A NEW ν - METRIC IN CONTROL THEORY

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Outline

(1) What is the metric on ?

$$d_\nu : X \times X \longrightarrow [0, \infty)$$

X = set of "unstable control systems"

(2) Why is it needed ?

Robust stabilization problem

(3) The "classical" ν-metric

G. Vinnicombe, 1993

(4) Our extension

(Linear, deterministic) Control Theory

$u : [0, \infty) \longrightarrow \mathbb{R}$ input

$y : [0, \infty) \longrightarrow \mathbb{R}$ output

$$\hat{y}(s) = g(s)\, \hat{u}(s)$$

transfer function

$$\left(g(s) = \frac{1}{s-1}, \quad g(s) = \frac{1}{s+1}, \quad g(s) = e^{-s} \frac{s}{s-1}, \quad \ldots \right)$$

Stable system

"nice" inputs $\longmapsto$ nice outputs

Classes of stable transfer functions

(1) $\quad RH^\infty := H^\infty \cap \mathbb{C}(s)$ $\qquad \left(u \in L^2[0,\infty) \Rightarrow y \in L^2[0,\infty) \right)$

H^∞

bounded and holomorphic in $\mathbb{C}_{>0}$

Examples : $\frac{1}{s+1}$, $\frac{s-1}{s+1}$, ...

(2) $\quad \mathcal{A}^+ := \left\{ \hat{\mu} : \begin{array}{l} \mu \text{ is a complex Borel measure on } \mathbb{R} \\ \text{such that } \operatorname{supp}(\mu) \subset [0,\infty), \text{ and} \\ \text{without a singular nonatomic part} \end{array} \right\}$

$u \in L^p[0,\infty) \Rightarrow y \in L^p[0,\infty) \quad (1 \le p \le \infty)$; Example : $e^{-s} \frac{s}{s+1}$.

(3) $\quad A(\mathbb{D}^n) \qquad$ polydisk algebra

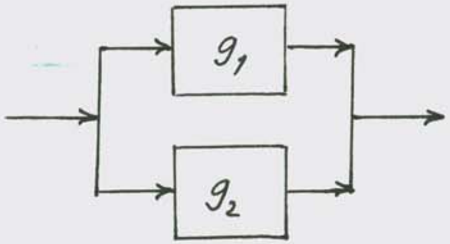
$\vdots$

Abstract approach

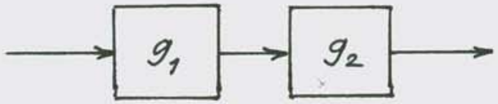
Stable control systems

R = ring of stable control systems

Why ring?



$g_1 + g_2$



$g_1 g_2$

Unstable control systems

$$\mathbb{F}(R) \text{ `='} \left\{ \frac{n}{d} : n, d \in R,\ d \neq 0 \right\}$$

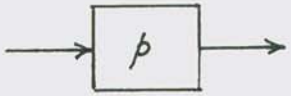
Example: $R = RH^\infty$

$$g(s) = \frac{1}{s-1} \notin RH^\infty$$

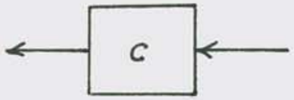
$$= \frac{\frac{1}{s+1}}{\frac{s-1}{s+1}} \in \mathbb{F}(RH^\infty).$$

Stabilization problem

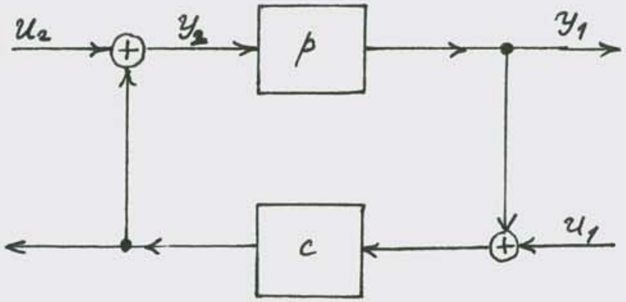
Given a $p \in \mathbb{F}(R)$



find a $c \in \mathbb{F}(R)$



such that their interconnection is stable, i.e.,



$$H(p,c) := \begin{bmatrix} \frac{-pc}{1-pc} & \frac{p}{1-pc} \\ \frac{-c}{1-pc} & \frac{1}{1-pc} \end{bmatrix} \in R^{2\times 2}$$

Closed loop transfer function

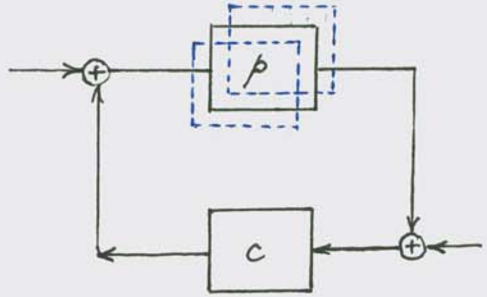
Solution to the stabilization problem.

$p \in \mathbb{F}(R)$ has a coprime factorization if $p = \frac{n}{d}$ $(n, d \in R,\ d \neq 0)$ and $\exists\ x, y \in R$ s.t. $nx + dy = 1$.

(Then $c := -\frac{x}{y}$ stabilizes p.)

Robust stabilization problem

In reality p is not known exactly ; e.g. $p = e^{-s\boxed{T}} \frac{s}{s-\boxed{a}}$



Want c to stabilize not only p, but all $\tilde{p}$s "near" p.

What is an appropriate notion of closeness between unstable plants?

Want : d which (1) is a metric on {stabilizable plants}

(2) is easy to compute

(3) makes stabilizability a robust property of the plant.

Classical ν-metric d_ν (G. Vinnicombe, 1993)

$R = RH^\infty \subset C(\mathbb{T})$

$X = \mathbb{F}(RH^\infty)$ $\quad p = \frac{n}{d}$ normalized coprime factorization $\quad n, d \in RH^\infty,\ d \neq 0$

$\exists\, x, y \in RH^\infty$ s.t.

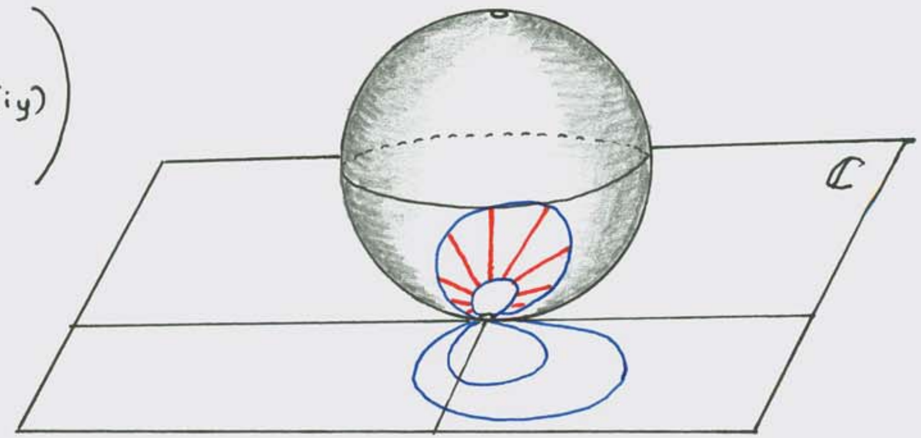
$nx + dy = 1$

$|n|^2 + |d|^2 = 1$ on $i\mathbb{R}$.

For $p_1, p_2 \in X$,
$$d_\nu(p_1,p_2) := \begin{cases} \| n_2 d_1 - n_1 d_2 \|_\infty & \text{if } n_1\bar{n}_2 + d_1\bar{d}_2 \in \operatorname{inv} C(\mathbb{T}) \\ & \text{and } \omega(n_1\bar{n}_2 + d_1\bar{d}_2) = 0, \\ 1 & \text{otherwise} \end{cases}$$

If $d_\nu(p_1,p_2) < 1$, then $d_\nu(p_1,p_2)$ = chordal distance.

$$d_\nu(p_1,p_2) = \sup_{y\in\mathbb{R}} \left(\begin{array}{l} \text{chordal distance} \\ \text{between } p_1(iy) \text{ and } p_2(iy) \\ \text{on Riemann sphere} \end{array} \right)$$



Why winding number constraint? Why not just use the chordal metric?

Stabilizability is not a robust property in the chordal metric.
($p \in RH^\infty$ is stable and is stabilized by $c = 0$;
but every neighbourhood of p in the chordal metric has unstable plants.)

Stability margin $\qquad \mu_{p,c} := \begin{cases} \dfrac{1}{\|H(p,c)\|_\infty} & \text{if } p \text{ stabilized by } c \\ 0 & \text{otherwise.} \end{cases}$

large $\mu_{p,c} \Rightarrow$ more stable ; better performance

$\mu_{p,c} > 0 \Leftrightarrow p$ stabilized by c.

Theorem (Vinnicombe, 1993) $\qquad \mu_{\tilde{p},c} \geq \mu_{p,c} - d_\nu(p, \tilde{p})$

What if $R \neq RH^\infty$?

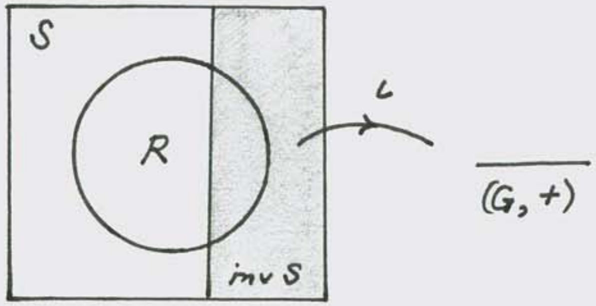
For example $R = \mathcal{A}^+$?

$$d_\nu(p_1, p_2) = ?$$

$$e^{-sT_1} \frac{s}{s-a_1} \qquad e^{-sT_2} \frac{s}{s-a_2}$$

Extension of the ν-metric

Abstract set-up:



R commutative integral domain with identity

S commutative complex semisimple Banach algebra with an involution $\cdot^*$ and with identity

$\text{inv}\, S :=$ set of invertible elements of S

$(G, +)$ Abelian group with identity 0

Index function $\iota : \text{inv}\, S \longrightarrow (G, +)$ s.t

(I1) $\iota(ab) = \iota(a) + \iota(b)$

(I2) $\iota(a^*) = -\iota(a)$

(I3) ι is locally constant (G has discrete topology)

(I4) $x \in R \cap (\text{inv}\, S)$ invertible in R iff $\iota(x) = 0$.

What is the extension of d_ν?

$$X := \{ p \in \mathbb{F}(R) : \quad p \text{ has a normalized coprime factorization} \}$$

$p = \dfrac{n}{d}$ s.t. (1) $n, d \in R$, $d \neq 0$

(2) $\exists\, x, y \in R$ s.t. $nx + dy = 1$

(3) $n^* n + d^* d = 1$ in S.

For $p_1, p_2 \in X$,
$$d_\nu(p_1, p_2) := \begin{cases} \|n_2 d_1 - n_1 d_2\|_\infty & \text{if } n_1 n_2^* + d_1 d_2^* \in \operatorname{inv} S \text{ and } \\ & \iota(n_1 n_2^* + d_1 d_2^*) = 0, \\ 1 & \text{otherwise} \end{cases}$$

$\|\cdot\|_\infty$? $\quad M(S)$ = maximal ideal space of the Banach algebra S

$x \in S$; $\quad \hat{x} \in C(M(S); \mathbb{C})$ Gelfand transform

$\hat{x}(\varphi) := \varphi(x) \quad (\varphi \in M(S))$

$$\|x\|_\infty := \sup_{\varphi \in M(S)} |\hat{x}(\varphi)| .$$

Theorem 1 d_ν is a metric on X.

Theorem 2 $\mu_{\tilde{p},c} \geqslant \mu_{p,c} - d_\nu(p,\tilde{p})$.

Here $\mu_{p,c} := \dfrac{1}{\|H(p,c)\|_\infty}$ if p is stabilized by c.

Examples

(1) $R = RH^\infty$

$S = C(\mathbb{T})$

$G = \mathbb{Z}$

ι = winding number $\omega : \operatorname{inv} C(\mathbb{T}) \longrightarrow \mathbb{Z}$

Then d_ν = classical ν-metric.

Also $R = A(\mathbb{D})$, $W^+(\mathbb{D})$, $\widehat{L^1(0,\infty)} + \mathbb{C}$, $\ldots$.

$$(2) \qquad R = \mathcal{A}^+ = \left\{ \hat{\mu} : \begin{array}{l} \mu \text{ complex Borel measure on } \mathbb{R}, \\ \operatorname{supp} \mu \subset [0,\infty), \text{ without singular nonatomic part} \end{array} \right\}$$

$$= \left\{ \hat{f}_a + \sum_{k \geq 0} f_k e^{-\cdot t_k} : \begin{array}{l} f_a \in L^1[0,\infty), \ (f_k)_{k\geq 0} \in \ell^1, \\ t_0 = 0 < t_1, t_2, t_3, \dots \end{array} \right\}$$

$$S = \mathcal{A} = \left\{ \hat{f}_a + \underbrace{\sum_{k \in \mathbb{Z}} f_k e^{-\cdot t_k}}_{F_{AP}} : f_a \in L^1(\mathbb{R}), \ (f_k)_{k \in \mathbb{Z}} \in \ell^1 \right\}$$

$$G = \mathbb{R} \times \mathbb{Z}$$

$$\iota \qquad \iota(F) = \iota(\hat{f}_a + F_{AP}) = \left(w_{av}(F_{AP}), \ w\left(1 + F_{AP}^{-1} \hat{f}_a\right)\right) \qquad F \in \operatorname{inv} \mathcal{A}.$$

Example
$$d_\nu\left(e^{-sT} \frac{s}{s-a_1}, \ e^{-sT} \frac{s}{s-a_2} \right) = \frac{|a_1 - a_2|}{\sqrt{2}\,(a_1 + a_2)} ;$$

$$d_\nu\left(e^{-sT_1} \frac{s}{s-a}, \ e^{-sT_2} \frac{s}{s-a} \right) = 1 .$$

(3) $R = A(\mathbb{D}^n)$ polydisk algebra

$S = C(\mathbb{T}^n) \times C(\mathbb{T})$

$f \in A(\mathbb{D}^n)$

$f_d : \overline{\mathbb{D}} \to \mathbb{C}$

$f_d(z) := f(z, \dots, z) \quad (z \in \overline{\mathbb{D}})$

$f_d \in C(\mathbb{T})$

$f|_{\mathbb{T}^n} \in C(\mathbb{T}^n)$

$$A(\mathbb{D}^n) \longrightarrow C(\mathbb{T}^n) \times C(\mathbb{T})$$
$$f \longmapsto (f|_{\mathbb{T}^n}, f_d)$$

$G = \mathbb{Z}$

$\iota = \big((g, h) \mapsto \omega(h) \big)$