It's too late now!

Jeremy Gray

What you might have done starting out a long time ago.

It's well known (in 1820) that

An algebraic curve is given by a polynomial equation in two variables.

Conic sections are given by second-degree equations.

There are some things Euler and Cramer discovered about cubic curves (degree 3) and quartic curves (degree 4).

For example, the folium of Descartes

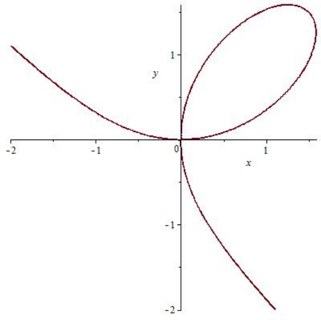

Figure :

$$x^3 + y^3 = 3xy.$$

Notice the singular point at the origin and the infinite branch or branches.

And there's a new idea from the 1820s: duality

Duality associates: to each point of the plane a line, and

to each line in the plane a point

in such a way that

collinear points are associated to concurrent lines, and

concurrent lines are associated to collinear points

Fix a conic.

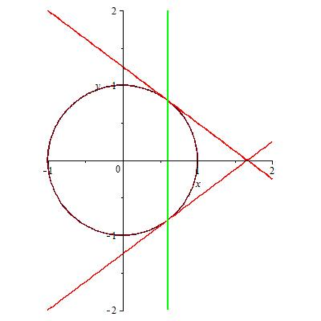

Figure : From an external point draw the two tangents to the conic. Draw the line that joins the points of tangency. Associate *this* line to *that* point.

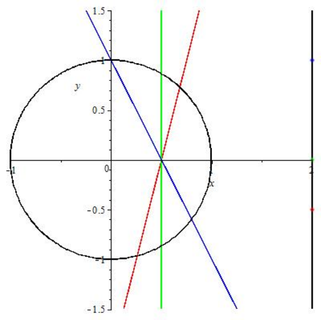

Figure : Let the external point move along a straight line, $x = 2$. The corresponding lines rotate about a point, $(\frac{1}{2}, 0)$.

This picture also tells you how to deal with points inside the conic.

Algebraically

To the point (a, b) other than the origin associate the line $ax + by + 1 = 0$.

OK, I've cheated – but it's still a duality, it's the conic that's gone weird.

The test for collinearity of three points is precisely the test for concurrence of three lines,

Theorem: the dual of the dual of a point P is P,

the dual of the dual of a line ℓ is ℓ.

True confessions: points, and a line, at infinity.

Recall that the conic sections are sections of a cone, and that mapping along the rays of the cone (the lines through the vertex) establishes a $1-1$ correspondence between one conic and another –

except for the infinitely distant point on the parabola –

and the infinitely distant points on the hyperbola – its infinite branches.

The projective plane in the 1820s

It is the usual plane with a line at infinity – convince yourself of this by drawing pictures.

Go to your National Gallery.

Dream of getting a job as a set designer in theatres (trompe l'oeil)

Or even creating backdrops for films (well, you can dream in the 1820s).

An imaginary city

Duality pictorially – but this will lead to problems

Fix a conic, Ω.

Pick a curve, C.

At each point P on C take the tangent t_P to C at P.

Use the conic Ω to associate t_P with a point P'.

The dual curve to C is the set of points P'.

Equivalently

Fix a conic, Ω.

Pick a curve, C.

To each point P on C associate its dual line ℓ_P with respect to Ω

The dual curve to C is the curve enveloped by the lines ℓ_P.

Curves and their tangents

Consider the curve $C : f(x, y) = 0$ of degree n.

Differentiating gives

$$f_x dx + f_y dy = 0, so \ \frac{dy}{dx} = -\frac{f_x}{f_y}.$$

Let (a, b) be a point on the curve, then the tangent at (a, b) has equation

$$xf_x(a, b) + yf_y(a, b) - (af_x(a, b) + bf_y(a, b)) = 0.$$

The constant term $(af_x(a,b) + bf_y(a,b))$

Consider only the terms of maximal degree in $f(x,y)$. Say they are of degree n, and that they define $f^{(n)}(x,y)$. So

$$f(a,b) = f^{(n)}(a,b) + g(a,b)$$

where g is of degree less than n.

$$x\frac{\partial}{\partial x}f^{(n)}(x,y) + y\frac{\partial}{\partial y}f^{(n)}(x,y) = nf^{(n)}(x,y),$$

so $f(a,b) = 0 \Rightarrow f^{(n)}(a,b) = -g(a,b)$, and

$$a\frac{\partial}{\partial x}f^{(n)}(a,b) + b\frac{\partial}{\partial y}f^{(n)}(a,b) = nf^{(n)}(a,b) = -ng(a,b),$$

and the constant term is of degree $n-1$ at most in a and b.

Tangents from a point to a curve

Suppose (u, v) lies on the tangent to the curve C at (a, b).

Then

$$uf_x(a, b) + vf_y(a, b) - (af_x(a, b) + bf_y(a, b)) = 0.$$

Which lines through (u, v) are tangents to C at some point?

Which lines through (u, v) are tangents to C at some point?

Consider

$$uf_x(a,b) + vf_y(a,b) - (af_x(a,b) + bf_y(a,b)) = 0$$

as an equation for *fixed* (u, v) and *varying* (a, b).

It is the equation of a curve C' in (a, b) coordinates (called a polar curve to C) and this curve meets C in the points where the line

$$uf_x(a,b) + vf_y(a,b) - (af_x(a,b) + bf_y(a,b)) = 0$$

is a tangent to C and passes through (u, v). The polar curve is of degree $n - 1$ in a and b.

Conclusion!

The curve C has degree n.

The polar curve C' has degree $n-1$.

So they meet in $n(n-1)$ points, by Bezout's theorem.

There are $n(n-1)$ (concurrent) tangents to a curve of degree n from a point P not on the curve.

So, by duality, the dual curve to C is met in $n(n-1)$ points by a line (the dual line to P).

So the dual curve has degree $n(n-1)$.

Disaster! The duality paradox

On the one hand the dual of the dual of a curve C is point for point the curve C.

On the other hand the dual of C has degree $n(n-1)$, so the dual of the dual of curve C is of degree

$$N = ([n(n-1)]\,([n(n-1)]-1)\,.$$

If $n > 2$ then $N > n$.

This is the research frontier.

Obstacles

It is not easy to calculate the equation of the dual of a curve.

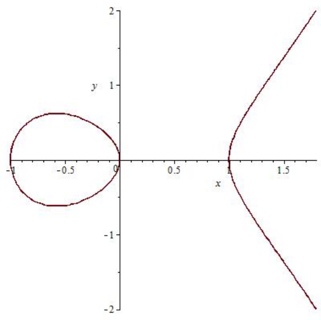

Figure : It seems impossible to draw 6 tangents to a cubic curve. Where are the 'missing' tangents to $y^2 = x(x^2 - 1)$?

Julius Plücker (1801–1868)

Julius Plücker's question and answer

What is a tangent to a curve?

A line that meets the curve in two coincident points.

What if the curve has a double point?

Example: the folium of Descartes

$$C: \quad x^3 + y^3 - 3xy = 0.$$

$$f_x = 3(x^2 - y);\ f_y = 3y^2 - x.$$

$$uf_x(a,b) + vf_y(a,b) - (af_x(a,b) + bf_y(a,b)) = 0$$

$$u3(a^2 - b) + v3(b^2 - a) - \left(a3(a^2 - b) + b3(b^2 - a)\right) = 0,$$

(a, b) is on the curve, so $a^3 + b^3 = 3ab$ and so

$$u(a^2 - b) + v(b^2 - a) - (3ab - 2ab) = 0,$$

$$u(a^2 - b) + v(b^2 - a) - ab = 0,$$

Notice that this polar curve is of degree 2 in a and b.

The polar curve always passes through the double point

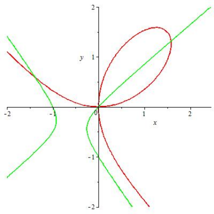

Figure : The polar curve with respect to (u, v) (switch now to (x, y) coordinates) has equation

$$u(x^2 - y) + v(y^2 - y) - xy = 0,$$

and always passes through the origin. Here $(u, v) = (2, -2)$.

The double point gives rises to a false tangent

The line from (u, v) to the origin is not a tangent

The polar curve meets the folium in two points at the double point, which must be discounted.

The degree of the dual curve is lowered by 2 for each double point on the curve.

Cusps points are a particular kind of double point

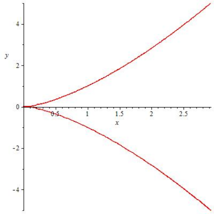

Figure : $y^2 = x^3$. All lines through the origin meet the cusp twice there, the horizontal tangent meets it *three* times

A polar curve to a cusp

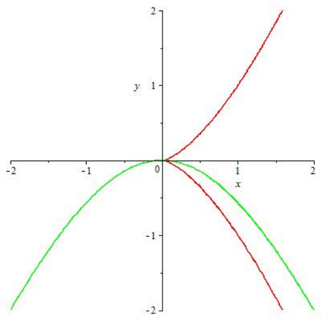

Figure : The polar curve $-3x^2u + 2vy + y^2 = 0$ has a 3-fold point of intersection with the origin with the cusp. Here $(u, v) = (2, -5)$

A cusp on the curve lowers the degree of its dual by 3.

The dual of the cubic $y^2 - x^3 = 0$

A point on this curve has coordinates $(a, b) = (t^2, t^3)$.

The equation of the tangent at (t^2, t^3) to the curve is

$$x(-3a^2) + y(2b) - (a(-3a^2) + b(2b)) = 0,$$

$$-3a^2x + 2by + b^2 = 0.$$

$$-3t^4x + 2t^3y + t^6 = 0.$$

$$-3t^{-2}x + 2t^{-3}y + 1 = 0.$$

The dual point to this has coordinates $(-3t^{-2}, 2t^{-3})$ so the dual curve is $(x/3)^3 + (y/2)^2 = 0$, which is indeed of degree 3.

Notice that its cusp corresponds to the point at infinity on the original curve ($t = \infty$).

Cusps and inflection points are dual

Consider the curves $y = x^3$ and $y^2 = x^3$.

In each case, the line $y = 0$ meets the curve where $x^3 = 0$, which has the three-fold solution $x = 0$, and so a triple point of contact.

In the first case, in an inflection point; in the second case at a cusp

Double points and bitangents are dual

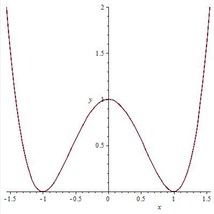

Figure : The x-axis is bitangent to the w-shaped curve

Cubic curves

The basic table

degree	n	n'	$n'' = n$
double points	d	b	d
cusps	c	j	c
inflection points	j	c	j
bitangents	b	d	b

$$n' = n(n-1) - 2d - 3c.$$

$$n'' = n'(n'-1) - 2b - 3j = n.$$

Non-singular cubics

$d = 0, c = 0$. For any cubic, $b = 0$.

degree	$n = 3$	n'	$n'' = n$
double points	0	0	0
cusps	0	j	0
inflection points	j	0	j
bitangents	0	0	0

In this case $n' = n(n-1) - 2d - 3c = 3.2 = 0 - 0 = 6.$

$n'' = 6.5 - 3j = 3$, so $j = 9$.

A non-singular cubic has 9 inflection points.

Cubics with a double point

$d = 1$. For any cubic, $b = 0$.

degree	$n = 3$	n'	$n'' = n$
double points	1	0	1
cusps	c	j	0
inflection points	j	c	j
bitangents	0	1	0

In this case $n' = 6 - 2d - 3$, so $c = 0$ and $n' = 4$

$n'' = 4.3 - 3j = 4$, so $j = 3$.

A cubic with a double point has 3 inflection points and no cusps.

Cubics with a cusp

$c = 1$. For any cubic, $b = 0$.

degree	n	n'	$n'' = n$
double points	d	0	d
cusps	1	j	1
inflection points	j	1	j
bitangents	0	1	0

In this case $n' = 6 - 2d - 3$, so $d = 0$ and $n' = 3$

$n'' = 3.2 - 3j = 3$, so $j = 1$.

A cubic with a cusp has 1 inflection point and no double points.

Inflection points

A non-singular curve of degree n is cut at its inflection points by a curve of degree $3(n-2)$

(this new curve is called the *Hessian* of the original curve).

This suggests that $j = 3n(n-2)$.

However, if the curve has a double point the Hessian makes 6-fold contact with the original curve at the double point, so

$j = 3n(n-2) - 6d$.

Notice that these results confirm our analysis in the case of cubics.

The non-singular quartic, $d = 0, c = 0$

$n = 4$ so, $j = 3.4.2 = 24$.

degree	n	n'	$n'' = n$
double points	0	b	0
cusps	0	24	0
inflection points	24	0	24
bitangents	b	0	b

$n' = n(n-1) - 2d - 3c = 4.3 - 0 - 0 = 12.$

$n'' = 12.11 - 2b - 3.24 = 132 - 2b - 72$, so $b = 28$.

The non-singular curve of degree 4 has 28 bitangents.

Can all 28 bitangents to a quartic curve be real?

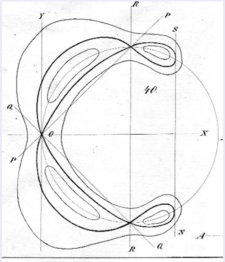

Figure : The curve composed of four beans has 28 real bitangents. (Plücker 1839)

Plücker's curve – the correct equation

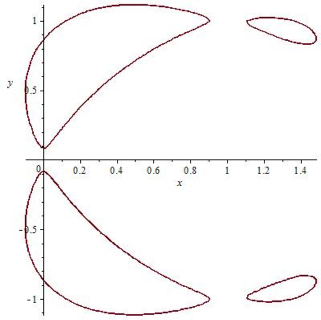

Figure : $(y^2 - x^2)(x-1)(x-\frac{3}{2}) - 2(y^2 + x(x-2))^2$

Some bitangents

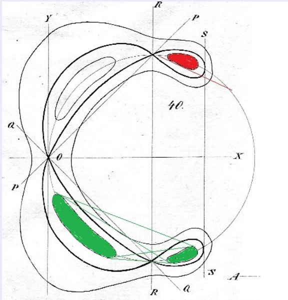

Figure : Each bean has its own bitangent, each of the 6 pairs has 4 bitangents

Non-singular curves of degree $n > 4$

$$N = ([n(n-1)]\,([n(n-1)] - 1)\,.$$

$$j = 3n(n-2) - 6d = 3n(n-2) - 0 = 3n(n-2)$$

The duality paradox will be resolved if

$$N = [n(n-1)]\,([n(n-1)] - 1) - 2b - 3j = n$$

$$n = [n(n-1)]\,([n(n-1)] - 1) - 2b - 9n(n-2)$$

or, as Plücker conjectured and Jacobi proved,

$$b = \frac{1}{2}n(n-2)(n^2-9).$$

What's left? The next round of research problems

Reducible curves: a conic and a line together do not behave like a cubic, but are defined by a cubic equation.

Higher singularities.

Bringing the algebra and the geometry into harmony – Bezout's theorem

A plane curve of degree k and another of degree m meet in mk points.

Problems:

A line parallel to the axis of a parabola meets the parabola only once – solution: points at infinity.

Two circles only meet in two points – solution: complex points of intersection.

A line tangent to a curve meets it (at least) twice there – solution: count carefully.

The familiar plane has a line at infinity

An ellipse does not meet the line at infinity;

A parabola touches the line at infinity;

A hyperbola crosses the line at infinity.

The projective plane is not orientable

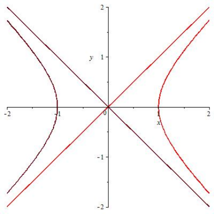

Figure : The asymptote is a tangent to the curve at infinity. The tangent cannot be an inflectional tangent – a hyperbola has degree 2. But as the hyperbola heads to infinity this asymptote is on its left, and when it comes back it is on the right.

Conclusion: the projective plane is not orientable.

Non-orientable surfaces

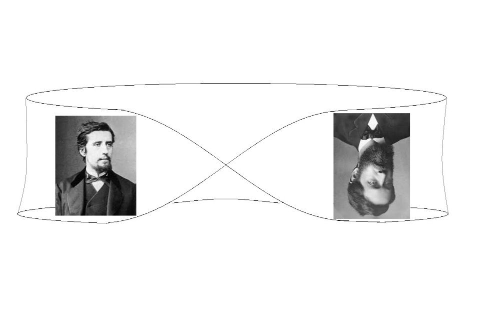

Figure : Darboux (left) and Klein (right) exploring a non-orientable surface. Non-orientability is not a local property – all finite parts of the projective plane are orientable.

Used earlier by Möbius and Plücker.

Write

$$x = \frac{X}{Z}, \quad y = \frac{Y}{Z},$$

$$f(x, y) = 0 \text{ as } F(X, Y, Z) = Z^k.f\left(\frac{X}{Z}, \frac{Y}{Z}\right) = 0,$$

where k is the least power of Z that clears the expression of fractions.

$F(X, Y, Z) = 0$ is the equation of a curve in projective space (even in the modern sense of the term, for $\mathbb{R}P^2$ or $\mathbb{C}P^2$.)

The line $Z = 0$ is the line at infinity, but by a coordinate change any line may be regarded as the line at infinity.

How can a real curve cross another in complex points?

Solution: think of the curve as a complex curve, a locus in $\mathbb{C} \times \mathbb{C}$.

Riemann (1857): An algebraic curve is a branched covering of the (Riemann) sphere, and conversely.

The surface defined by

$$w^2 = (1 - x^2)(1 - k^2x^2)$$

is a covering of the sphere branched over 4 points $z = \pm 1, \pm \frac{1}{k}$

so its genus $\tilde{\chi} = 2\times$ genus of sphere $-4 = 0$;

it is a torus.

Cayley, 1878 – a curve can be thought of as a set of points in $\mathbb{C} \times \mathbb{C}$

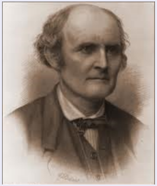

Figure :

I was under the impression that the theory was a known one; but I have not found it anywhere set out in detail.

Counting carefully, or, how bad is my singular point?

Consider the nonsingular cubic curve in space, parameterised by $(t^2-1, t(t^2-1), t)$.

It projects onto the (x, y) plane as the curve parameterised by $(t^2-1, t(t^2-1))$,

which has equation $y^2 = x^2(x+1)$, a cubic with a double point at the origin.

It projects onto the (y, z) plane as the curve $y = z(z^2-1)$ – a non-singular cubic.

This suggests that one might see a singular cubic in the plane as a projection of a non-singular curve in space.

Cremona transformations – "going up and coming down"

The quadratic Cremona transformation (a birational map from P^2 to P^2). Send $[x, y, z]$ to $[1/x, 1/y, 1/z] = [yz, zx, xy]$. Not defined at $[1, 0, 0], [0, 1, 0], [0, 0, 1]$. Otherwise maps the line $z = 0$, $[s, t, 0]$, to the point $[0, 0, st] = [0, 0, 1]$.

The transformation is of period 2.

It blows the lines $x = 0, y = 0, z = 0$ down to the opposite points, and blows up those points to lines.

The fate of a double point

Consider the line $y = mx$ through $[0, 0, 1]$, parameterized by $[t, mt, s]$.

It is sent to $[mts, st, mt^2] = [ms, s, mt]$ or $x = my$, a line that meets $z = 0$ at the point $[0, 0, m]$.

So a curve with a double point and distinct tangents at the origin is sent to a curve, and the branches that crossed at the double point meet the line $z = 0$ at distinct points.

Generally, the degree of the image of a curve is affected by what happens at the singular points of the transformation.

Max Noether's theorem (1870)

Figure : Every Cremona transformation of the projective plane is a product of quadratic transformations.
A different and more rigorous proof is (Veronese 1885).
Then (Castelnuovo 1901).

The situation for P^3 is unclear – see (Hilda Hudson 1927).

What you might hope

Given a curve with a higher order point, to reduce them to lower order points by repeated transformations, until they are no longer singular.

Given a curve with many singular points, to reduce them all at once!

What you get

You can always resolve a singular point of whatever kind.

But the triangle you use may cross the curve, and so produce new singular points.

All you can hope for is that the singularities you introduce are not as bad as the ones you are eliminating.

Result: A singular curve can be reduced in this fashion to a curve with finitely many double points and no other singular ones.

Rudolf Friedrich Alfred Clebsch (1833–1872)

Figure : Cremona transformations, and 1–1 transformations of a curve with d double points, c cusps, and no other singular points do not change the number $p = \frac{1}{2}(n-1)(n-2) - d - c$. He called this number the genus of the curve, and identified it with Riemann's (much more) topological concept.

The Riemann–Roch theorem

On a Riemann surface, there is a formula connecting the dimension of the space of functions having poles at a certain number of points, with the dimension of the space of 1-forms having zeros at the given points, the genus of the curve, and the number of points.

Was it proved by Riemann (or Roch, who sorted out the contribution of the 1-forms)?

Riemann gave a very convincing argument, but only for complex curves (Riemann surfaces) having at most some double points.

What was still to be done?

The resolution of singularities of plane algebraic curves.

The proof of the Riemann-Roch theorem for singular curves.

Leaving the plane for higher dimensions.

Francis Sowerby Macaulay, Charlotte Angas Scott, Hilda Hudson

(Macaulay, Scott) A careful study of the resolution of singularities.

(Macaulay) The Riemann–Roch theorem for curves with arbitrary singularities.

(Macaulay) Singularities of surfaces

(Hudson) Cremona transformations of space ($\mathbb{C}P^3$).

Among the next generation

Figure : Francis Sowerby Macaulay, Charlotte Angas Scott, Hilda Hudson