Scaling limit of the Erdős-Rényi random graph and associated random walk

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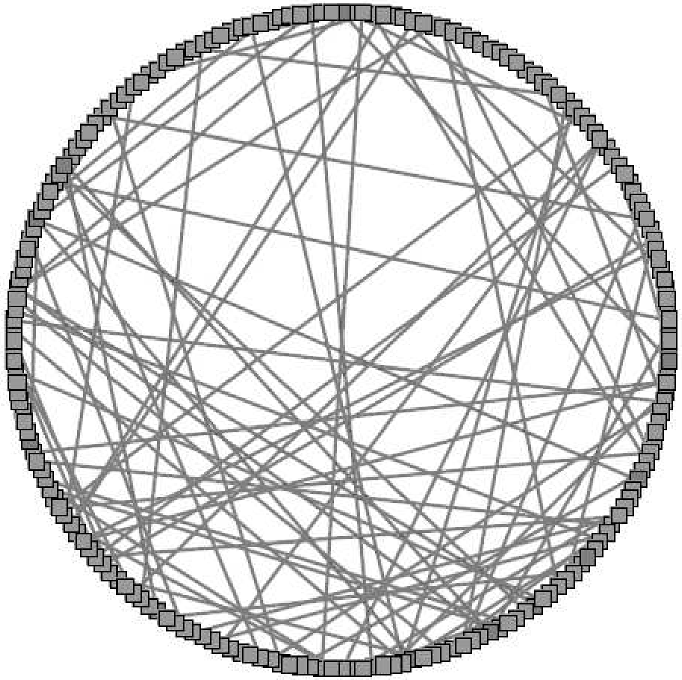
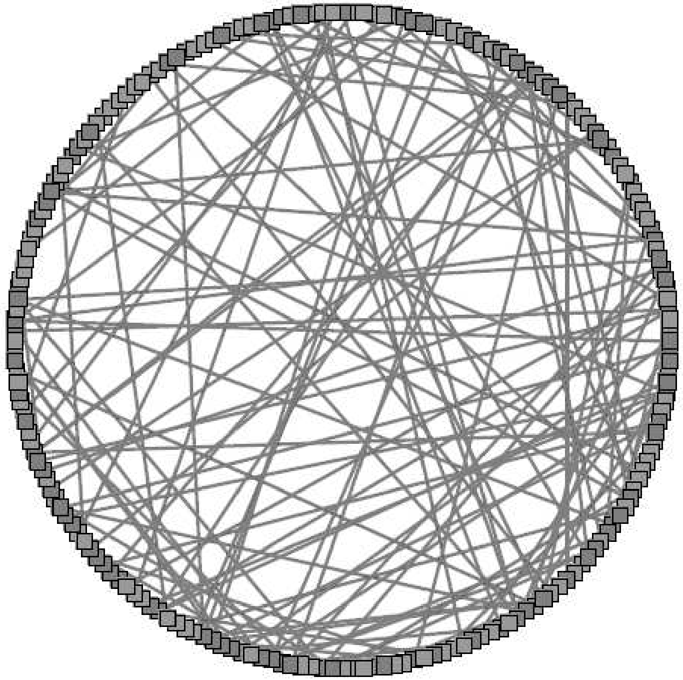
ERDŐS-RÉNYI RANDOM GRAPH

$G(n,p)$ is obtained via bond percolation with parameter p on the complete graph with n vertices. Often it is convenient to parameterise $p = c/n$.

e.g. $n = 200$, $c = 0.8$



$n = 200$, $c = 1.2$



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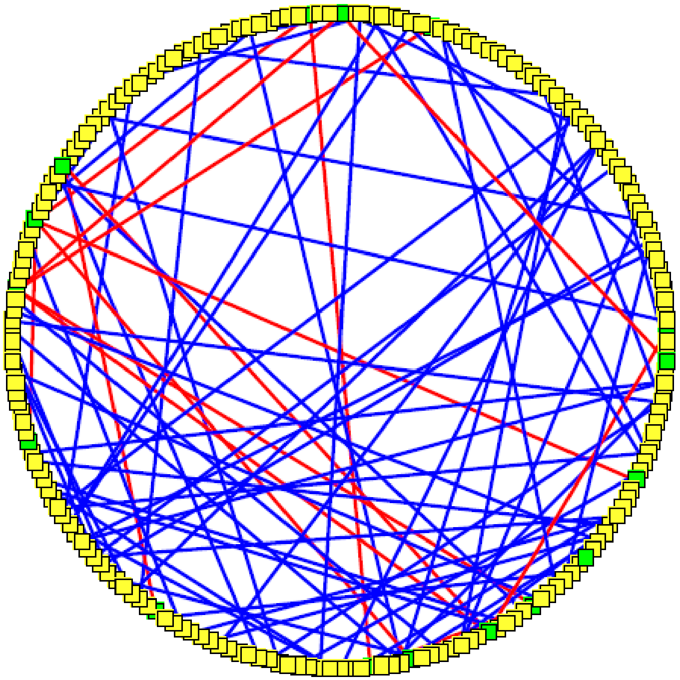

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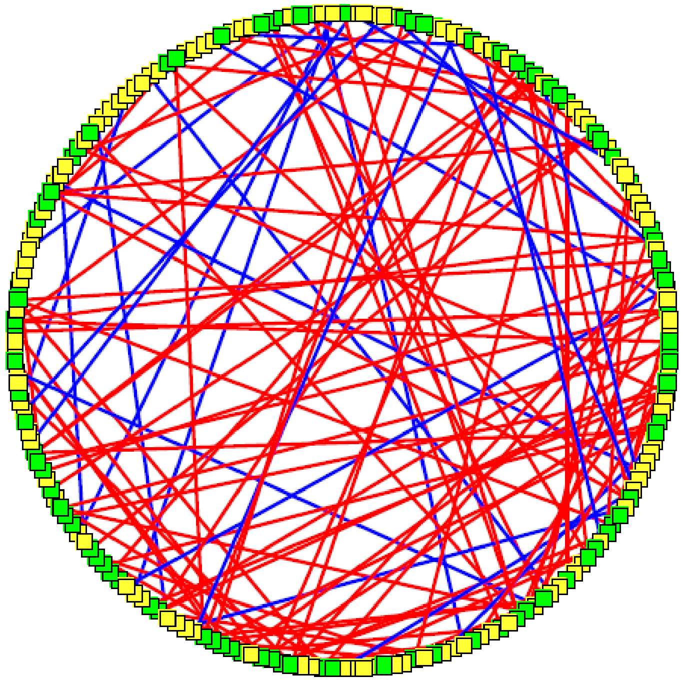

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PHASE TRANSITION

For $c > 1$, the largest connected component $\mathcal{C}_1^n$ has $\Theta(n)$ vertices, and diameter of $\Theta(\ln n)$.

For $c < 1$, $\mathcal{C}_1^n$ has $\Theta(\ln n)$ vertices.

Branching process intuition: A particular vertex, v say, has

$$\mathrm{Bin}(n-1, c/n) \approx \mathrm{Po}(c)$$

neighbours. Iterating this, those vertices within a graph distance k of v are approximately the first k generations of a $\mathrm{Po}(c)$ branching process (for k not too big).

SCALING LIMIT AT CRITICALITY

[Aldous] In the critical case, when $p = n^{-1}$, the largest connected component has size $\Theta(n^{2/3})$. Moreover, $\mathcal{C}_1^n$ also has a $\Theta(1)$ surplus.

NB. The surplus of a component is equal to $\#E - (\#V - 1)$, which is the number of edges more than a spanning tree it has.

[Addario-Berry, Broutin, Goldschmidt] Considering the largest connected component as a metric space,

$$n^{-1/3}\mathcal{C}_1^n \to \mathcal{M}_1,$$

where $\mathcal{M}_1$ is a random metric space.

RANDOM WALK ON CRITICAL RANDOM GRAPH

For each fixed realisation of $\mathcal{C}_1^n$, we can define a corresponding discrete time simple random walk

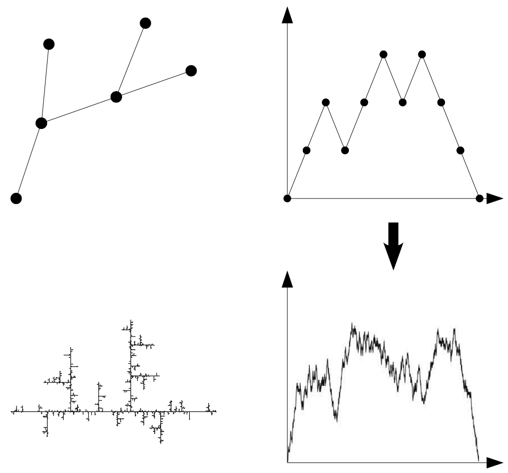
$$\left(X_t^{\mathcal{C}_1^n}\right)_{t\geq 0}.$$

Given the results of the previous slide, an obvious next step is to determine how this process should be rescaled to yield a diffusion on the metric space scaling limit $\mathcal{M}_1$ (if indeed it can be).

Plan:
- Description of metric space scaling limit $\mathcal{M}_1$.
- Random walk scaling limit.
- Asymptotic properties of random walks.

SCALING RANDOM TREES

n-vertex ordered graph tree, $T_n \leftrightarrow$ contour process, rescaled by $n^{-1/2}$ in space and $(2n-2)^{-1}$ in time



CRT, $\mathcal{T} \leftarrow$ normalised Brownian excursion

Examples:

- Conditioned finite variance branching processes [Aldous].
- Combinatorial random trees [Aldous].
- Connections with critical percolation clusters in high dimensions [Hara/Slade].

CONDITIONING $\mathcal{C}_1^n$ ON ITS SIZE

For $m \in \mathbb{N}$, can construct $\mathcal{C}_1^n | \{\#\mathcal{C}_1^n = m\}$ as follows: first, choose an m-vertex random labelled tree T_m^p according to

$$\mathbf{P}(T_m^p = T) \propto (1-p)^{-a(T)},$$

where $a(T)$ is the number of extra edges 'permitted' by T. Then, add extra edges independently with probability p to form G_m^p.

If G is a connected graph with depth-first tree T and surplus s,

$$\begin{aligned} \mathbf{P}(G_m^p = G) &\propto (1-p)^{-a(T)} p^s (1-p)^{a(T)-s} = (p/(1-p))^s \\ &\propto p^{m-1+s}(1-p)^{\binom{m}{2}-m+1-s} = \mathbf{P}(G(m,p) = G). \end{aligned}$$

Finally, observe $\mathcal{C}_1^n | \{\#\mathcal{C}_1^n = m\} \sim G(m,p) | \{G(m,p) \text{ connected}\}$.

TILTING VIA THE EXCURSION AREA

In the discrete setting, the ‘permitted’ extra edges correspond to lattice points under the depth-first walk of the graph tree; the total number of them is (nearly) the area below this function.

In the continuous setting, an analogous construction of $\mathcal{M}_1$ is possible: first, choose a random excursion $\tilde{e}$ according to the tilted measure

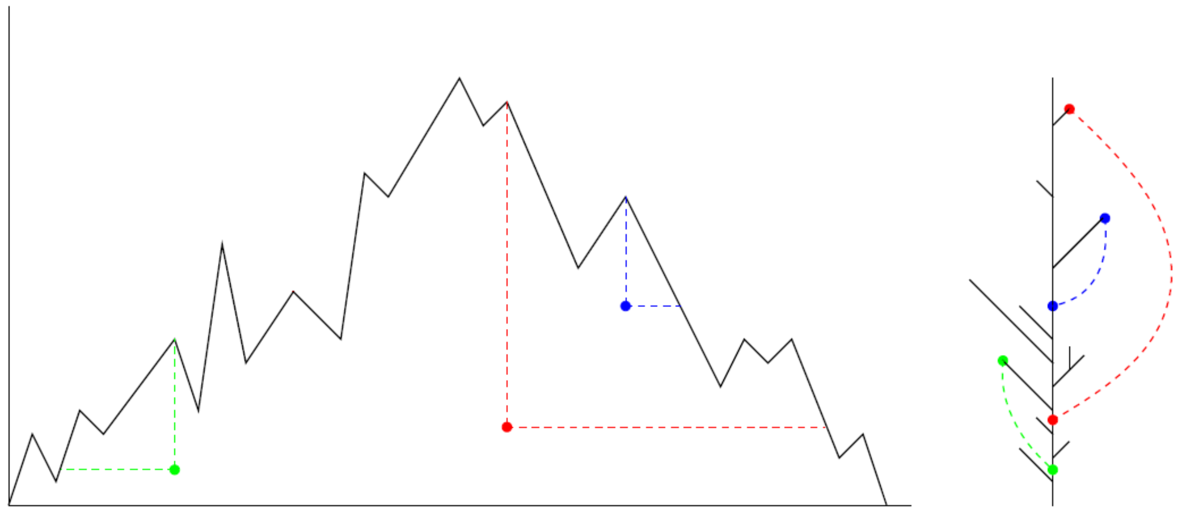
$$\mathbf{P}(\tilde{e} \in df) \propto \mathbf{P}(e \in df)\exp\left(\int_0^1 f(t)dt\right),$$

where e is the normalised Brownian excursion.

Define $\tilde{\mathcal{T}} := \mathcal{T}_{\tilde{e}}$.

POINT PROCESS DESCRIBING CONNECTIONS

Let $\mathcal{P}$ be a unit intensity Poisson process on the plane. Points of $\mathcal{P}$ that lie below the excursion $\tilde{e}$ describe pairs of vertices to 'glue' together.

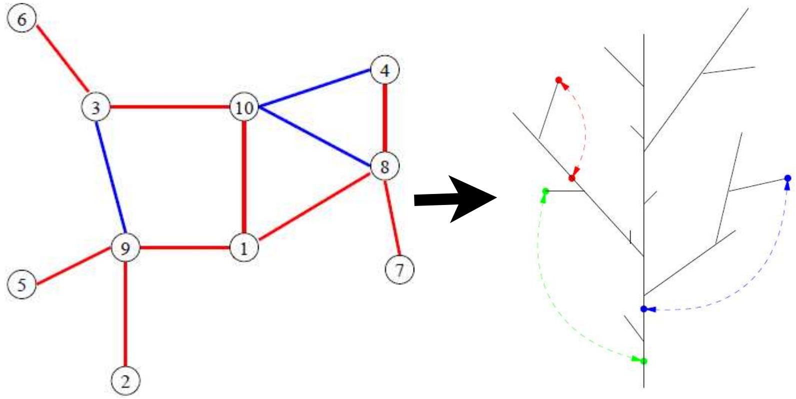


Picture produced by Christina Goldschmidt.

A point at (t, x) identifies the vertex v at height $\tilde{e}(t)$ with the vertex at distance x along the path from the root to v.

CRITICAL RANDOM GRAPH SCALING LIMIT [Addario-Berry, Broutin, Goldschmidt]

The random metric space scaling limit $\mathcal{M}_1$ of the rescaled largest connected component of the critical random graph $n^{-1/3}\mathcal{C}_1^n$ is defined, up to a random scaling factor Z_1, by gluing of pairs of vertices of $\tilde{\mathcal{T}}$ according to $\mathcal{P}$.



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SCALING LIMIT FOR RANDOM WALKS ON GALTON-WATSON TREES

Let $(T_n)_{n\geq 1}$ be a family of Galton-Watson trees such that:

T_n - has critical (mean 1), finite variance offspring distribution.
- is conditioned to have n vertices.

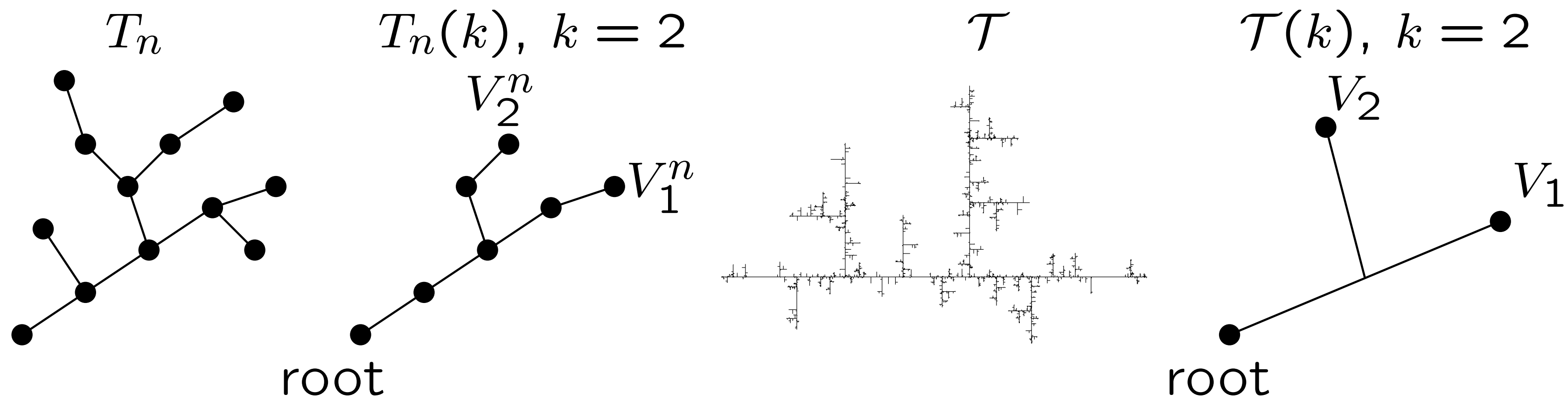
We can randomly isometrically embed all the graph trees $(T_n)_{n\geq 1}$ and the continuum random tree $\mathcal{T}$ into a common metric space, E say, so that

$$\left(n^{-1/2} X^{T_n}_{\lfloor tn^{3/2} \rfloor}\right)_{t\geq 0} \to \left(X^{\mathcal{T}}_t\right)_{t\geq 0},$$

in distribution in $D(\mathbb{R}_+, E)$.

PROOF IDEA

Let $T_n(k)$ be minimal sub-tree of T_n spanning root and k uniform random vertices.



Similarly, given $\mathcal{T}$ and $\mu^{\mathcal{T}}$, let $\mathcal{T}(k)$ be minimal sub-tree of $\mathcal{T}$ spanning root and k $\mu^{\mathcal{T}}$-random vertices.

Step 1: Show processes on graph subtrees converge for each k.
Step 2: Show these are close to processes of interest as $k \to \infty$.

INTUITION FOR TIME SCALING FACTOR

For a simple random walk on a graph tree T, an elementary calculation yields the commute time identity

$$\mathbf{E}_x\tau_y + \mathbf{E}_y\tau_x = 2\#E(T)d_T(x,y).$$

(In fact, with resistance distance this holds for all graphs).

In particular, we might reasonably expect that

$$\begin{aligned} \text{time scaling} &= \text{mass scaling} \times \text{distance scaling} \\ &= n \times n^{1/2} \\ &= n^{3/2}. \end{aligned}$$

ADAPTING TO RANDOM GRAPHS

Essentially the same argument works:
- select subgraphs consisting of a finite number of line segments.
- prove convergence on these.
- show these are close to processes of interest.

For the largest component of the critical random graph, the time scaling becomes

$$\begin{aligned} \text{time scaling} &= \text{mass scaling} \times \text{distance scaling} \\ &= n^{2/3} \times n^{1/3} \\ &= n. \end{aligned}$$

NB. This has been seen before in the mixing time of the random walk [Nachmias/Peres].

SCALING LIMIT FOR RANDOM WALKS ON CRITICAL RANDOM GRAPHS

Let $\mathcal{C}_1^n$ be the largest component of random graph at criticality window, $p = n^{-1}$, then

$$\left(n^{-1/3} X^{\mathcal{C}_1^n}_{\lfloor tn \rfloor}\right)_{t \geq 0} \to \left(X_t^{\mathcal{M}_1}\right)_{t \geq 0},$$

in distribution in both a quenched (for almost-every environment) and annealed (averaged over environments) sense.

SOME COROLLARIES

Convergence of maximum commute time:

$$N^{-1} \max_{x,y \in \mathcal{C}_1^n} (\mathbf{E}_x \tau_y + \mathbf{E}_y \tau_x) \to 2 Z_1 \mathrm{diam}_R \mathcal{M}_1,$$

in distribution.

Convergence of mixing times: for fixed $\varepsilon > 0$,

$$N^{-1} t_{\mathrm{mix}}(\mathcal{C}_1^n) \to t_{\mathrm{mix}}(\mathcal{M}_1),$$

in distribution, where

$$t_{\mathrm{mix}}(\mathcal{C}_1^n) := \inf \left\{ t : \frac{1}{2} \sum_{x \in \mathcal{C}_1^n} \left| \mathbf{P}_\rho (X_t^{\mathcal{C}_1^n} = x) - \pi^n(x) \right| \leq \varepsilon \right\}.$$