi \rangle_{\Omega_t} - \langle \boldsymbol{u}, \dot{\chi} \rangle_{\Omega_t} + \langle \boldsymbol{D}\nabla \boldsymbol{u}, \nabla \chi \rangle_{\Omega_t} = \langle \boldsymbol{f}(\boldsymbol{u}), \chi \rangle_{\Omega_t}$$

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Reference FE scheme

$$\left\langle \frac{(J^n \boldsymbol{U}^n - J^{n-1} \boldsymbol{U}^{n-1})}{\tau}, \psi \right\rangle_{\hat{\Omega}} + \langle D(B^n : \nabla \boldsymbol{U}^{\boldsymbol{n}}), \nabla \psi \rangle_{\hat{\Omega}} = \left\langle J^n \boldsymbol{f}(\boldsymbol{U}^n, \boldsymbol{U}^{n-1}), \psi \right\rangle_{\hat{\Omega}}$$

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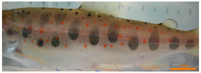
Moving FE scheme (affine domain evolution)

$$\left\langle \frac{\boldsymbol{U}^n}{\tau}, \phi^n \right\rangle_{\Omega_{t^n}} + \langle D \nabla \boldsymbol{U}^n, \nabla \phi^n \rangle_{\Omega_{t^n}} = \left\langle \boldsymbol{f}(\boldsymbol{U}^n, \boldsymbol{U}^{n-1}), \phi^n \right\rangle_{\Omega_{t^n}} + \left\langle \frac{\boldsymbol{U}^{n-1}}{\tau}, \phi^{n-1} \right\rangle_{\Omega_{t^{n-1}}}$$

Numerical simulation

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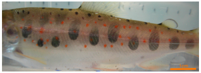

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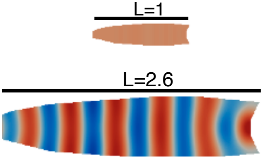
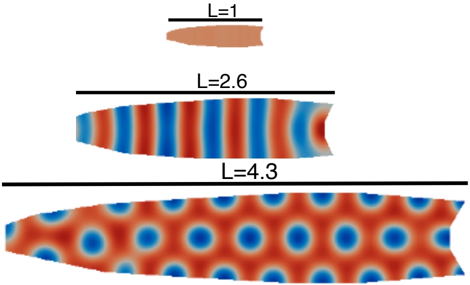


Numerical simulation



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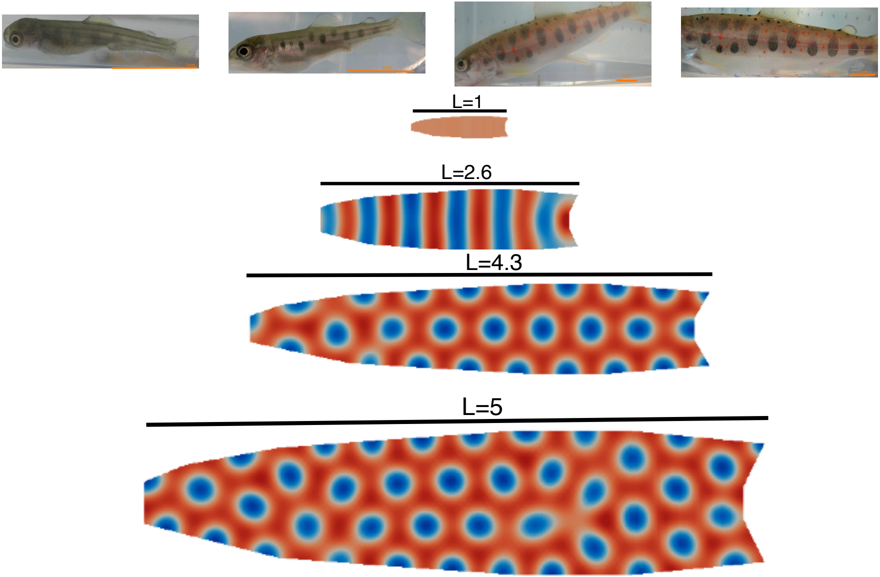

Peridoic evolution: $\boldsymbol{x}(t) = (1 + 7sin(\pi \frac{t}{T}))\boldsymbol{x}(0)$

Nonlinear evolution: $\boldsymbol{x}(t) = \boldsymbol{x}(0) + 7sin(\pi \frac{t}{T})\boldsymbol{x}(0)^2$

Future work

- Existence results for more general evolution

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- Adaptivity

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- Adaptivity
- Concentration driven evolution

References

S. Hollis, R. Martin Jr, and M. Pierre. Global existence and boundedness in reaction-diffusion systems. *SIAM Journal on Mathematical Analysis*, 18:744, 1987.

J. Morgan. Global existence for semilinear parabolic systems. *SIAM Journal on Mathematical Analysis*, 20:1128, 1989.

J. Smoller. *Shock waves and reaction-diffusion equations*. Springer, 1994.

A. Turing. The chemical basis of morphogenesis Phil. In *R. Soc*, pages 37–72, 1952.