Suppression of bad news in markets:
Equilibrium analysis of correlated optimal data censors

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Filtering with selectively censored data (news)

Averaging, bandwagon and quality effects from correlation
Motivation: a disclosure game

1. At the first ‘ex-ante date’ Nature selects a probabilistic strategy (‘action’) \( X \) from a known space of actions. Actions are represented by a family of distributions.

2. At the interim date (a known later date), as a result of an independent draw with some probability \( q \), this action is observed noisily by an agent (‘observer’).

3. At the ‘terminal date’ (a still later date), there is a publicly observed vector of outcomes \( F_i \) dependent on the action \( X \).

The ‘public’ comprises the agents and a disjoint set of principals (e.g. investors).
At the interim date a **pre-assessment**/evaluation of the outcome $F_i$ may be formed from the observation.

What is the disclosure game? What is news? Answer ($T$ for a transform):

$$T_i = T(X, Y_i) = \text{private signal about } X \text{ involving the observer’s noise } Y_i,$$

received at the ‘interim date’ prior to public (common) knowledge of $X$ at the terminal date.

The effect of $X$ is to yield an outcome, e.g. via

$$F_i := f_i \cdot T(X, Z_i) = \text{effect of } X \text{ with uncertainties from } Z_i.$$  

Leads to a public interim re-assessment of any disclosed signals from the agents.

This could be the evaluation of some underlying complex system based on partial noisy observation.
The ex-ante assessment is modelled as

$$\mathbb{E}[F_i] = f_i \cdot \mathbb{E}[T(X, Z_i)].$$

**Game objective:** maximization at the interim date of the re-assessment of $F_i$.

**Disclosure option:** opportunity to suppress the reporting of the signal $T_i$, if

$$\mathbb{E}[F_i \mid \text{report } T_i] < \mathbb{E}[F_i \mid \text{no report/no disclosure}]$$

equivalently, on using $F_i = f_i \cdot T(X, Z_i)$,

$$\mathbb{E}[T(X, Z_i) \mid \text{report } T_i] < \mathbb{E}[T(X, Z_i) \mid \text{ND}],$$

*assuming* there is a positive probability that the observer is unable to observe $T_i$. 
A basic question: When is a censor $\gamma$ optimal?

Answer: it is the ‘indifferent censor’ $\gamma$: indifference as to reporting when $T = \gamma$.

Note for later that

$$\mathbb{E}[T|\text{ND using } \gamma] := \frac{(1 - q)\mathbb{E}[T] + q\mathbb{E}[T \cdot 1_{T<\gamma}]}{(1 - q) + q\mathbb{E}[1_{T<\gamma}]}$$

We assume:

(i) $0 < q < 1$ and $q$ is public (common) knowledge,

(ii) the observer does not lie, and cannot directly announce credibly absence of an observation.
The Equity-valuation model

Take \( X = Y_0, Y_i, Z_i \) all log-normal with unit-mean, so in stochastic-exponential format:

\[
Y_i = e^{\sigma_i v_i - \frac{1}{2} \sigma_i^2}, \quad \text{for } i = 0, 1, 2, \ldots, n,
\]

with \( v_i \) all independent, standard normal, and

\[
T_i = X Y_i \quad \text{and} \quad F_i = f_i X Z_i.
\]

The observers are called firm-managers and identified with \( Y_i \).

Easy to include individual dependency loading index \( \alpha_i \) of firm \( i \) on \( X \):

\[
T_i = X Y_i \quad \text{and} \quad F_i = f_i X^{\alpha_i} Z_i.
\]
Corollaries of the model:

1. \( T_i = e^{\sigma_0 w_i - \frac{1}{2}\sigma_0^2}, \) with \( \sigma_0 w_i = \sigma_0 v_0 + \sigma_1 v_i \) and \( \sigma_0^2 = \sigma_0^2 + \sigma_1^2. \)

So \( v_0 \) is the only source of all the correlation.

Useful to refer to \( p_i = 1/\sigma_i^2, \) the \textbf{precision} of \( Y_i. \)

2. 

\[ \mathbb{E}[F_i|\text{data}] = f_i \mathbb{E}[X|\text{data}]. \]
Noiseless Dye Cutoff: the Censor equation

For $T = X$, i.e. true value rather than a noisy signal is observed

*Dye indifference* equation, or *Dye Censor Equation* is

$$\gamma = \mathbb{E}[X|ND(\gamma)].$$

It is equivalent to:

$$\lambda(m_X - \gamma) = \mathbb{E}[(\gamma - X)^+], \text{ with odds } \lambda = \frac{1 - q}{q},$$

where

$$\mathbb{E}[(\gamma - X)^+] = \int (\gamma - t)^+ dF_X(t) = \int_{t\leq \gamma} F_X(t) dt.$$
Alternative characterizations of the Dye censor: Minimized valuation consistent with available information:

$$\gamma = \arg \min_{\gamma} \mathbb{E}[X | ND(\gamma)].$$

No-arbitrage valuation: $\gamma$ such that $\mathbb{E}[X]$ values $X$ consistently with the possibility of further $\gamma$-censored information becoming available later.
The hemi-mean function

This put-payoff is valued under an expectation, and we call

\[ H_X(\gamma) := \int_{t \leq \gamma} F_X(t) dt, \]

the *hemi-mean function* of \( X \). Since \( H'' = f_X \geq 0 \) that itself is an increasing convex function of \( \gamma \) and so has a smoothed out hockey-stick shape: it looks like the valuation of a call (dual to the put). Examples below! Dye equation standardizes to:

\[ \lambda(1 - \gamma) = H_X(\gamma). \]
The Normal Censor

The pink/red intersection identifies the normal Dye censor (here $\lambda = 1$).

A corresponding dual call payoff $(X - x)^+$ is in green.
**Location-scale cutoff standardization theorem.** For the location and scale family of distributions $\Phi_F\left(\frac{x - \mu}{\sigma}\right)$, with mean $\mu$ and variance $\sigma^2$, the Dye cutoff $\gamma(\mu, \sigma, \lambda)$ satisfies

$$\gamma(\mu, \sigma, \lambda) = \mu - \sigma \xi(\lambda).$$

So:

$$p_{\text{Low}} < p_{\text{High}} \implies \gamma(p_{\text{Low}}) < \gamma(p_{\text{High}}),$$

i.e. more disclosure from the low-precision firm.

This will be altered by the presence of additional information sources.
**Location-scale cutoff standardization theorem.** Let \( \Phi_F(x) \) be an arbitrary zero-mean, unit-variance, cumulative distribution for \( F \) defined on \( \mathbb{R} \). For the location and scale family of distributions \( \Phi_F(x-\mu)/\sigma \), with mean \( \mu \) and variance \( \sigma^2 \), the Dye cutoff \( \gamma(\mu, \sigma, \lambda) \) satisfies

\[
\gamma(\mu, \sigma, \lambda) = \mu - \sigma \xi(\lambda), \text{ where } \lambda = \frac{1 - q}{q},
\]

so that

\[
\xi(\lambda) = -\gamma(0, 1, \lambda) < 0
\]

is the cutoff when standardizing to zero mean and unit variance and is a function only of the odds \( \lambda \). The standardized cutoff \( \xi(\lambda) \) is a convex and decreasing function of \( \lambda \) satisfying

\[
\lambda = H_F(-\xi)/\xi,
\]

where \( H_F(x) = \int_{-\infty}^{x} \Phi_F(t)dt \) is the corresponding hemi-mean function.
Black-Scholes Censor

The red-pink intersection identifies the log-normal Dye censor (for $\lambda = 1$).
Green indicates the dual call payoff.
Noisy Dye Cutoff: Estimator-Censor equation

For $T = T(X, Y)$, put

$$\mu_X(t) : = \mathbb{E}[X | T = t],$$

the regression function,

$$S : = \mu_X(T),$$

the estimator, or $X^{\text{est}}$.

Since

$$\mathbb{E}[F] = f_i \mathbb{E}[X],$$

then, provided $\mu_X(.)$ is strictly increasing, the *Dye Equation* holds in the form:

$$\mu_X(\gamma_T) = \gamma_S = E[S | ND(\gamma_S)],$$

where $\gamma_S$ is the censor for $S$ and $\gamma_T$ is the equivalent censor for $T$. 
Equivalently, as $S$ is an unbiased estimator of $X$ one has

$$\lambda(m_X - \gamma_S) = H_S(\gamma).$$

By the conditional mean formula (tower law/iterated expectation):

$$\mathbb{E}[S] = \mathbb{E}[\mathbb{E}[X|T]] = \mathbb{E}[X] = m_X.$$

So the hemi-mean function rules OK.
Multi-Censor Equilibrium equation

One has $n$ simultaneous equations corresponding to a simultaneous interim-report date:

$$
\mathbb{E}[X|T_j] = \gamma_j \text{ for all } j = \mathbb{E}[X|ND_i(\gamma)],
$$
with $\gamma = (\gamma_1, \ldots, \gamma_n)$ and $ND_i = \text{only } i \text{ makes no disclosure}.$

We call these the *Marginal Dye equations.*
Log-normal Marginal Dye equations

Recall the Estimator version of the Dye equation:

\[ \lambda(m_X - \gamma_S) = H_S(\gamma). \]

Conditioning on the other disclosures, yields for some \( K \) and \( \kappa_i = p_i/p \)

\[ \mu_X(\gamma_1, \ldots, \gamma_n) = E[X|T_i = \gamma_i \text{ all } i] = K \gamma_1^{\kappa_1} \ldots \gamma_n^{\kappa_n}, \]

(see below). Change of random variable, and change of variable:

\[ S := \mu_X(T_1, \gamma_2, \ldots, \gamma_n), \text{ and } s = \mu_X(\gamma, \gamma_2, \ldots, \gamma_n) \]

yields a conditioned format, in which \( m_{S|\gamma_2,\ldots} \) replaces \( m_S \):

\[ \lambda(\mathbb{E}[S|\gamma_2,\ldots,\gamma_n] - s) = H_S(s|\gamma_2,\ldots,\gamma_n). \]
Principal findings for the Equity Valuation case:

**Preparatory Step.** Replace the $n$ firm-managers $Y_i$ by $n$ hypothetical observers/managers $\hat{Y}_i$ which are uncoupled – acting as though all the competitors had vanished – but with refined precision parameters

$$\kappa_i \sigma_{0i} \sqrt{1 - \rho_i^2}, \text{ with } \kappa_i := \frac{p_i}{p} \text{ and } \sigma_{0i}^2 = \sigma_0^2 + \sigma_i^2,$$

and

$$p = p_0 + ... + p_n, \text{ total precision.}$$

Here $\rho_i$ measures the dependence of $T_i$ on the remaining $T_j$ (more properly: **partial co-variance** of $w_i$ on the remaining $w_j$).
Conclusion. If the corresponding Dye censors for $\hat{T}_i = X\hat{Y}_i$ are $\hat{\gamma}_i$, then the true managers have censors $\gamma_i$ given by the weighted average:

$$
\log \gamma_i = \frac{\log g_i}{\kappa_{-i}} + \frac{1}{\kappa_0} \left( \frac{\kappa_1}{\kappa_{-1}} \log g_1 + \frac{\kappa_2}{\kappa_{-2}} \log g_2 + \ldots + \frac{\kappa_n}{\kappa_{-n}} \log g_n \right),
$$

with

$$
\kappa_{-i} = p_i/(p - p_i),
$$

and where $g_j$ is the hypothetical firm-$j$ censor.
In fact

\[ g_i = \log \left( \hat{\gamma}_{\text{LN}} \left( \lambda_i, \kappa_i \sigma_0, \sqrt{1 - \rho_i^2} \right) L_{-i} \right), \quad \lambda_i = \frac{1 - q_i}{q_i} \]

\[ L_{-i} = \exp \left( -\frac{n - 1}{2(p - p_i)} - \frac{1}{2} \right) = \exp \left( \frac{1}{2} \left( \frac{1}{p_{av,-i}} - \frac{1}{p_{av}} \right) \right), \]

where \( L_{-i} \) is a mean adjustment.
Bandwagon effect

Bandwagon Inflator Theorem. The presence of correlation increases the precision parameter of the cutoff and hence raises the cutoff:

\[ \hat{\gamma}_{LN}(\lambda_i, \sigma_0) < \hat{\gamma}_{LN}(\lambda_i, \kappa_i \sigma_0) < \hat{\gamma}_{LN} \left( \lambda_i, \kappa_i \sigma_0 \sqrt{1 - \rho_i^2} \right). \]

Proof. Clear since \( \hat{\gamma}_{LN}(\lambda, .) \) is increasing in precision, and also \( \rho_i^2 \) is increasing in \( p_i \).
Estimator-quality effect

Estimator-Quality Theorem. The mean-adjustor for firm $i$ is increasing in $p_i$ with

$$\exp\left(-\frac{1}{2(p - p_i)}\right) < L_i < \exp\left(\frac{1}{2p_{av, -i}}\right),$$

and in particular

$$L_i < L_j \iff p_i < p_j.$$  

The adjustor is a strict deflator, i.e. $L_i < 1$, iff $p_i$ is below the sector average, equivalently below the competitor average, i.e.

$$p_i < \frac{p}{n}, \text{ equivalently } p_i < \frac{p - p_i}{n - 1}.$$
Tools:

**Basic Tools:** Isomorphism. Equity a log-normal variate, but it is easy to move back and forth from log-normal to normal via the isomorphism $\exp : (\mathbb{R}, +) \to (\mathbb{R}_+, \cdot)$

Explicit Normal and Black-Scholes **put-option formulas**.

**Main Tools:** Linear regression easily computed via a Hilbert space approach: view $\mathbb{E}[..]$ as a projection and use $P$ the **precision matrix**.

**Strategy:** Uncoupling the co-dependency and solving the uncoupled censor equations via $P$. 
Some simple algebra: the precision matrix

Put

\[
P_n := \begin{bmatrix}
p_1 & p_2 & \cdots & p_n \\
p_1 & p_2 & \cdots & p_n \\
\vdots & \vdots & \ddots & \vdots \\
p_1 & p_2 & \cdots & p_n
\end{bmatrix}.
\]

and

\[
P_n(x) = P_n - xI.
\]

Recall that for \( \sigma_i^2 \) a variance parameter, \( p_i = 1/\sigma_i^2 \) is the precision parameter.

**Proposition 1.** For any \( n \), the characteristic function of the matrix \( P_n \) is

\[
\det(P_n - xI) = (-1)^n x^{n-1} (x - p_1 - \cdots - p_n),
\]
Proof. Easy exercise. [Hint: $P_n$ has nullity $n - 1$.]

Proposition 2. For any non-zero parameter $q$ such that $p_q := q + p_1 + \ldots + p_n \neq 0$, the simultaneous system of equations

$$(P_n + qI)x = s,$$

i.e.

$$p_1x_1 + \ldots + (p_i + q)x_i + \ldots + p_nx_n = s_i,$$

has the unique solution

$$x_i = \frac{s_i}{q} + c, \text{ with } c = \frac{1}{qpq}(p_1s_1 + \ldots + p_ns_n).$$
Proof. Easily checked; by Prop. 1, \( \det(P_n + qI) = q^{n-1}(p_1 + \ldots + p_n + q) \neq 0 \), so the solution is unique. \(\square\)
*Example 1: Normal put-option formula

Notation

\[ F_X(t) := \Pr[X \leq t] \]

Cases: \( X = u \sim N(0, \sigma^2) \) normal

\[ \Phi(t) = F_u(t) := \Pr[u \leq t]. \]

with density \( \varphi(t) = \Phi'(t) \). Here

\[ \mathbb{E}[(t - X)^+] = t \Phi \left( \frac{t + \frac{1}{2} \sigma^2}{\sigma} \right) + \varphi(t/\sigma^2). \]
*Example 2: Black-Scholes put-option formula

For $X$ log-normal

$$X = e^{\sigma u - \frac{1}{2} \sigma^2} \text{ with } u \sim N(0, 1),$$

$$\mathbb{E}[(t - X^+)^+] = t \Phi \left( \frac{\log t + \frac{1}{2} \sigma^2}{\sigma} \right) - \Phi \left( \frac{\log t - \frac{1}{2} \sigma^2}{\sigma} \right).$$
Simplification:

Again use the conditional mean formula:

\[
\mathbb{E}[S|\gamma_2, \ldots, \gamma_n] = \mathbb{E}[\mathbb{E}[X|\gamma_2, \ldots, \gamma_n]|\gamma_2, \ldots, \gamma_n], \text{ defn of } S \\
\]

\[
= \mathbb{E}[\mathbb{E}[\mathbb{E}[X|T_1, \gamma_2, \ldots, \gamma_n]|\gamma_2, \ldots, \gamma_n]|\gamma_2, \ldots, \gamma_n], \text{ refine} \\
\]

\[
= \mathbb{E}[KT_1^{\kappa_1}\gamma_2^{\kappa_2}\ldots\gamma_n^{\kappa_n}|\gamma_2, \ldots, \gamma_n], \text{ apply formula} \\
\]

\[
= K\gamma_2^{\kappa_2}\ldots\gamma_n^{\kappa_n} \cdot \mathbb{E}[T_1^{\kappa_1}|\gamma_2, \ldots, \gamma_n].
\]
Theorem (Conditional hemi-mean formula).

\[ \mathbb{E}[T_1^{\kappa_1}|T_2, \ldots, T_n] = L_{-1} T_2^{\bar{h}_2^{\kappa_2}} \cdots T_n^{\bar{h}_n^{\kappa_n}}, \]

where, with \( p = p_0 + \ldots + p_n \) the total precision,

\[ L_{-1} = \exp \left( \frac{n - 1}{2(p - p_1)} \right) \exp \left( -\frac{n}{2p} \right), \text{ and } \bar{h}_j = \frac{p_j}{p - p_1}, \text{ for } j > 1. \]

Proof uses conditional mean formula and yields \( L_{-1} = K_{-1}/K \).
Uncoupling Theorem

**Uncoupling Theorem.** The substitution

\[ y_1 = \frac{\gamma_1^{\kappa_1}}{L - \gamma_2^{\kappa_2}} \cdots \gamma_{n}^{\kappa_n} \]

reduces the marginal Dye equation, namely

\[ \lambda_1 (\mathbb{E}[X|\gamma_2, \ldots, \gamma_n] - \mu_X(\gamma, \gamma_2, \ldots, \gamma_n)) = \int_{t_1<\gamma_1} \left[ \mu_X(\gamma_1, \gamma_2, \ldots, \gamma_n) - \mu_X(t_1, \gamma_2, \ldots, \gamma_n) \right] dF_{T_1}(t_1|\gamma_2, \ldots, \gamma_n), \]

to the standard form

\[ \lambda_1(1 - y_1) = H_{LN}(y_1, \kappa_1 \sigma_{01} \sqrt{1 - \rho_1^2}), \]

where \( 1 - \rho_1^2 \) is the partial covariance, or Schur complement, of \( w_1 \) given \( w_1, \ldots, w_n \).
Notational convention for shifting from LN to N

\[ \eta_i = \log Y_i + \frac{1}{2} \sigma_i^2 = \sigma_i \nu_i \] the underlying normal variate, etc
Background: a little linear regression

Proposition (Geometric weighted-average)

\[ E[X | T_1 = t_1, \ldots, T_n = t_n] = K t_1^{\kappa_1} \cdots t_n^{\kappa_n}, \text{ with } \kappa_i = \frac{p_i}{p}, \text{ and } \]

\[ K = e^{2p} = \exp \left( \frac{1}{2p_{av}} \right) t_1^{\kappa_1} \cdots t_n^{\kappa_n}, \text{ with } p_{av} := \frac{p_0 + \cdots + p_n}{n}. \]

Sketch Proof. Put \( \xi = \log X, \tau_i = \log T_i \) (+ take off constants), do classical linear regression with normal variates, transform back via \( \exp \), finally compute the constant \( K \) using the tower law.
Remarks. 1. The preceding shows why log-normals are as easy as normals.

2. The normal regression arguments need only $P$, so some simple algebra.
Reprise: a little linear regression

Lemma (Arithmetic weighted-average). One has

\[ \mathbb{E}[\xi|\tau_1, \tau_2] = \kappa_1 \tau_1 + \ldots + \kappa_n \tau_n, \text{ with } \kappa_i = \frac{p_i}{p}. \]

Proof. Method: write

\[ \xi^{\text{est}} = \mathbb{E}[\xi|\tau_1, \ldots, \tau_n] = \kappa_1 \tau_1 + \ldots + \kappa_n \tau_n. \]

By the conditional mean formula,

\[ \mathbb{E}[\tau_1 \xi^{\text{est}}] = \mathbb{E}[\tau_1 \mathbb{E}[\xi|\tau_1, \ldots, \tau_n]] = \mathbb{E}[\mathbb{E}[\tau_1 \xi | \tau_1, \ldots, \tau_n]] \\
= \mathbb{E}[\tau_1 \xi] \]
Recall, $v_i$ independent so $E[v_i v_j] = \delta_{ij}$ and

$$\tau_i = (v_0 + v_i)$$

Compute to obtain

$$\mathbb{E}[\tau_1 \xi^{\text{est}}] = \mathbb{E}[\tau_1 \xi]$$

equivalent to:

$$\kappa_1(\sigma_0^2 + \sigma_1^2) + \kappa_2\sigma_0^2 + \ldots + \kappa_n\sigma_0^2 = \sigma_0^2.$$ 

Setting $k_i = \kappa_i/p_i$, obtain

$$k_1(p_0 + p_1) + k_2p_2 + \ldots + k_np_n = 1.$$ 

More generally,

$$k_1p_1 + \ldots + k_i(p_0 + p_i) + \ldots + k_np_n = 1.$$ 

Solution now obviously: $k_i = 1/(p_0 + \ldots + p_n)$. 
Covariance: the Hilbert space view

Recall that each $w_i$ has mean-zero and that

$$
\mathbb{E}[w_i w_i] = 1, \quad \text{and} \quad \mathbb{E}[w_i w_j] = \frac{\sigma_0^2}{\sigma_{0i} \sigma_{0j}} > 0.
$$

So any combination of $w_1, \ldots, w_n$ has mean zero, i.e. they span a vector space $W$. For $w, w' \in W$ write

$$
\langle w, w' \rangle := \text{cov}(w, w') = \mathbb{E}[ww'].
$$

This is an inner product (so $W$ is a Hilbert space under $\langle ., . \rangle$) iff the following covariance matrix is non-singular

$$
Q = (\rho_{ij}) \text{ where } \rho_{ij} = \mathbb{E}[w_i w_j].
$$
It turns out that $Q$ is related to the precision matrix.

**Theorem.** *For* $p_i > 0$ *the covariance matrix is non-singular and*

$$
det Q = (p_0 + p_1) \ldots (p_0 + p_m) \det [P + p_0 I]
$$

$$
= \bar{p} p_0^{m-1} (p_0 + p_1) \ldots (p_0 + p_m).
$$
Appendix: the Schur complement: 1

Aim: find the variance of $\mathbb{E}[w_i|w_j \forall j \neq i]$. NB. Requires first to solve e.g.

$$\mathbb{E}[w_n|w_1, \ldots, w_{n-1}] = \sum_{j<n} \beta_j w_j.$$  

Answer: put $\tilde{Q}_i = Q$ omitting the $i$-th row and column; likewise, $\tilde{\rho}_i = i$-th row $(\rho_{i1}, \ldots, \rho_{in})$ omitting $i$-th entry.

The Schur complement (of $\tilde{Q}_i$ in $Q$) is given by

$$\rho_{ii} - \tilde{\rho}_i \tilde{Q}_i^{-1} \tilde{\rho}_i^T.$$
Putting

\[ \rho_i := \sqrt{\bar{\rho}_i \bar{Q}_i^{-1} \bar{\rho}_i^T}, \]

the Schur complement becomes

\[ 1 - \rho_i^2. \]

(This notation permits specialization to the \( n = 2 \) case to yield \( \bar{Q}_i = (1) \) and \( \bar{\rho}_i = (\rho) \), so that \( \rho_i = \rho = \rho_{12} \).)

The conditional distribution of \( w_i \) given all the \( w_j \) for \( j \neq i \) is normal with variance given by the Schur complement.
The Schur complement: 2

Consider the distribution of \( \mathbb{E}[T_n|T_1, \ldots, T_{n-1}] \), or equivalently that of \( E[w_n|w_1, \ldots, w_{n-1}] \). Recall that

\[
T_i = e^{\sigma_0 i w_i - \frac{1}{2} \sigma_0^2}, \quad \text{with} \quad \sigma_0 i w_i = \sigma_0 w_0 + \sigma_i v_i.
\]

Put

\[
w_n^{n-1} = \mathbb{E}[w_n|w_1, \ldots, w_{n-1}] = \sum_{j<n} \beta_j w_j.
\]

Then, by definition and by the conditional mean formula,

\[
\rho_{in} = \mathbb{E}[w_i w_n] = \mathbb{E}[w_i w_n^{n-1}] = \sum_{j<n} \beta_j \rho_{ij}.
\]

We solve the system of \( m := n - 1 \) equations for \( i < n \)

\[
\sum_{j<n} \rho_{ij} \beta_j = \rho_{in},
\]
or, in matrix form with \( \tilde{\rho}_n := (\rho_{1n}, \ldots, \rho_{n-1,n}) \)

\[
Q_{n-1}\beta = \tilde{\rho}_n,
\]

by computing explicitly \( \beta = Q_{n-1}^{-1}\tilde{\rho}_n \). Here we have denoted the principal submatrix of the covariance matrix \( Q_n \) by:

\[
Q_{n-1} = (\rho_{ij})_{i,j<n}.
\]

Using the precision matrix one may easily find the \( \beta_j \) explicitly. WE have an important corollary.
Monotonicity Theorem (Own precision refined by presence of others) The Schur complement

\[ 1 - \rho_n^2, \]

corresponding to conditioning \( w_n \) on \( w_1, \ldots, w_{n-1} \) as a factor in the conditional variance, acts to increase the precision; increasing the precision of the competitors refines one’s own conditional precision. Indeed, one has the explicit formula with \( m = n - 1 \) and \( \bar{p} = p - p_n = p_0 + \ldots + p_{n-1}, \)

\[
\rho_n^2 = \frac{p_m}{p_0\bar{p}(p_0 + p_m)} \left[ \sum_{i=1}^{m} p_i(\bar{p} - p_i) + \sum_{i<j\leq m} (p_i + p_j) \sqrt{\frac{p_i p_j}{(p_0 + p_i)(p_0 + p_j)}} \right],
\]

which is increasing in \( p_i \) for each \( i < n \), and so the Schur complement itself decreases with \( p_i \).
In fact one has:

**Theorem 1.** Provided all the precisions $p_i$ are finite and positive, the regression equations

$$E[w_n|w_1, \ldots, w_{n-1}] = \beta_1 w_1 + \ldots + \beta_{n-1} w_{n-1},$$

which are equivalent to the solution of the system $Q_{n-1} \beta = \tilde{\rho}_n$, have non-singular matrix $Q_{n-1}$ and the equivalent system of equations, for $i = 1, 2, \ldots, m =: n - 1$,

$$\rho_{i1} \beta_1 + \ldots + \beta_i + \ldots = \rho_{in},$$

has the unique solution:

$$\beta_i = \frac{p_i + p_0}{p} \rho_{in}.$$
In the setting of the Uncoupling Theorem, the equations

\[ \gamma_i^{\kappa_i} = \tilde{\gamma}_i L_{-i} \prod_{j \neq i} \gamma_j^{\bar{h}_j^{i} - \kappa_j}, \]

imply

\[ x_i - \sum_{j \neq i} \bar{h}_j^i \ x_j = B_i := \frac{1}{\kappa_i} \log (\tilde{\gamma}_i L_{-i}) = \frac{p}{p_i} \log (\tilde{\gamma}_i L_{-i}), \]

with

\[ x_i = \log \gamma_i. \]
Proof. Cross-multiply take logs and note

\[
\bar{h}_j^i \kappa_i = \bar{h}_j^i - \kappa_j
\]

\[
= \frac{p_j}{p - p_i} - \frac{p_j}{p} = p_j \frac{p - (p - p_i)}{p(p - p_i)} = \frac{p_i}{p} \frac{p_j}{p - p_i}.
\]

The more revealing re-statement is

\[
(k_i - 1)x_i + \sum_{j \neq i} \kappa_j x_j = b_i := \frac{(p_i - p)}{p_i} \log (\gamma_i L_{-i}).
\]
Conditional hemi-mean formula

The following identifies the hemi-mean function.

**Theorem (Conditional hemi-mean formula).**

\[
\mathbb{E}[T_1^{\kappa_1}|T_2, \ldots, T_n] = L_{-1}T_2^{\bar{h}_2-\kappa_2} \cdots T_n^{\bar{h}_n-\kappa_n}, \text{ where } L_{-1} = \exp\left(\frac{n - 1}{2(p - p_1)}\right) \exp\left(-\frac{n}{2p}\right)
\]

and \( \bar{h}_j = \frac{p_j}{p - p_1}, \) for \( j > 1. \)

Hence, for any \( \gamma, \)

\[
\mathbb{E}[T_1^{\kappa_1}1_{T_1<\gamma}|(T)_{-1}] = L_{-1} \prod_{j>1} T_j^{\bar{h}_j-\kappa_j} \Phi_{\text{LN}}(\gamma^{\kappa_1}/L_{-1} \prod_{j>1} T_j^{\bar{h}_j-\kappa_j}, \kappa_1\sigma_{01}\sqrt{1 - \rho_1^2}).
\]
Proof. The random variable $S = T_1^{\kappa_1}$ has mean $m = m(\kappa_1, \sigma_{01})$ and volatility $\kappa_1 \sigma_{01}$. Hence, by the Exponent Effect Theorem,

$$H_S(\gamma^{\kappa}) = m H_{LN}(\gamma^{\kappa_1}/m, \kappa_1 \sigma_{01}).$$

The distribution of $S$ conditional on $T_2 = t_2, \ldots, T_2 = t_n$ (for any $t_2, \ldots, t_n$) has a mean $\xi = \xi_{-1}$ (depending on $t_2, \ldots, t_n$ to be determined below) and a volatility $\kappa_1 \sigma_{01} \sqrt{1 - \rho_n^2}$, with $1 - \rho_n^2$ the ‘Schur complement’ of $T_n$ in $(T_2, \ldots, T_n)$, because that is the effect on normal variates of conditioning (see Bingham & Fry (2010)).
Thus putting $\eta = \eta_{-1} := m\xi_{-1}$ we have for any $\gamma > 0$ that

$$H_{S|t_2...}(\gamma^{k_1}) = E[(\gamma^{k_1} - T_1^{k_1})1_{T_1 < \gamma}|T_2 = t_2, ..., T_n = t_n]$$

$$= m\xi H_{LN}(\gamma^{k_1}/m\xi, \kappa_1\sigma_{01}\sqrt{1 - \rho_n^2})$$

$$= \gamma^{k_1}\Phi_N \left( \frac{\log(\gamma^{k_1}/\eta) + \frac{1}{2}\kappa_1^2\sigma_{01}^2\rho_n}{\kappa_1\sigma_{01}\sqrt{1 - \rho_n^2}} \right)$$

$$- \eta\Phi_N \left( \frac{\log(\gamma^{k_1}/\eta) - \frac{1}{2}\kappa_1^2\sigma_{01}^2\rho_n}{\kappa_1\sigma_{01}\sqrt{1 - \rho_n^2}} \right).$$

(CH)

This leaves open the determination of the ‘constant’ $\eta = \eta_{-1}$. But minus the second term has the value

$$E[T_1^{k_1}1_{T_1 < \gamma}|T_2 = t_2, ..., T_n = t_n].$$
So taking the limit as $\gamma \to +\infty$ we obtain

$$\eta = \eta_{-1} = E[T_1^{\kappa_1} | T_2 = t_2, \ldots, T_n = t_n].$$

Now, by the conditional mean formula, with

$$\bar{h}_i = \bar{h}_i^1 = \frac{p_i}{p - p_1}$$

$$H^{-1}t_2^{\bar{h}_2} \ldots t_n^{\bar{h}_n} = E[X | T_2 = t_2, \ldots, T_n = t_n]$$

$$= E[E[X | T_1, T_2 = t_2, \ldots, T_n = t_n] | T_2 = t_2, \ldots, T_n = t_n]$$

$$= E[K \alpha T_1^{\kappa_1} t_2^{\kappa_2} \ldots t_n^{\kappa_n} | T_2 = t_2, \ldots, T_n = t_n]$$

$$= K t_2^{\kappa_2} \ldots t_n^{\kappa_n} E[T_1^{\kappa_1} | T_2 = t_2, \ldots, T_n = t_n]$$
and so

\[
\eta_{-1} = (H_{-1}K^{-1})h_{2-k_2}...h_{n-k_n} \\
= \exp \left( \frac{n - 1}{2(p_0 + p_2 + ... + p_n)} \right) \exp \left( -\frac{n}{2p} \right) h_{2-k_2}...h_{n-k_n} \\
= \exp \left( \frac{n - 1}{2(p - p_1)} \right) \exp \left( -\frac{n}{2p} \right) h_{2-k_2}...h_{n-k_n},
\]

as required. The rests is now clear from (CH) above.
Postscript: Log-normal vs normal: standardization

**Normal** $x$ with mean $m$ and variance $\sigma^2$ transforms to $v = (x - m)/\sigma \sim N(0, 1)$, i.e. zero-mean unit-variance. Note the moment generating function for $x \sim N(0, 1)$ is

$$E[e^{sx}] = e^{\frac{1}{2}s^2}.$$

**General log-normal**

$$X = m x e^{\sigma x - \frac{1}{2}\sigma^2} \text{ with } x \sim N(0, 1).$$

Consider now the power transformation $Y = X^{\kappa}$ for $0 < \kappa < 1$, then with $s = \kappa \sigma$

$$Y = e^{\kappa \sigma x - \frac{1}{2}\kappa \sigma^2} = e^{\frac{1}{2}\kappa(\kappa - 1)\sigma^2} e^{sx - \frac{1}{2}s^2}$$

$$= e^{\frac{1}{2}\kappa(\kappa - 1)\sigma^2} Z.$$
That is, the new variable has reduced mean

\[ m = m(\kappa, \sigma) := e^{\frac{1}{2}(\kappa-1)\sigma^2}. \]

(Smart reason: derive this from from Ito’s Lemma! via the second derivative of \( y^\kappa \).)

Log-normal \( X \) with mean \( m_X \) and variance \( \sigma^2 \) transforms using \( \kappa = 1/\sigma \) to \( Y = X^\kappa \) with unit variance and mean

\[ m_Y = m_X e^{\frac{1}{2}(1-\sigma)} \]

and so we arrive at \( Z = X^\kappa/m_Y = (Y/m_Y) \sim LN(1, 1) \), i.e. unit-mean unit-variance.