Planar dynamical systems perturbed by heavy-tailed Lévy noise

Ilya Pavlyukevich (joint work with Michael Högele, U Potsdam)

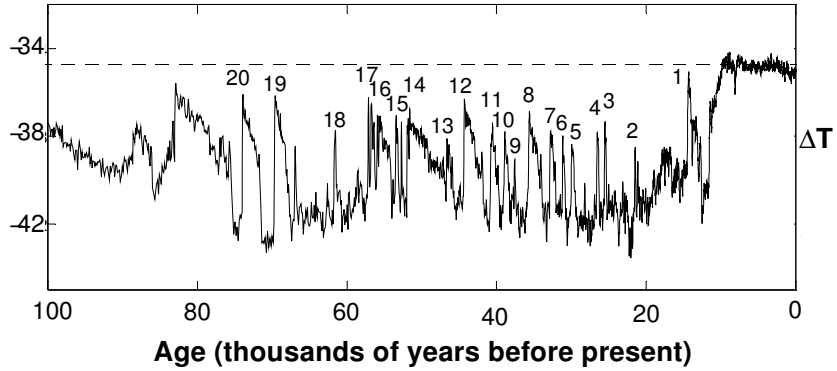
Friedrich–Schiller–Universität Jena
Institut für Stochastik

seit 1558

EPSRC Workshop: Topics in Control
30 November – 2 December 2011, Warwick, UK

1. Example: climate extreme events

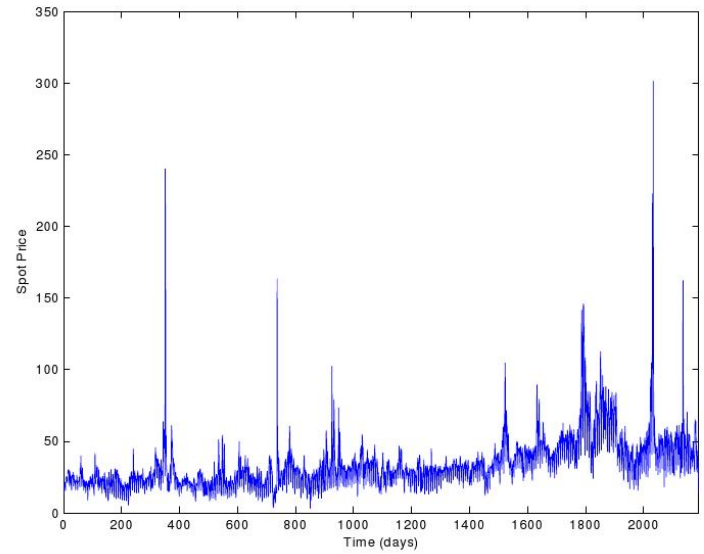
Earth's global temperature



- P. Ditlevsen (U Copenhagen) Geophys. Res. Lett. 26, 1999 Phys. Rev. E 60, 1999
- Langevin equation for the climate dynamics:
- $dX(t) = -U'(X(t))dt + \varepsilon\, dL(t)$
- L — α-stable symmetric Lévy process $+$ maybe a BM
- U — double-well potential, wells correspond to the climate states
- Statistical analysis: $\alpha \approx 1.75$ (or $\alpha \approx 0.7$?)

2. Example: electricity prices

$\Longrightarrow$



- Mean-reversion models
  Pyndick; Borovkova&Permana; R. Weron

- Stochastic differential equation for energy prices

- $dX(t) = -U'(X(t))dt + \varepsilon\, dL(t)$

- L — positively skewed (α-stable) Lévy process $+$ maybe a BM

- U — 'parabolic' potential, e.g. $U(x) = M(x-a)^2$.

- Goal: good model for the spikes

3. Motivation: Duffing oscillator

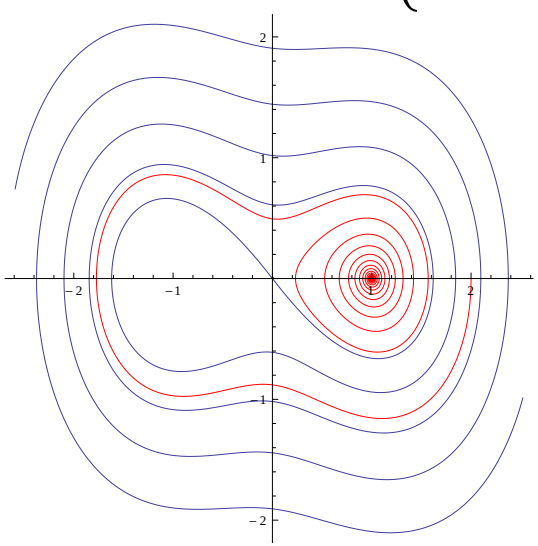
Perturbed Duffing oscillator with friction.

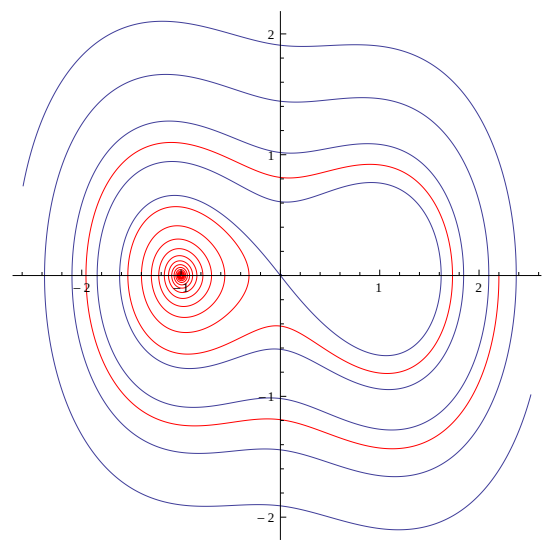
$\ddot{x} + \delta\dot{x} + U'(x) = \varepsilon\dot{L}$, $\varepsilon \to 0$, $\delta > 0$.

U double-well potential, e.g. $U(x) = x^4/4 - x^2/2$.

$\varepsilon = 0$: two attractors, say ± 1 and a saddle $x = 0$, domains of attraction $\Omega_\pm$, separatrix Γ

Equation in phase space $\begin{cases} \dot{x} = v, \\ \dot{v} = -\delta v - U'(x) + \varepsilon\dot{L} \end{cases}$





Noise induces transitions

4. Motivation: van der Pol oscillator

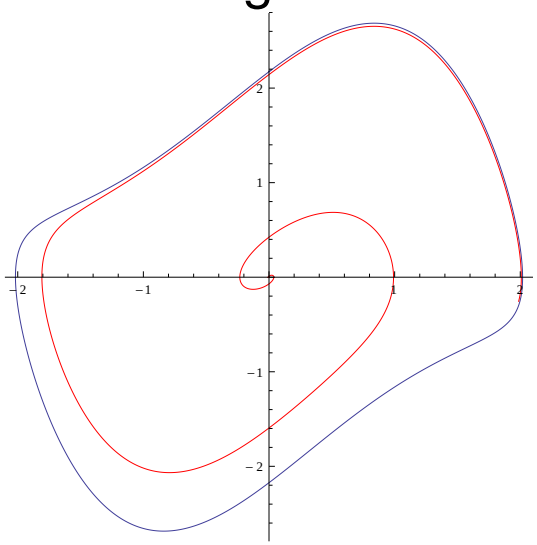
van der Pol oscillator: harmonic oscillator with non-linear friction

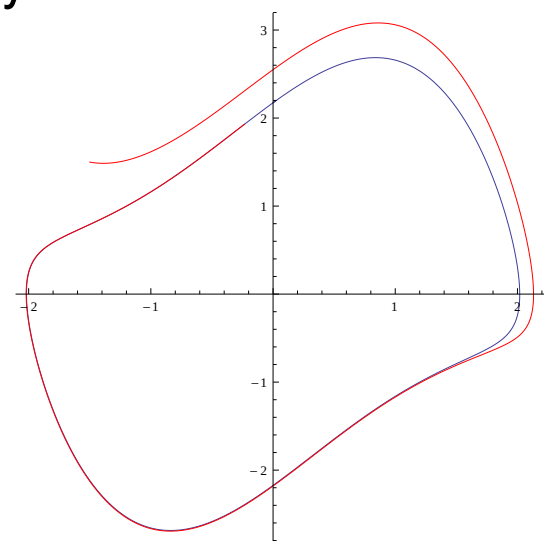
$\ddot{x} - \delta(1 - x^2)\dot{x} + x = \varepsilon \dot{L}$, $\varepsilon \to 0$, $\delta > 0$.

$\varepsilon = 0$: a stable limit cycle and an repulsive focus $x = 0$

Equation in phase space $\begin{cases} \dot{x} = v, \\ \dot{v} = \delta(1 - x^2)v - x + \varepsilon \dot{L} \end{cases}$

Noise: exit from a neighbourhood of a limit cycle





5. Multiple limit cycles in electric circuits

Y. A. Saet and G. L. Viviani: Mathematical Modelling, 7, pp. 377–384, 1986

Saet and Viviani (~ 1985): Liénard equation $\ddot{x} + \mu W(x)\dot{x} + x = 0$, $\mu > 0$.

One can construct a polynomial $W(x)$ such that the dynamical system has limit cycles with predefined amplitides.

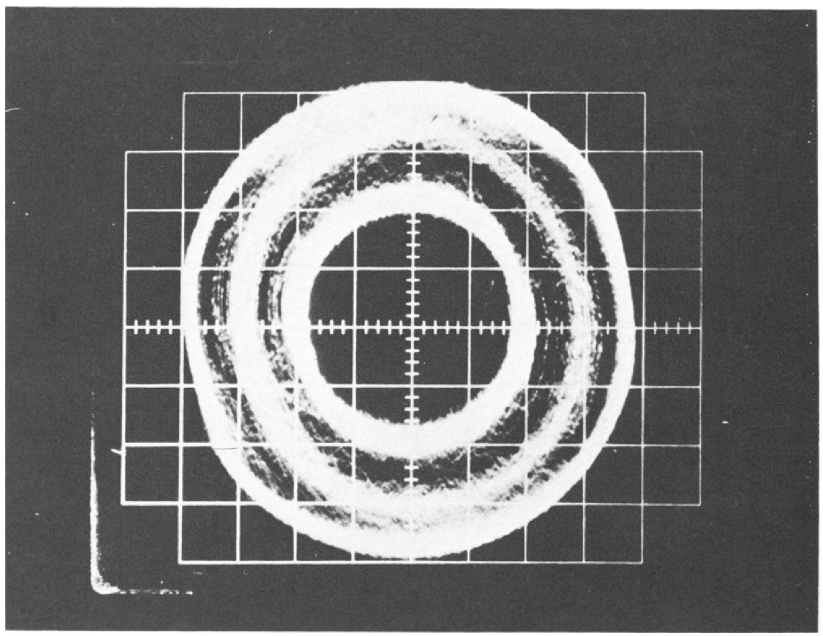
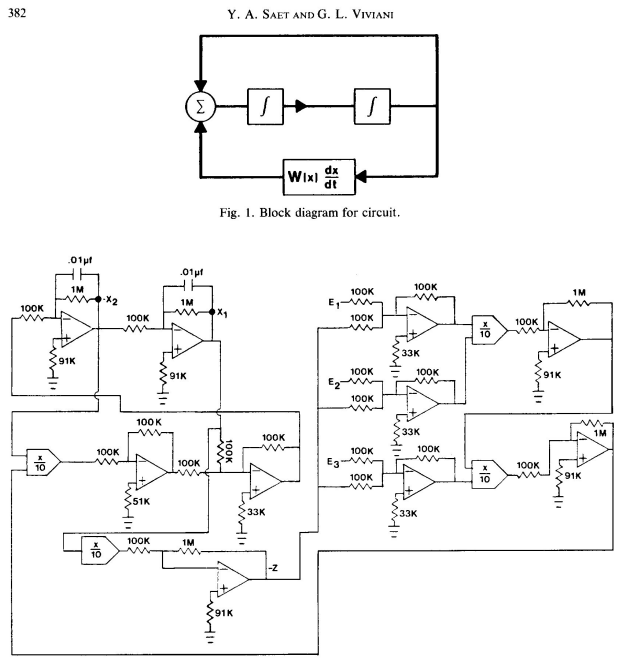


Fig. 4. Single-exposure photograph of white-noise excited oscillator with three stable limit cycles.



L. N. Epele, H. Fanchiotti, A. Spina, and H. Vucetich: Physical Review A, 31(4) 1985

6. Multiple limit cycles in chemical reactions

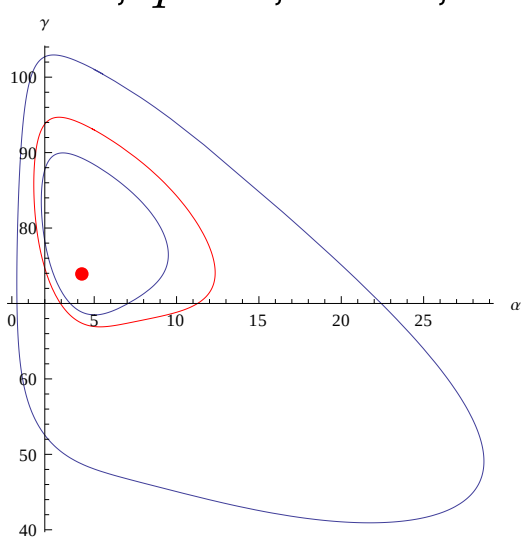
A. Goldbeter and F. Moran: European Biophysics Journal, 15, 1988

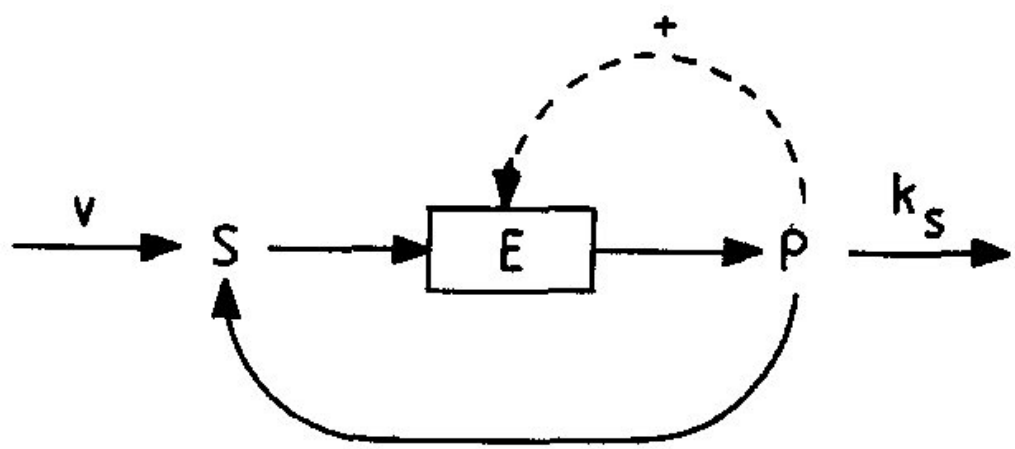
Autocatalytic enzyme reaction with input of substrate α both from a constant source $v > 0$ and from non-linear recycling of product γ into substrate.

$$\frac{d\alpha}{dt} = v + \frac{\sigma_i \gamma^n}{K^n + \gamma^n} - \sigma_M \frac{\alpha(1+\alpha)(1+\gamma)^2}{L + (1+\alpha)^2(1+\gamma)^2}$$

$$\frac{d\gamma}{dt} = q\sigma_M \frac{\alpha(1+\alpha)(1+\gamma)^2}{L + (1+\alpha)^2(1+\gamma)^2} - k_s\gamma - \frac{q\sigma_i \gamma^n}{K^n + \gamma^n}$$

$v = 0.255$, $q = 1$, $n = 4$, $K = 10$, $L = 5 \cdot 10^6$, $k_s = 0.06$, $\sigma_i = 1.3$; $\sigma_M = 10$.

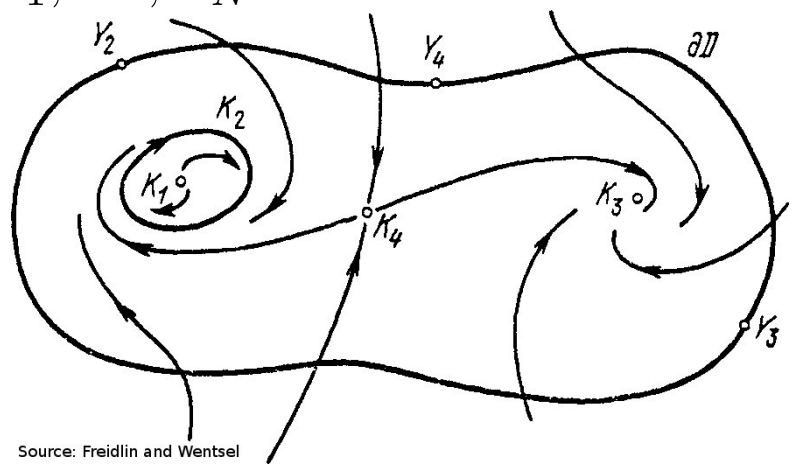




7. Setting

Planar structurally stable dynamical system $\dot{u} = f(u)$ with $K_1, \ldots, K_N$ attractors which are either exponentially stable foci or orbitally exponentially stable limit cycles.

Domains of attraction $\Omega_1, \ldots, \Omega_N$.



Source: Freidlin and Wentsel

Perturbed system

$$X_t = X_0 + \int_0^t f(X_s)\, ds + \varepsilon \int_0^t F(X_{s-})\, dZ_s$$

$Z \in \mathbb{R}^m$-dimensional Levy Process with heavy tails

$F\colon \mathbb{R}^m \to \mathbb{R}^2$ sufficiently smooth function, $\varepsilon \downarrow 0$
Goal: long term behaviour of X.

8. Source of randomness: Lévy process Z

Z is a **Lévy process** if $Z_0 = 0$, is stochastically continuous and has

independent stationary increments

(and right continuous paths with left limits).

$$Z = \underbrace{\text{Brownian motion} \ + \ \text{drift}} + \underbrace{\text{jumps}} = Z^c + Z^d$$

Lévy–Khintchine formula for $Z \in \mathbb{R}^m$: $\langle x, y \rangle = \sum_{i=1}^m x_i y_i$

$$\mathbf{E}\mathrm{e}^{\mathrm{i}\langle Z_t, \lambda \rangle} = \exp\Big[\underbrace{-\frac{t}{2}\langle A\lambda, \lambda \rangle}_{\text{Brownian motion}} + \underbrace{\mathrm{i}t\langle \lambda, \mu \rangle}_{\text{drift}} + \underbrace{t \int \Big(\mathrm{e}^{\mathrm{i}\langle \lambda, y \rangle} - 1 - \frac{\mathrm{i}\langle \lambda, y \rangle}{1 + \|y\|^2} \Big) \nu(dy)}_{\text{jumps}} \Big]$$

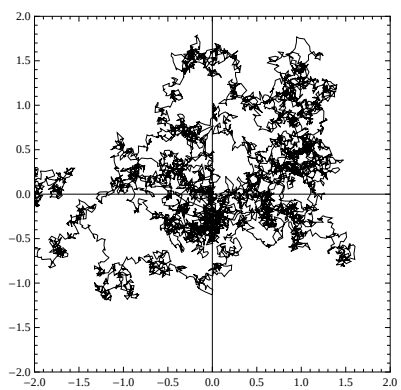
9. α-stable Lévy process (Lévy flights)

Isometric α-stable LP in $\mathbb{R}^m$:

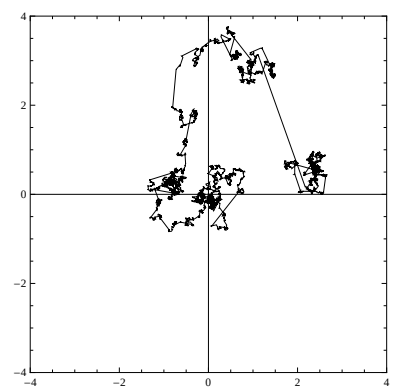
$$\mathbf{E}\mathrm{e}^{\mathrm{i}\langle Z_t,\lambda\rangle} = \exp\Big[-tc_{m,\alpha}\|\lambda\|^\alpha\Big], \quad \alpha\in(0,2), \quad c_{m,\alpha} = \frac{\pi^{m/2}}{2^\alpha}\frac{\Gamma(-\frac{\alpha}{2})}{\Gamma(\frac{m+\alpha}{2})}$$

Jump measure: $\nu(dy) = \dfrac{dy}{\|y\|^{\alpha+m}}$, $\alpha\in(0,2)$

Cauchy process: $\alpha = 1$, probability density $p(x) \sim \dfrac{1}{1+\|x\|^2}$



Brownian motion



1.50-stable Lévy process

$\dfrac{\mathbf{P}(\|Z_1\|\geqslant u)}{\nu(y\colon \|y\|\geqslant u)} \to 1$, $u\to\infty$. (see Kolokoltsov (LNM 1724) for expansions)

10. Heavy tails

ν is a regularly varying at ∞: there is a non-zero Radon measure m on $\mathcal{B}(\overline{\mathbb{R}}^m \backslash \{0\})$ with $m(\overline{\mathbb{R}}^m \backslash \mathbb{R}^m) = 0$ with the scaling property, $a > 0$,

$$m(aA) = \lim_{u \to +\infty} \frac{\nu(auA)}{\nu(\{\|z\| \geqslant u\})} = \frac{1}{a^r} \lim_{u \to +\infty} \frac{\nu(uA)}{\nu(\{\|z\| \geqslant u\})} = \frac{1}{a^r} m(A),$$

$0 \notin \overline{A}$, $m(\partial A) = 0$, for some $r > 0$. In particular, $u \mapsto \nu(\{\|z\| \geqslant u\})$ is regularly varying at infinity with index $-r$, $\nu(\{\|z\| \geqslant u\}) = u^{-r} l(u)$.

Isometric α-stable LP in $\mathbb{R}^m$

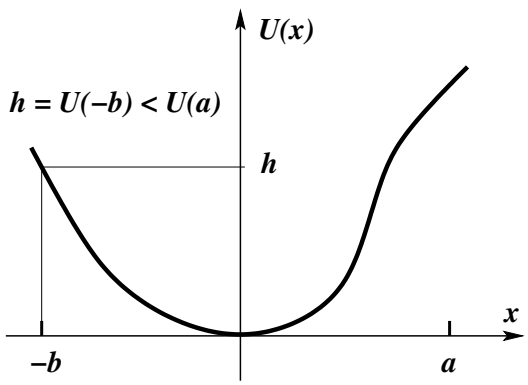
$$\nu(dy) = C(\alpha, m) \frac{dy}{\|y\|^{\alpha+m}} \qquad \nu(\{\|z\| > u\}) = \frac{1}{u^\alpha}, \quad m = \nu$$

Weakly tempered α-stable LP in $\mathbb{R}^m$:

$$\nu(dy) = \frac{dy}{\|y\|^{\alpha+m}} \frac{C(\alpha, \beta, m)}{(1 + \|y\|^2)^{\beta/2}}, \qquad \alpha \in (0, 2),\ \beta \geqslant 0,$$

$$\nu(\{\|z\| > u\}) \approx \frac{1}{u^{\alpha+\beta}}, \quad u \to \infty, \quad m(dy) = C(\alpha, \beta, m) \frac{dy}{\|y\|^{\alpha+\beta+m}}$$

11. One-dimensional case with additive α-stable noise



$$X_t = x - \int_0^t U'(X_s)\,ds + \varepsilon W_t$$

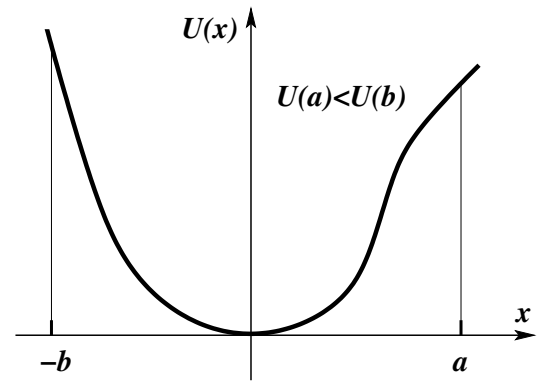
Kramers' law
(Freidlin, Wentzell, Bovier):

$$\mathbf{E}_x\tau \approx \frac{\varepsilon\sqrt{\pi}}{|U'(-b)|\sqrt{U''(0)}}\mathrm{e}^{2h/\varepsilon^2}$$

Exponential exit (Day, Bovier)

$$\mathbf{P}_x\Big(\frac{\tau}{\mathbf{E}_x\tau} > u\Big) \to \mathrm{e}^{-u}$$

Diffusion 'climbs up and out'



$$X_t = x - \int_0^t U'(X_s)\,ds + \varepsilon Z_t,$$
$$\nu(dy) = |y|^{-1-\alpha}dy$$

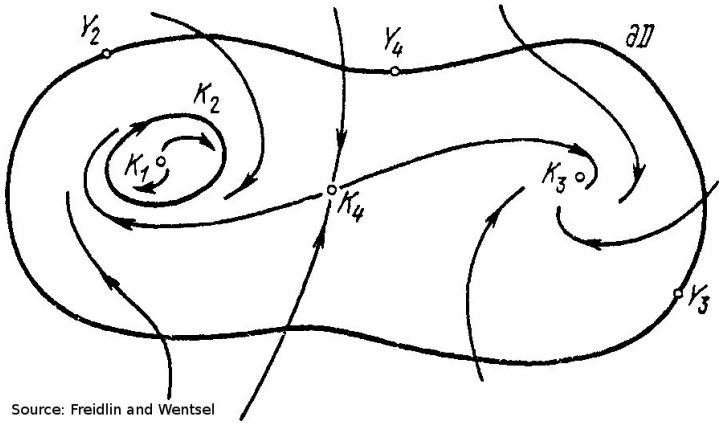
(Godovanchuk, 1981; Imkeller, P.)

$$\mathbf{E}_x\tau \approx \frac{\alpha}{\varepsilon^\alpha}\Big[\frac{1}{a^\alpha} + \frac{1}{b^\alpha}\Big]^{-1} = \frac{\nu([-b,a]^c)}{\varepsilon^\alpha}$$

$$\mathbf{P}_x\Big(\frac{\tau}{\mathbf{E}_x\tau} > u\Big) \to \mathrm{e}^{-u}$$

Lévy motion driven SDE 'jumps out'

12. Gaussian perturbations



Source: Freidlin and Wentsel

Freidlin-Wentzell theory: Action functional

$$S_{0T}(\varphi) = \begin{cases} \frac{1}{2}\int_0^T \|\dot{\varphi}_s - f(\varphi_s)\|^2\, ds, & \text{if } \varphi\colon [0,T] \to \mathbb{R}^2 \text{ absolutely continuous,} \\ +\infty & \text{else.} \end{cases}$$

Quasipotential

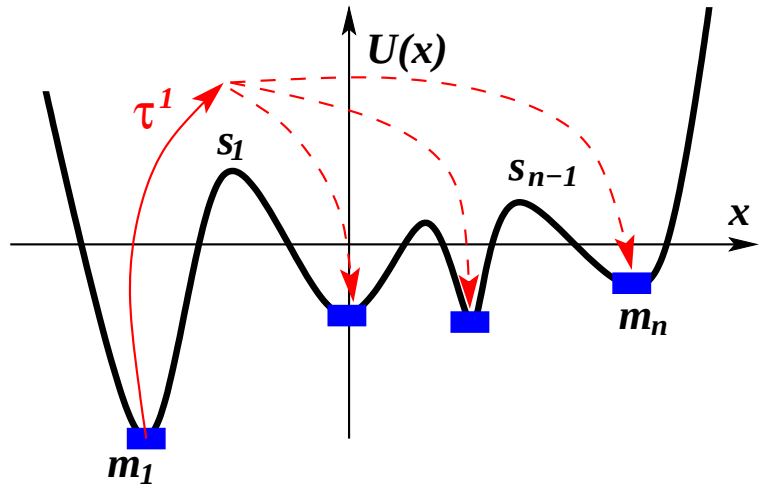
$$\tilde{V}(K_i, K_j) = \inf\{S_{0T}(\varphi)\colon \varphi_0 \in K_i,\ \varphi_T \in K_j,\ \varphi_t \in G \cup \partial G \setminus \bigcup_{k \neq i,j} K_k,\ t \in (0,T)\}.$$

All points of limit cycles are equivalent, exponentially long transition times between the attractors (see Wentzell; Kolokoltsov, . . .)

13. Heavy tail perturbations. Transitions

Let $\nu(dy) = |y|^{1+\alpha}dy$. For small $\Delta > 0$ denote $\mathbf{B}_i = \{y : |y - m_i| \leqslant \Delta\}$ and

$$\tau_x^i(\varepsilon) = \inf\{t \geqslant 0 : X_x^\varepsilon(t) \in \cup_{j\neq i} B_j\}$$



Theorem 1. [I.&P. 2008] *For $x \in B_i$ the following holds as $\varepsilon \to 0$:*

$$\mathbf{E}\mathrm{e}^{-u\varepsilon^\alpha q_i \tau_x^i(\varepsilon)} \to \frac{1}{1+u}, \ u > -1,$$

$$\mathbf{E}\tau_x^i(\varepsilon) \approx \frac{1}{\varepsilon^\alpha q_i}, \qquad q_i = \int_{\mathbb{R}\setminus(s_{i-1},s_i)} \frac{dy}{|y-m_i|^{1+\alpha}},$$

$$\mathbf{P}(X^\varepsilon(\tau_x^i(\varepsilon)) \in B_j) \to \frac{q_{ij}}{q_i}, \qquad q_{ij} = \int_{(s_{j-1},s_j)} \frac{dy}{|y-m_i|^{1+\alpha}}, \quad i \neq j.$$

14. Metastability

Theorem 2. *Let $x \in (s_{j-1}, s_j)$ and $0 < t_1 < \cdots < t_k$. Then for any bounded continuous function f*

$$\mathbf{E} f\left(X_x^\varepsilon\left(\frac{t_1}{\varepsilon^\alpha}\right), \ldots, X_x^\varepsilon\left(\frac{t_k}{\varepsilon^\alpha}\right)\right) \to \mathbf{E} f\Big(Y_{m_j}(t_1), \ldots, Y_{m_j}(t_k)\Big),$$

where Y is a Markov chain on $\{m_1, \ldots, m_n\}$ with a generator $Q = (q_{ij})$, $q_{ii} = -q_i$.

Remark. Y has the invariant measure

$$\pi(dy) = \sum_{j=1}^{n} \pi_j \delta_{m_j}(dy),$$

$$\pi_j > 0,$$

where $Q^T \pi = 0$.

15. Separation of time scales

$$Z_t = \underbrace{Z_t \mathbb{I}(\|\Delta Z_t\| \geqslant \varepsilon^{-\rho})}_{:=\eta_t^\varepsilon} + \underbrace{(Z_t - \eta_s^\varepsilon)}_{:=\xi_t^\varepsilon}, \quad \rho \in (0,1)$$

- η_t^ε is a compound Poisson process
- jump measure $\nu_\eta(A) = \nu(A \cap \{\|x\| \geqslant \varepsilon^{-\rho}\})$; $\beta_\varepsilon = \nu_\eta(\mathbb{R}^m) \sim \varepsilon^{\alpha\rho}$
- arrival times $0 = \tau_0 < \tau_1 < \tau_2 < \ldots$; $\tau_k - \tau_{k-1} \overset{\text{iid}}{\sim} \mathscr{E}(\beta_\varepsilon)$
- iid jump sizes $(W_k)_{k\geqslant 1}$, $\mathbf{P}(W_1 \in A) = \frac{\nu(A\cap\{\|x\|\geqslant\varepsilon^{-\rho}\})}{\beta_\varepsilon}$.

$$X_t = X_0 + \int_0^t f(X_s)\,ds + \varepsilon \int_0^t F(X_{s-})\,d\xi_s^\varepsilon, \quad t \in [0, \tau_1)$$

$$X_{\tau_1} = X_{\tau_1-} + \varepsilon F(X_{\tau_1-}) W_1$$

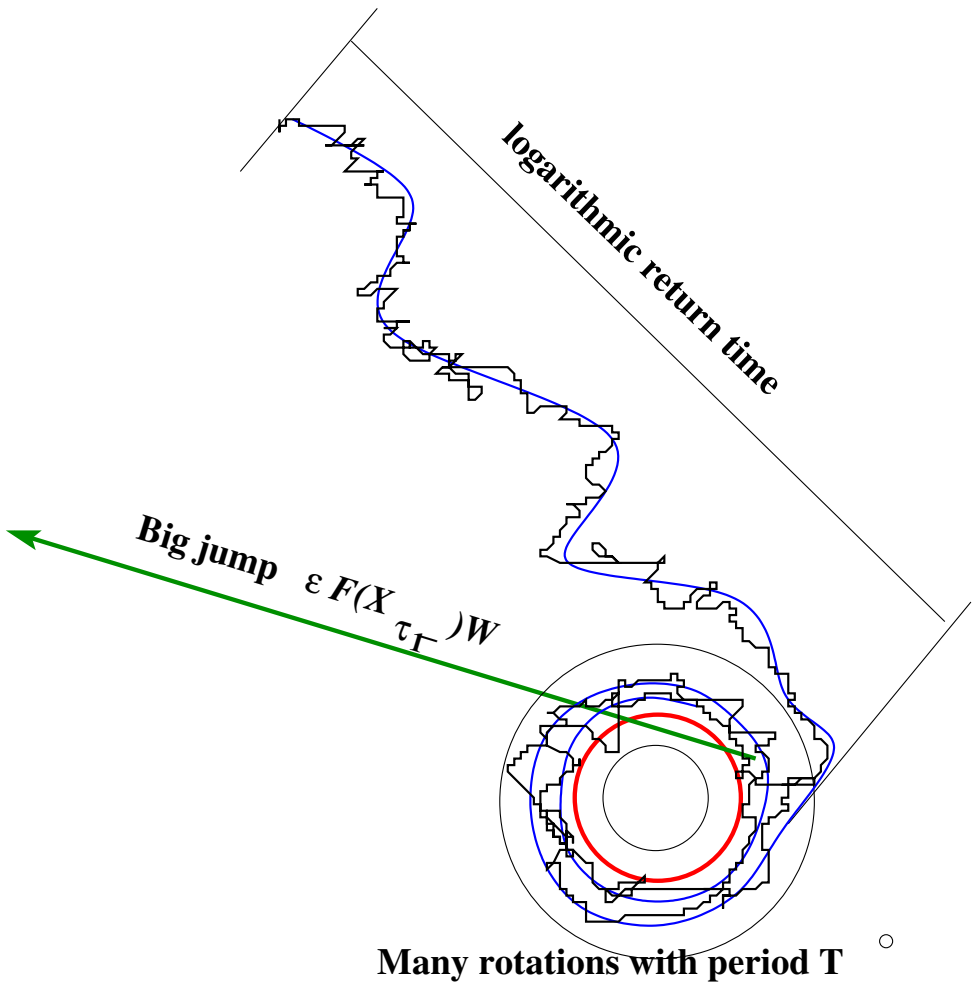
$$X_t = X_{\tau_1} + \int_{\tau_1}^{\tau_1+t} f(X_s)\,ds + \varepsilon \int_0^t F(X_{s-})\,d\xi_s^\varepsilon, \quad t \in [0, \tau_2 - \tau_1)$$

$$X_{\tau_2} = X_{\tau_2-} + \varepsilon F(X_{\tau_2-}) W_2$$

$$\ldots$$

16. Dynamics between big jumps

$$X_t = X_0 + \int_0^t f(X_s)\,ds + \varepsilon \int_0^t F(X_{s-})\,d\xi_s^\varepsilon, \quad t \in [0, \tau_1)$$



Return time to a small ε-dependent neigbourhood of $K \sim \ln \frac{1}{\varepsilon}$.

Mean waiting time before a big jump $\sim \frac{1}{\varepsilon^{\alpha\rho}}$

Mean number of rotations $\sim \frac{1}{T^\circ \varepsilon^{\alpha\rho}}$

17. Orbital stability

A solution $\varphi(t)$ of the system $\dot{x} = f(x)$ is called exponentially stable if $\delta > 0$, $C > 0$ and $\alpha > 0$ exist such that

$$\|x(t; x_0) - \varphi(t)\| \leqslant C\mathrm{e}^{-\alpha t}\|x_0 - \varphi(0)\|, \quad t \geqslant 0$$

as soon as $\|x_0 - \varphi(0)\| \leqslant \delta$.

For $x \in \mathbb{R}^2$ and a limit cycle K define a distance

$$d_K(x) := \inf_{y \in K} \|x - y\| = \inf_{t \leqslant T^\circ} \|x - \varphi(t)\|.$$

The function $x \mapsto d_K(x)$ is well defined in a small neighbourhood of K.

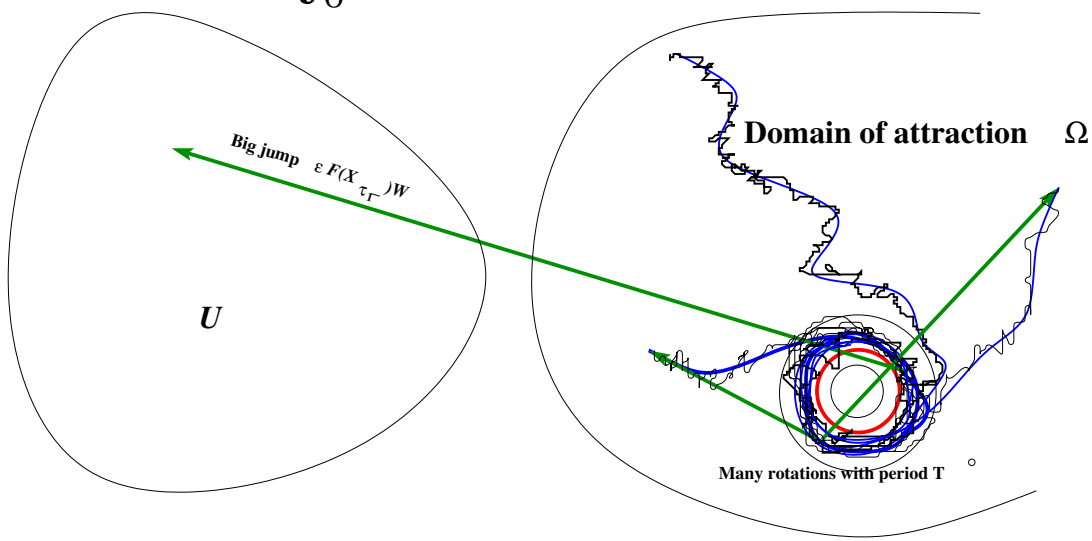
A periodic solution $\varphi(t)$ of the system $\dot{x} = f(x)$ is called exponentially orbitally stable if $\delta > 0$, $C > 0$ and $\alpha > 0$ exist such that

$$d_K(x(t; x_0)) \leqslant C\mathrm{e}^{-\alpha t} d_K(x_0), \quad t \geqslant 0$$

as soon as $d_K(x_0) \leqslant \delta$.

18. Transitions

$$X_t = X_0 + \int_0^t f(X_s)\,ds + \varepsilon \int_0^t F(X_{s-})\,d\xi_s^\varepsilon + \varepsilon F(X_{\tau_1-})W_1 \mathbb{I}_{\{t=\tau_1\}}, \quad t \in [0, \tau_1]$$



$$\widehat{\Omega^c}(y) := \{w \in \mathbb{R}^m \colon y + F(y)w \in \Omega^c\}, \quad \widehat{U}(y) := \{w \in \mathbb{R}^m \colon y + F(y)w \in U\},$$

$$\tau_x^\varepsilon := \inf\{t \geqslant 0 \colon X_t^\varepsilon \notin \Omega\}$$

$$q_U = \int_0^{T^\circ} m(\hat{U}(\varphi(s)))\,ds, \quad q = \frac{1}{T^\circ}\int_0^{T^\circ} m(\widehat{\Omega^c}(\varphi(s)))\,ds,$$

$$\mathbf{E}\Big[\mathrm{e}^{-\theta q h(\varepsilon)\tau_x^\varepsilon}\mathbb{I}(X_{\tau_x^\varepsilon} \in U)\Big] \overset{\varepsilon\downarrow 0}{\to} \frac{1}{1+\theta}\cdot\frac{q_U}{q}$$

19. Asymptotics I

Characteristic time scale of X^ε: $h_\varepsilon = \nu(\{\|x\| \geqslant \frac{1}{\varepsilon}\}) \sim \varepsilon^\alpha$;

Characteristic time scale of a 'big jumps process': $\beta_\varepsilon = \nu(\{\|x\| \geqslant \frac{1}{\varepsilon^{-\rho}}\}) \sim \varepsilon^{\alpha\rho}$;

The first 'big jump time' $\mathbf{P}(T > u) = \mathrm{e}^{-\beta u}$

$$\mathbf{E}\,\mathrm{e}^{-\theta q h_\varepsilon T}\mathbb{I}\Big(X_t \in \Omega, t < T, X_{T-} + \varepsilon F(X_{T-})W \notin \Omega\Big)$$

$$\Big[\!\Big[\text{use that } \mathbf{P}\Big(\sup_{t \leqslant T \wedge \ln\frac{1}{\varepsilon}} \|X_t - u_t(x_0)\| \leqslant d_\varepsilon \text{ and } \sup_{t \geqslant T \wedge \ln\frac{1}{\varepsilon}} d_K(X_t) \leqslant d_\varepsilon\Big) \approx 1\Big]\!\Big]$$

$$\approx \int_0^\infty \beta_\varepsilon \mathrm{e}^{-(\theta q h_\varepsilon + \beta_\varepsilon)t}\mathbf{P}\Big(X_t + \varepsilon F(X_t)W \notin \Omega\Big)\,dt$$

$$\approx \sum_{k=0}^\infty \int_{kT^\circ}^{(k+1)T^\circ} \beta_\varepsilon \mathrm{e}^{-(\theta q h_\varepsilon + \beta_\varepsilon)t}\mathbf{P}\Big(\varphi_t^{(k)} + \varepsilon F(\varphi_t^{(k)})W \notin \Omega\Big)\,dt$$

$$\approx \sum_{k=0}^\infty \mathrm{e}^{-(\theta q h_\varepsilon + \beta_\varepsilon)kT^\circ}\beta_\varepsilon \int_0^{T^\circ} \mathbf{P}(\varepsilon W \in \widehat{\Omega^c}(u_t))\,dt$$

20. Asymptotics II

$$\approx \sum_{k=0}^{\infty} \mathrm{e}^{-(\theta q h_\varepsilon + \beta_\varepsilon) k T^\circ} \beta_\varepsilon \int_0^{T^\circ} \mathbf{P}(\varepsilon W \in \widehat{\Omega^c}(u_t))\, dt$$

$$\approx \frac{\beta_\varepsilon}{1 - \mathrm{e}^{-(\theta q h_\varepsilon + \beta_\varepsilon) T^\circ}} \int_0^{T^\circ} \frac{1}{\beta_\varepsilon} \nu(\widehat{\Omega^c}(u_t))\, dt$$

$$\approx \frac{h_\varepsilon}{\theta q h_\varepsilon + \beta_\varepsilon} \cdot \frac{1}{T^\circ} \int_0^{T^\circ} m(\widehat{\Omega^c}(u_t))\, dt = \frac{h_\varepsilon}{\theta q h_\varepsilon + \beta_\varepsilon} q$$

Analogously,

$$\mathbf{E}\, \mathrm{e}^{-\theta q h_\varepsilon T} \mathbb{I}\Big(X_t \in \Omega, t < T, X_{T-} + \varepsilon F(X_{T-}) W \in \Omega \Big) \approx \frac{\beta_\varepsilon}{\theta q h_\varepsilon + \beta_\varepsilon} - \frac{h_\varepsilon}{\theta q h_\varepsilon + \beta_\varepsilon} q$$

21. Asymptotics III

For $x \in \Omega_i$:

$$\mathbf{E}\Big[\mathrm{e}^{-\theta q h_\varepsilon \tau_x^\varepsilon}\mathbb{I}(X_{\tau_x^\varepsilon} \in \Omega_j)\Big] \approx \sum_{k=1}^{\infty} \mathbf{E}\Big[\mathrm{e}^{-\theta q h_\varepsilon \tau_x^\varepsilon}\mathbb{I}(X_{\tau_x^\varepsilon} \in \Omega_j)\mathbb{I}(\tau_x^\varepsilon = \tau_k)\Big]$$

$$\approx \sum_{k=1}^{\infty} \Big[\frac{\beta_\varepsilon}{\theta q h_\varepsilon + \beta_\varepsilon} - \frac{h_\varepsilon}{\theta q h_\varepsilon + \beta_\varepsilon} q\Big]^{k-1} \frac{h_\varepsilon}{\theta q h_\varepsilon + \beta_\varepsilon} q_{\Omega_j}$$

$$= \frac{1}{1+\theta}\frac{q_{\Omega_j}}{q},$$

$$q = \sum_{j \neq i} q_{\Omega_j}$$

22. Metastability for a system with limit cycles

For a limit cycle K_i denote T_i° its period and $\varphi^{(K_i)} = \varphi^{(K_i)}(t)$, $t \in [0, T_i^\circ)$ any its parameterization, $\dot{\varphi}^{(K_i)} = f(\varphi^{(K_i)}(t))$, $\varphi^{(K_i)}(0) \in K_i$.

Attractors $K_1, \ldots, K_N$ given, consider the probability measures $\pi_1, \ldots, \pi_N$ on $\mathbb{R}^2$ with

$$\pi_i(A) = \begin{cases} \delta_{K_i}(A), & \text{if } K_i \text{ is a stable fixed point,} \\ \dfrac{1}{T_i^\circ} \displaystyle\int_0^{T_i^\circ} \mathbb{I}_{A \cap K_i}(\varphi^{(K_i)}(t))\, dt, & \text{if } K_i \text{ is a stable limit cycle.} \end{cases}$$

Clearly, $\operatorname{supp} \pi_i = \{K_i\}$. For $x \in \mathbb{R}^2$ denote the set of "jumps leading to transition to Ω_j from x" (Itô-SDE)

$$\widehat{\Omega_j}(x) := \{w \in \mathbb{R}^m \colon x + F(x)w \in \Omega_j\}.$$

23. Measure-valued Markov chain

We construct a Markov chain $Y = (Y_t)_{t \geqslant 0}$ on $\{\pi_1, \dots, \pi_N\}$ with the generator $Q = (q_{ij})_{i,j=1}^N$ defined by

$$
\begin{aligned}
q_{ij} &= \int_{K_i} m(\widehat{\Omega_j}(y))\, \pi_i(dy) \\
&= \begin{cases} m(\widehat{\Omega_j}(K_i)), & \text{if } K_i \text{ is a stable fixed point ,} \\ \dfrac{1}{T_i^\circ} \displaystyle\int_0^{T_i^\circ} m\Big(\widehat{\Omega_j}(\varphi^{(K_i)}(s))\Big)\, ds, & \text{if } K_i \text{ is a stable limit cycle} \end{cases}
\end{aligned}
$$

for $i \neq j$ and $q_i = \sum_{j \neq i} q_{ij}$.

Theorem. For $x \in \Omega_i$, $k \geqslant 1$, $0 < t_1 < \dots < t_k$ and $A_1, \dots, A_k$ with $\pi_i(\partial A_j) = 0$, $i = 1, \dots, N$, $j = 1, \dots, k$,

$$
\mathbf{P}_x\Big(X\Big(\frac{t_1}{H(\varepsilon)}\Big), \dots, X\Big(\frac{t_k}{H(\varepsilon)}\Big)\Big) \to \mathbf{E}_{\pi_i}[Y_{t_1}(A_1) \cdots Y_{t_k}(A_k)].
$$

24. Simulations