1. Review of the frictionless case

- filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \le t \le T}, \mathbb{P})$
- vector of discounted security prices $(S_t)_{t=0,\dots,T}$ (adapted)
- $\mathbb{Q}$ is an EMM if
  (i) $\mathbb{Q} \sim \mathbb{P}$, and
  (ii) S is a vector martingale under $\mathbb{Q}$.

There are then 3 possibilities:

1) No EMM exists $\Leftrightarrow$ $\exists$ an arbitrage
2) $\exists$ a unique EMM $\tilde{\mathbb{P}}$ $\Leftrightarrow$ (no arbitrage and) every claim $X \in L^1(\tilde{\mathbb{P}})$ is hedgeable (i.e. $\exists$ a self-financing investment strategy with set-up cost of $\mathbb{E}_{\tilde{\mathbb{P}}} X$ and final value X).
3) $\exists$ multiple EMMs $\Leftrightarrow$ no arbitrage but some claims are not hedgeable, but can be superhedged.

Remarks:

(i) we can stick to $\mathbb{P}$ throughout if we use densities: $\Lambda_t^{\mathbb{Q}} \stackrel{def}{=} \left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t} = \mathbb{E}_{\mathbb{P}}\left[\Lambda_T^{\mathbb{Q}} \middle| \mathcal{F}_t\right]$

and S is $\mathbb{Q}$-m.g $\Leftrightarrow$ $\Lambda^{\mathbb{Q}} S$ is a $\mathbb{P}$ m.g.

(ii) the superhedging price in case (3) is given by

$$\sup_{\text{EMMs } \mathbb{Q}} \mathbb{E}_{\mathbb{Q}}[X] = \sup \mathbb{E}_{\mathbb{P}}[\Lambda_T^{\mathbb{Q}} X].$$

(iii) the superhedging price $\pi(X)$ is <u>defined</u> by

$$\pi(X) = \inf\{c \in \mathbb{R} : \exists \text{ a self-financing strategy } \phi \text{ with } \phi_{-1} = c e_1 \text{ and } \phi_T \cdot S_T \geq X\}$$

(where asset 1 is cash and $e_1, \ldots, e_d$ are the canonical basis vectors).

Now we rephrase things so that the case with transaction costs is clearer:

Let $K_t \stackrel{def}{=} \{ \xi \in L^0(\mathcal{F}_t, \mathbb{R}^d) : \xi_t . S_t \leq 0 \text{ a.s} \}$
(can trade to ξ from $\underline{0}$ at time t prices [with consumption]).

Set $A = \sum_{t=0}^{T} K_t$, and say ϕ is self-financing if $\Delta \phi_t \in K_t$ for each t.

A is a convex cone and we define

$$\pi(X) = \inf \{ c : X e_1 \in c e_1 + A \} \text{ or}$$
$$= \inf \{ c : (X - c) e_1 \in A \}.$$

Equivalently, if $\underline{Y} \in L^0(\mathcal{F}_T, \mathbb{R}^d)$ is a desired final holding:

$$\pi(\underline{Y}) = \inf \{ c : \underline{Y} - c e_1 \in A \}$$

An arbitrage is $Y \in A$: $Y \geq 0$ a.s and $\mathbb{P}(Y \neq 0) > 0$.

1. FTAP either (i) A contains an arbitrage or (ii) A is closed and arb. free

2. (ii) $\iff \exists$ an EMM

3. If (ii) holds then $Y \in A \iff \mathbb{E}_{\mathbb{P}}[\Lambda_T^{\mathbb{Q}} S_T . Y] \leq 0$ for all EMMs $\mathbb{Q}$.

2. Transaction costs

Might as well just specify bid-ask prices:

π is a bid-ask matrix if

$\pi^{ii} = 1$ for each i, $\pi^{ij} > 0 \ \forall i,j$

and

$$\pi^{i_1 i_2} \cdots \pi^{i_{k-1} i_k} \geq \pi^{i_1 i_k}$$

(in particular, $\pi^{ij}\pi^{ji} \geq 1$)

π^{ij} represents no. of units of asset i traded for 1 unit of asset j.

A bid-ask process $(\pi_t ; 0 \leq t \leq T)$ is an adapted process with each π_t being a bid-ask matrix (the time t trading prices).

Now we can define the self-financing trades at time t:

$$K_t = \Big\{ \xi : \xi = \sum_{i,j} a_t^{ij} (e_j - \pi_t^{ij} e_i) - \sum_k b_t^k e_k : a_t^{ij}, b_t^k \in L_+^0(\mathcal{F}_t) \Big\};$$

K_t is generated by trades and consumption.

Easy exercise:

define $K_t^* = \{ z \in L^0(\mathcal{F}_t, \mathbb{R}^d) : z_j \leq \pi_t^{ij} z_i \ \forall ij \}$

then $\xi \in K_t \iff \xi \in L^0(\mathcal{F}_t, \mathbb{R}^d)$ and

$$z.\xi_t \leq 0 \quad \forall z \in K_t^*.$$

This agrees with frictionless case where $K_t^* = \{ \lambda S_t : \lambda \in L^0_+(\mathcal{F}_t) \}$.

Now set $A = \sum_{t=0}^{T} K_t$ (as before!)

We can price a claim $\underline{X}$ by setting

$p_i(x) = \inf \{ c : \underline{X} - c e_i \in A \}$.

Natural questions:

1) Do we still have FTAP?

2) How do we recognise an element of A?

Consistent price processes nearly answer these questions.

Z is a consistent price process if
Z is a $\mathbb{P}$-martingale $(\in \mathbb{R}^d)$ and
$Z_t \in K_t^*$ and $Z_t > 0$ a.s. for each t

<u>Theorem</u> (i) $\bar{A}$, the closure in L^0 of A, is arbitrage-free iff $\exists$ a consistent price process;

(ii) if $\bar{A}$ is arb. free then

$$X \in \bar{A} \iff E\, X . Z_T \leq 0 \text{ for every}$$

consistent price proc. Z s.t. $X . Z_T \in L^1(\mathbb{P})$.

So the big question is whether

A arbitrage free $\Rightarrow$ A closed ?

3. Coherent risk measures

A coherent risk measure is a monetary measure of risk.

Def^n: $\rho : L^\infty(\mathcal{F}_T) \to \mathbb{R}$ is a coherent risk measure if it has the following properties:

1) Monotonicity: $X \leq Y \Rightarrow \rho(X) \leq \rho(Y)$
2) Subadditivity: $\rho(X+Y) \leq \rho(X) + \rho(Y)$
3) translation invariance: $\rho(c+X) = c + \rho(X)$
4) +ve homogeneity: $\rho(\lambda X) = \lambda \rho(X)$ for $\lambda \geq 0$.

Theorem (under technical cond^{n}s)

i) The acceptance set $A_\rho \stackrel{def}{=} \{X : \rho(X) \leq 0\}$ is closed and arb. free.

ii) $\exists$ $\mathcal{Q}$, a collection of p.m.s. $Q : Q \ll P$ and

$$\rho(X) = \sup_{Q \in \mathcal{Q}} \mathbb{E}_Q X. \qquad (*)$$

Conversely, if we define ρ by $(*)$, then ρ is a CRM.

3.2

Remarks:

1) I have reversed the sign on X from the standard defn. of a CRM

2) A_ρ is a cone and we can 'price' by noting that

$$\rho(X) = \inf\{c : X - c \in A_\rho\}.$$

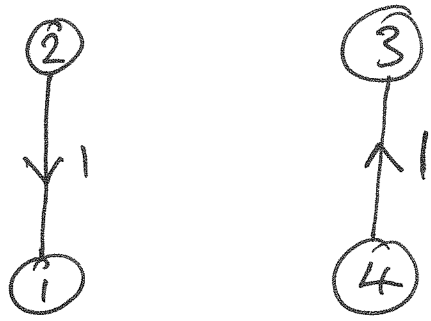
4. Counterexample [Schachermayer]

Take $\Omega = \{0, 1, \ldots\}$ $\mathbb{P}(\omega) = 1/2^{\omega+1}$

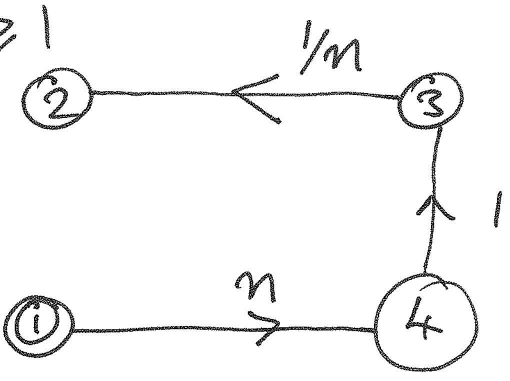
$\mathcal{G}_0$ is trivial and $\mathcal{G}_1 = 2^{\Omega}$.

Showing prices in direction of arrows:

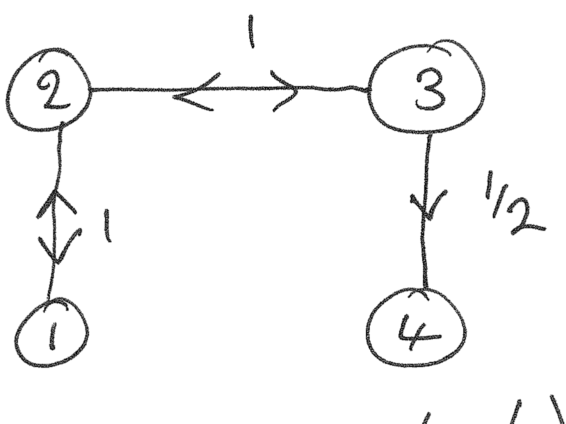
$t=0$



$t=1, \omega = n \geq 1$



$t=1$ $\omega = 0$



All other prices are (effectively) infinite.

Initial trade is to $(N, -N, 1, -1)$.

If $\omega = 0$, can trade to $(0, 0, \frac{1}{2}, 0)$ at $t=1$.

If $1 \leq \omega \leq N$, can trade to $\underline{0}$, but if $\omega > N$ cannot.

It's fairly easy to check A is arb. free but clearly $\bar{A}$ is not (take limit of situation above).

Thus FTAP fails!

5. Rescuing FTAP

Idea: adjust bid-ask prices to $\tilde{\pi}_t^{ij}$ so that

(i) prices are tighter i.e. $\tilde{\pi}_t^{ij} \leq \pi_t^{ij}$ so that $\tilde{A} \supseteq A$

(ii) while ensuring that $c(e_j - \tilde{\pi}_t^{ij} e_i) \in \bar{A}$ ~~so that~~ for all $c \in L^0_+(\mathcal{F}_t)$ so that $\tilde{A} \subseteq \bar{A}$

(iii) while ensuring that $\tilde{A}$ satisfies FTAP i.e. $\tilde{A}$ arb. free $\Rightarrow$ $\tilde{A}$ closed (and hence $\tilde{A} = \bar{A}$).

To achieve (i) and (ii), we define $\tilde{\pi}$ as follows:

$$\tilde{\pi}_t^{ij} = \operatorname{ess\,inf}\{p : c(e_j - pe_i) \in \bar{A} \text{ for all } c \in L^\infty_+(\mathcal{F}_t)\}.$$

Clearly $\tilde{\pi}_t^{ij} \leq \pi_t^{ij}$ since

$$c(e_j - \pi_t^{ij} e_i) \in A \quad \text{for all } c \in L^0_+(\mathcal{F}_t).$$

To establish (iii) 5.2

we use result of Kabanov, Rasonyi and Stricker : given a sequence of cones K_t; $0 \le t \le T$. with K_t closed in $L^0(\mathcal{F}_t, \mathbb{R}^d)$, define the null strategies

$$N = \{ (\xi_t)_{0 \le t \le T} : \xi_t \in K_t \text{ for } 0 \le t \le T \text{ with } \sum \xi_t = 0 \text{ a.s.} \}.$$

Theorem : suppose the K_t are as above and are closed under multiplication by elements of $L^\infty_+(\mathcal{F}_t)$ then $A = \sum_t K_t$ is closed if N is a vector space.

We establish (iii) by showing that if $\tilde{A}$ is arb. free then $N \equiv N(\tilde{K}_0 \times \ldots \times \tilde{K}_T)$ is a vector space.

Sketch : suppose $\xi \in N$ with

$$\xi_t = \sum_{i,j} a_t^{ij} (e_j - \tilde{\pi}_t^{ij} e_i) - \sum b_t^k e_k$$

and suppose $\exists\, k, t$ with $\mathbb{P}(b_t^k > 0) > 0$.

Define $\hat{\xi}_s = \xi_s$ for $s \neq t$, $\hat{\xi}_t = \xi_t + b_t^k e_k \in \tilde{K}_t$

then $\sum \hat{\xi}_s = \sum \xi_s + b_t^k e_k = b_t^k e_k$

so $\hat{\xi}$ gives an arbitrage in $\tilde{A}$.

Now assume $\tilde{A}$ is arb. free, ξ is null

and $\xi_t = \sum_{i,j} a_t^{ij}(e_j - \tilde{\pi}_t^{ij} e_i)$.

Fix i, j, t and let $F = (a_t^{ij} > 0)$.

Now take bounded $c \in L_+^\infty(\mathcal{F}_t)$ with $c \leq B$

and let $\bar{\xi}_s = B\xi_s \quad s \neq t, \quad \bar{\xi}_t = B\xi_t - c 1_F a_t^{ij}(e_j - \tilde{\pi}_t^{ij} e_i)$

$\sum_s \bar{\xi}_s \in \tilde{A}$ (since $B \geq c$) and

$\bar{A} \ni \sum \bar{\xi}_s = -c 1_F a_t^{ij}(e_j - \tilde{\pi}_t^{ij} e_i)$ (since ξ null)

so... on F, $\tilde{\pi}_t^{ji} \leq \frac{1}{\tilde{\pi}_t^{ij}} \Rightarrow$ must be equal

... $\Rightarrow -\xi \in N$, and so N a v.s.

6. Relationship with CRMs

Suppose $\bar{A}$ is arb. free and w.l.o.g. A is closed; further suppose that the last period of trading is frictionless:

$$\pi^{i,j}_T \, \pi^{j,i}_T = 1 \quad \forall i,j.$$

Denote by A° the collection of consistent price processes and notice that

$Z \in A^\circ \Rightarrow Z^i_T / Z^j_T = \pi^{j,i}_T$. Set $S^i_T = Z^i_T / Z^1_T = \pi^{1,i}_T$

and $\Lambda^Z = Z^1_T / Z^1_0$. Since Z is +ve m.g.

we can define a p.m. $\mathbb{Q}^Z$ by $\frac{d\mathbb{Q}^Z}{d\mathbb{P}} = \Lambda^Z$

and then

$$X \in A \cap L^\infty(\mathcal{F}_T, \mathbb{R}^d) \Longleftrightarrow \mathbb{E}_{\mathbb{P}} X . Z_T \le 0 \;\; \forall Z \in A^\circ$$

$$\Longleftrightarrow \mathbb{E}_{\mathbb{P}} (X . S_T) \Lambda^Z \le 0 \;\; \forall Z \in A^\circ$$

$$\Longleftrightarrow \mathbb{E}_{\mathbb{Q}^Z} (X . S_T) \le 0 \;\; \forall Z \in A^\circ$$

and so $A \cap L^\infty$ is the acceptance region of the CRM corresponding to

$\{ \mathbb{Q}_Z : Z \in A^\circ \}$!

6.2

Q: What happens if last period is not frictionless?

A: we can 'add' a period of trading which is frictionless (at time $T+1$); where trades are so potentially disadvantageous that we preserve the previous result.