Quasi-randomness and the Spectrum

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Quasi-randomness and the spectrum

Outline

- Discrepancy and eigenvalues.
- *Proof:* duality and Grothendieck.
- Regular partitions.

Discrepancy

Intuition

- Think of a (more or less) *regular* graph.
- *Goal:* express that ‘edges are spread uniformly’.
- *To be generalized later.*

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- $p = |E|\binom{n}{2}^{-1}$ is the *density* of G.
- G has *ε-discrepancy* ("ε-Disc") if

$$\left|e(S) - |S|^2 p/2\right| < \varepsilon|E| \text{ for all } S \subset V.$$

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- Entails that G shares many properties of $G(n, p)$ (Krivelevich, Sudakov 2006).
- But difficult to *check*.

Eigenvalue separation

Definition of ε-Eig

- G has n vertices and density p.
- Let M be the $V \times V$-matrix

$$M = p \cdot \mathbf{1} - \text{adjacency matrix},$$

whose entries are

$$m_{uv} = p - \begin{cases} 1 & \text{if } u, v \text{ are adjacent} \\ 0 & \text{otherwise} \end{cases},$$

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Proposition (folklore)

$\varepsilon\text{-Eig} \Rightarrow \varepsilon\text{-Disc}.$

Eig $\Rightarrow$ Disc

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ε-Eig $\Rightarrow$ ε-Disc.

Proof.

Let $\xi = \mathbf{1}_S$. Then

$$\begin{aligned} |\langle M\xi, \xi\rangle| &= \left|2e(S) - |S|^2 p\right|, \\ |\langle M\xi, \xi\rangle| &\leq \|M\| \cdot \|\xi\|^2 \leq \varepsilon np|S| \leq \varepsilon n^2 p \leq 2\varepsilon|E|. \end{aligned}$$

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Question

ε-Disc $\Rightarrow$ $f(\varepsilon)$-Eig?

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In general: *no.*

Example: $G = G(n, p)$ for $p = 1/\varepsilon n$.

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Dense case (Chung, Graham, Wilson 1989)

If $p > \delta$ for a *fixed* $\delta > 0$, then ε-Disc $\Rightarrow f(\varepsilon)$-Eig.

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Bilu, Linial 2007

Assume G is *d-regular* and that

$$\forall S \subset V : |e(S) - |S|^2 p/2| \le \varepsilon d|S|/2.$$

Then $f(\varepsilon)$ – Eig holds with $f(\varepsilon) = O(-\varepsilon \ln \varepsilon)$.

New result

'Essential' eigenvalue separation

G has ε-ess-Eig if there is $W \subset V$, $|W| \geq (1-\varepsilon)n$ such that the minor M_W on $W \times W$ satisfies

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Theorem (Alon, ACO, Han, Kang, Rödl, Schacht 2010)

There is a constant $\gamma > 0$ such that

$$(\gamma\varepsilon^2) - \mathrm{Disc} \Rightarrow \varepsilon - \text{ess-Eig}.$$

The ε^2 is best possible.

Grothendieck's inequality

The cut norm...

...of $B = (b_{uv})_{u,v \in V}$ is

$$\|B\|_\Box = \max \sum_{u,v \in V} b_{uv} \xi_u \zeta_v$$
$$\text{s.t.} \quad \xi_u, \zeta_u \in \{-1, 1\} \text{ for all } u \in V.$$

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The cut norm...

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A relaxation

- Allow *unit vectors* $\xi_u, \zeta_v \in \mathbb{R}^V$.
- That is, let

$$\mathrm{SDP}(B) = \max \sum_{u,v \in V} b_{uv} \langle \xi_u, \zeta_v \rangle$$

$$\text{s.t.} \quad \|\xi_u\| = \|\zeta_u\| = 1 \text{ for all } u \in V.$$

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Theorem (Grothendieck 1953)

There is a *constant* $\theta > 1$ such that $\|B\|_\square \le \mathrm{SDP}(B) \le \theta \|B\|_\square$.

The dual problem

- Let $\mathrm{SDP}(B) = \max \sum_{u,v \in V} b_{uv} \langle \xi_u, \zeta_v \rangle$ s.t. $\|\xi_u\| = \|\zeta_u\| = 1$.
- This is a *linear* problem on the cone of PSD matrices:

$$\mathrm{SDP}(B) = \max \frac{1}{2} \left\langle \begin{pmatrix} 0 & B \\ B & 0 \end{pmatrix}, X \right\rangle \quad \text{s.t.} \quad \mathrm{diag}(X) = \mathbf{1},\ X \geq 0.$$

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- The *dual* problem reads

$$\mathrm{DSDP}(B) = \begin{array}{ll} \min & \langle \mathbf{1}, y \rangle \\ \text{s.t.} & \frac{1}{2} \begin{pmatrix} 0 & B \\ B & 0 \end{pmatrix} \leq \mathrm{diag}(y), \; y \in \mathbb{R}^{2n}. \end{array}$$

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Proposition

$$\mathrm{DSDP}(B) \quad = \quad n \cdot \min_{z \in \mathbb{R}^n,\, z \perp \mathbf{1}} \lambda_{\max}\left[\begin{pmatrix} 0 & B \\ B & 0 \end{pmatrix} - \mathrm{diag}\begin{pmatrix} z \\ z \end{pmatrix}\right].$$

Resuming the proof

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Goal

$\mathrm{Disc}(\gamma\varepsilon^2) \Rightarrow \text{ess-Eig}(\varepsilon)$.

Proof outline

1. G has $\mathrm{Disc}(\gamma\varepsilon^2) \Rightarrow \|M\|_{\square} \le 10\gamma\varepsilon^2 n^2 p$ *(easy)*.
2. Grothendieck $\Rightarrow \mathrm{SDP}(M)$ is small.
3. SDP duality $\Rightarrow \mathrm{DSDP}(M)$ is small.
4. $\mathrm{DSDP} \rightsquigarrow G$ has $\text{ess-Eig}(\varepsilon)$.

$\mathrm{Disc}(\gamma\varepsilon^2) \Rightarrow \text{ess-Eig}(\varepsilon)$

- If G has $\mathrm{Disc}(\gamma\varepsilon^2)$...
- ...then $\mathrm{SDP}(M) \leq 10\theta \cdot \gamma\varepsilon^2 n^2 p$ (Grothendieck).

$\mathrm{Disc}(\gamma\varepsilon^2) \Rightarrow \text{ess-Eig}(\varepsilon)$

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The dual SDP

We have

$$\mathrm{SDP}(M) = n \cdot \min_{z \in \mathbb{R}^n,\, z \perp \mathbf{1}} \lambda_{\max}\left[\begin{pmatrix} 0 & M \\ M & 0 \end{pmatrix} - \mathrm{diag}\begin{pmatrix} z \\ z \end{pmatrix}\right].$$

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$$\mathrm{SDP}(M) = n \cdot \min_{z \in \mathbb{R}^n, z \perp \mathbf{1}} \lambda_{\max}\left[\begin{pmatrix} 0 & M \\ M & 0 \end{pmatrix} - \mathrm{diag}\begin{pmatrix} z \\ z \end{pmatrix}\right].$$

Hence, *there is* $z \perp \mathbf{1}$ such that

$$\lambda_{\max}\left[\begin{pmatrix} 0 & M \\ M & 0 \end{pmatrix} - \mathrm{diag}\begin{pmatrix} z \\ z \end{pmatrix}\right] < \frac{\varepsilon^2 np}{64}.$$

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There is $z \perp \mathbf{1}$ such that $\lambda_{\max}\left[\begin{pmatrix} 0 & M \\ M & 0 \end{pmatrix} - \mathrm{diag}\binom{z}{z}\right] < \varepsilon^2 np/64$.

- Let $W = \{v : |z_v| < \varepsilon np/8\}$.
- Goal: $\|M_W\| \leq \varepsilon np$ and $|W| \geq (1-\varepsilon)n$.

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$$\frac{1}{4}\|M_W\| \quad \leq \quad \lambda_{\max}\begin{pmatrix} 0 & M_W \\ M_W & 0 \end{pmatrix}$$

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&\leq \lambda_{\max}\left[\begin{pmatrix} 0 & M \\ M & 0 \end{pmatrix} - \mathrm{diag}\begin{pmatrix} z \\ z \end{pmatrix}\right] + \|\mathrm{diag}(z_W)\| \leq \frac{\varepsilon np}{4}.
\end{aligned}$$

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- Let $W = \{v : |z_v| < \varepsilon np/8\}$.
- To do: Show that $|W| \geq (1-\varepsilon)n$.
- Let $S = \{v : z_v < 0\}$ and $\xi = \mathbf{1}_S$.
- Then

$$\begin{aligned}
\frac{\varepsilon^2 np|S|}{32} &= \frac{\varepsilon^2 np}{64}\left\|\begin{pmatrix}\xi \\ \xi\end{pmatrix}\right\|^2 \\
&\geq \left\langle \left[\begin{pmatrix} 0 & M \\ M & 0 \end{pmatrix} - \mathrm{diag}\begin{pmatrix} z \\ z \end{pmatrix}\right] \cdot \begin{pmatrix}\xi \\ \xi\end{pmatrix}, \begin{pmatrix}\xi \\ \xi\end{pmatrix} \right\rangle \\
&= 2\langle M\xi, \xi\rangle - 2\sum_{v \in S} z_v
\end{aligned}$$

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- To do: Show that $|W| \geq (1-\varepsilon)n$.
- Let $S = \{v : z_v < 0\}$ and $\xi = \mathbf{1}_S$.
- Hence,

$$\sum_{v \in S} |z_v| \leq \frac{\varepsilon n^2 p}{64} + |\langle M\xi, \xi\rangle| \leq \frac{\varepsilon n^2 p}{32},$$

because $|\langle M\xi, \xi\rangle| \leq \|M\|_\Box \leq \frac{\varepsilon n^2 p}{64}$.

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- To do: Show that $|W| \geq (1-\varepsilon)n$.
- Let $S = \{v : z_v < 0\}$ and $\xi = \mathbf{1}_S$.
- Since $z \perp \mathbf{1}$, we get $\sum_{v \in V} |z_v| \leq \frac{\varepsilon n^2 p}{16}$.
- Thus, $|V \setminus W| \leq \varepsilon n/2$ $\square$

Extension: general degree distributions

- Let $G = (V, E)$ be a graph with degree sequence $(d_v)_{v \in V}$.
- For $S \subset V$ define $\mathrm{vol}(S) = \sum_{v \in S} d_v$.

Discrepancy revisited

G has $\mathrm{Disc}(\varepsilon)$ if for all $S \subset V$ $\left| e(S) - \frac{\mathrm{vol}^2(S)}{2\mathrm{vol}(V)} \right| \leq \varepsilon \mathrm{vol}(V)/2.$

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Intuition

$\frac{\mathrm{vol}^2(S)}{2\mathrm{vol}(V)}$=*expected* number of edges in a *random* graph.

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The normalized Laplacian...

...of G is the matrix $L = (\ell_{vw})_{v,w \in V}$ with

$$\ell_{vw} = \begin{cases} 1 & \text{if } v = w \text{ and } d_v \geq 1, \\ -1/\sqrt{d_v d_w} & \text{if } v, w \text{ are adjacent,} \\ 0 & \text{otherwise;} \end{cases}$$

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Eigenvalues revisited

G has $\mathrm{ess\text{-}Eig}(\varepsilon)$ if there is $W \subset V$, $\mathrm{vol}(W) \geq (1-\varepsilon)\mathrm{vol}(V)$ such that

$$1 - \varepsilon \leq \lambda_2(L_W) \leq \lambda_{\max}(L_W) \leq 1 + \varepsilon.$$

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Theorem (Alon, ACO, Han, Kang, Rödl, Schacht 2010)

There is a constant $\gamma > 0$ such that $\mathrm{Disc}(\gamma\varepsilon^2)$ implies $\text{ess-Eig}(\varepsilon)$.

Summary

- A spectral characterization of *discrepancy*.
- Off-spin: an eigenvalue *minimization* problem.
- Extends to *general degree distributions*.
- Corresponding *regular partitions* approximate a given graph by *'a few'* quasi-random ones.
- They exist on *bounded* graphs.