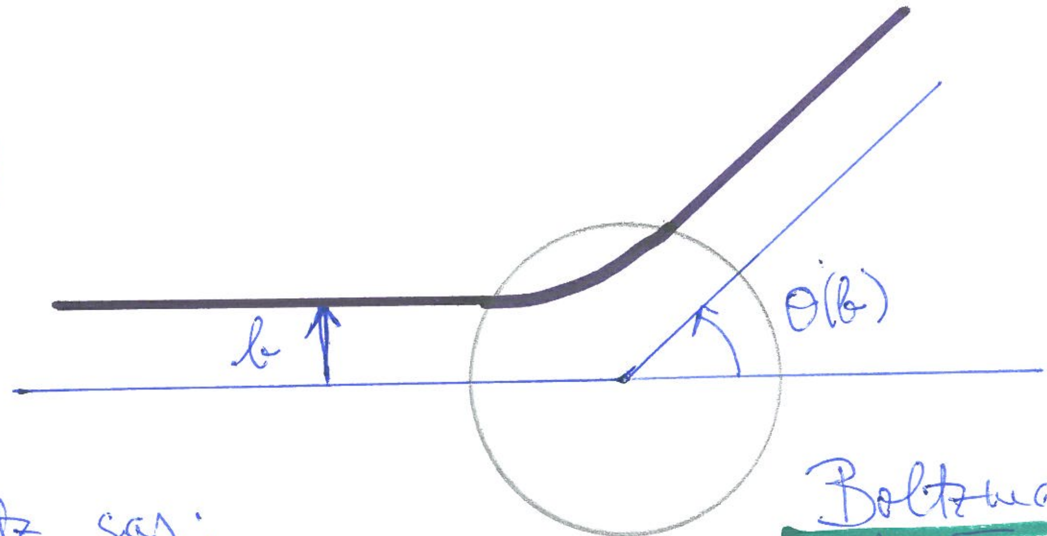
Setup

periodic $\mathbb{Z}^d$-based Lorentz gas with finite range, spherically symmetric scatterers

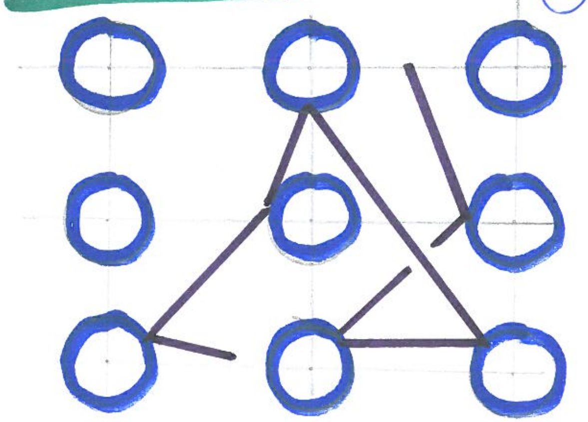
Collision mechanism



$b \in [0,1]$ impact parameter

$\Theta(b)$: scattering angle

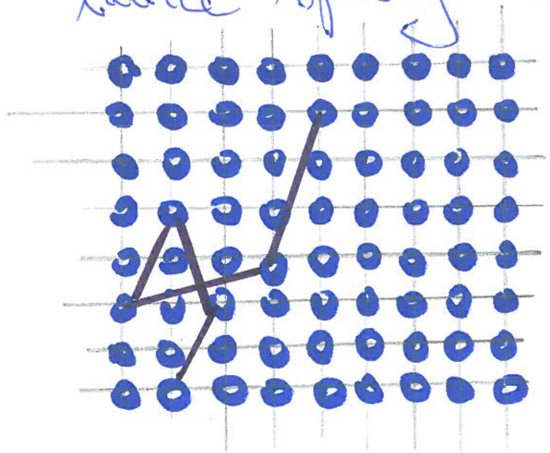
Periodic Lorentz gas:



Boltzmann-Grad Limit

radius of scatterers $r \to 0$

lattice spacing $a \sim r^{(d-1)/d} \to 0$



free flights of order 1

Markloj - Strömbergsson (2010 - 2011)

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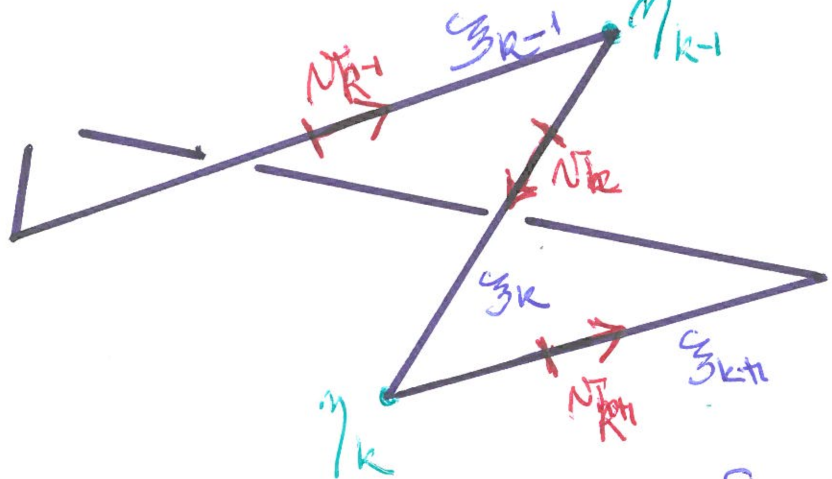
The Boltzmann - Grad limit /

The limit process :

$\boxed{d = 2}$



$v_k \in S^1$

$\xi_k \in (0, \infty)$

$\eta_k \in [-1, 1]$

$(\eta_k, \xi_k)_{k=0}^{\infty}$ Markov chain on $[-1,1] \times (0, \infty)$ with

transition kernel

no dependence on y

$$\mathbb{P}\left(\eta_{k+1} \in (z, z+dz), \xi_{k+1} \in (x, x+dx) \,\Big|\, \eta_k = w, \xi_k = y\right) = \Phi(w, z; x)\, dx\, dz$$

with Φ explicit in $d=2$, (qualitatively understood in $d \geq 3$) ③

$$v_{k+1} = S(\eta_k) v_k \quad ; \quad v_k = S(\eta_{k-1}) S(\eta_{k-2}) \cdots S(\eta_0) v_0$$

$$Q_n = \sum_{k=1}^{n} v_k \xi_k$$

<u>Mind</u> $(\eta_k)_{k=0}^{\infty}$ Markov chain on its own with transition kernel

$$\mathbb{P}\left(\eta_{k+1} \in (z, z+dz) \,\middle|\, \eta_k = w\right) = K_0(w,z)\, dz$$

$$K_0(w,z) = \int_0^{\infty} \phi(w,z;x)\, dx$$

<u>and</u> $\boxed{\text{given}}$ the sequence $(\eta_k)_{k=0}^{\infty}$ the <u>sequence of</u> free flights $(\xi_k)_{k=1}^{\infty}$ are $\boxed{\text{independent}}$

$$\mathbb{E}\left(\prod_{k=1}^{m} f_k(\xi_k) \,\Big|\, \underline{\underline{\eta}}\right) =$$

$$\prod_{k=1}^{m} \mathbb{E}\left(f_k(\xi_k) \,\Big|\, \eta_{k-1}, \eta_k\right)$$

$\boxed{d=3}$ $\quad \vartheta_k \in S^2, \quad \xi_k \in (0,\infty), \quad \eta_k \in B^2$ $\quad$ 5

(η_k, ξ_k) Markov chain on $B^2 \times (0,\infty)$

$$\mathbb{P}\left(\eta_{k+1} \in dz, \xi_{k+1} \in dx \,\middle|\, \eta_k = w, \xi_k = y\right) = \Phi(w, z; x)\, dx\, dz$$

$$\vartheta_k = R(w_0)\, S(\eta_0)\, S(\eta_1) \cdots S(\eta_{k-1}) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\underbrace{Q_n = \sum_{k=1}^{n} \vartheta_k \xi_k}$$

Mind: $(\eta_k)_{k=0}^{\infty}$ is MC on its own

$$\mathbb{P}\left(\eta_{k+1} \in dz \,\middle|\, \eta_k = w\right) = K_0(w,z)\, dz; \quad K_0(w,z) = \int_0^\infty \Phi(w,z;x)\, dx$$

The limit theorems

$\tau_n := \sum_{k=1}^{n} \xi_k$ (time to n-th collision)

$\nu_t := \max\{n : \tau_n \leq t\}$ (no. of collisions till t)

$v_n :=$ velocity vector between $(n-1)$th and n-th collision

$= R(v_0) S(\eta_0) S(\eta_1) \cdots S(\eta_{n-1}) \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

$Q_n := \sum_{k=1}^{n} v_k \xi_k$ discrete time displacement

$X_t := Q_{\nu_t} + (t - \nu_t) v_{\nu_t + 1}$ continuous time displacement

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Thm

$$\frac{Q_n}{\sigma_d \sqrt{n \lg n}} \Rightarrow \mathcal{N}(0, I_d) \qquad \sigma_d^2 = \frac{2^{1-d}}{d^2 (d+1)\, \zeta(d)}$$

$$\frac{X_t}{\tilde{\sigma}_d \sqrt{t \lg t}} \Rightarrow \mathcal{N}(0, I_d) \qquad \tilde{\sigma}_d^2 = \sigma_d^2 \cdot \underbrace{\frac{\pi^{(d+1)/2}}{\Gamma\left(\frac{d+1}{2}\right)}}_{\omega_{d-1}}$$

Some facts about Φ.

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$\boxed{d=2}$ explicit formula ; $\boxed{d=3}$ qualitative

$$\Phi(z,w;x) = \Phi(w,z;x) \ ;$$

$K_0(z,w) = K_0(w,z) \Rightarrow (\eta_k)$ is reversible with uniform stationary

$K_0(z,w)$ bounded from below $\Rightarrow (\eta_k)$ exponentially mixing with gap in spectrum

$$K_1(z,w) := \int_0^\infty x\, \phi(z,w;x)\, dx$$

$$K_2(z,w) := \int_0^\infty x^2 \phi(z,w;x)\, dx$$

$$\mu_k := \frac{K_1(\eta_{k-1}, \eta_k)}{K_0(\eta_{k-1}, \eta_k)} = \mathbb{E}\left(\xi_k \,\middle|\, \underline{\underline{\eta}}\right)$$

$$\beta_k^2 := \frac{K_2(\eta_{k-1}, \eta_k)}{K_0(\eta_{k-1}, \eta_k)} = \mathbb{E}\left(\xi_k^2 \,\middle|\, \underline{\eta}\right)$$

$$\alpha_k^2 := \beta_k^2 - \mu_k^2 = \mathrm{Var}\left(\xi_k \,\middle|\, \underline{\eta}\right)$$

$$\mathbb{E}(\mu_k) < \infty, \quad \mathbb{E}(\beta_k^2) = \infty = \mathbb{E}(\alpha_k^2)$$

$$\mathbb{E}(\mu_k^2) \begin{cases} = \infty & d=2 \\ < \infty & d=3 \end{cases}$$

Cutoffs needed

Cutoff length: $r_k = \sqrt{k (\log k)^{\delta}}$

$1 < \delta < 2$ fixed

$$\xi'_k := \xi_k \mathbb{1}(\xi_k \leq r_k)$$

$$\xi''_k := \xi_k \mathbb{1}(\xi_k > r_k)$$

$$m_k := E(\xi'_k \mid \underline{\eta}) = K_0(\eta_{k-1}, \eta_k)^{-1} \int_0^{r_k} x \phi(\eta_{k-1}, \eta_k; x)\,dx$$

$$b_k^2 := E((\xi'_k)^2 \mid \underline{\eta}) = K_0(\eta_{k-1}, \eta_k)^{-1} \int_0^{r_k} x^2 \phi(\eta_{k-1}, \eta_k; x)\,dx$$

$$a_k^2 := \mathrm{Var}(\xi'_k \mid \underline{\eta}) = b_k^2 - m_k^2$$

Decomposition and main steps of proof

$$Q_n = \sum_{k=1}^{n} \nu_k \xi_k = \underbrace{\sum_{k=1}^{n} \nu_k \tilde{\xi}_k}_{\mathrm{I}} + \underbrace{\sum_{k=1}^{n} \nu_k m_k}_{\mathrm{II}} + \underbrace{\sum_{k=1}^{n} \nu_k \xi''_k}_{\mathrm{III}}$$

III a.s. all but finitely many ξ''_k-s are zero

II $$E\left(\left(\sum_{k=1}^{n} \nu_k m_k\right)^2\right) = \begin{cases} O(n \log\log n) & d=2 \\ O(n) & d=3 \end{cases}$$

I Conditional Lindeberg

Tail estimates follow from refined analysis of Φ 12

Without cutoff: $\mathbb{P}(\xi_k > u) = \int_u^\infty \int_{B^{d-1}} \int_{B^{d-1}} \Phi(w, z, x)\, dw\, dz\, dx \sim \frac{C}{u^2}$

$\boxed{d=2}$

$$\mathbb{P}(\mu_k > u) \sim \frac{3}{4\pi^2} \frac{1}{u^2 \log u}$$

$$\mathbb{P}(\alpha_k > u) \sim \frac{1}{2\pi^2} \cdot \frac{1}{u^2}$$

$$\mathbb{P}(\beta_k > u) \sim \frac{1}{2\pi^2} \cdot \frac{1}{u^2}$$

$$\mathbb{E}(\mu_k) = \mathbb{E}(\xi_k) < \infty \qquad \mathbb{E}(\alpha_k^2) = \infty$$

$$\mathbb{E}(\mu_k^2) = \mathbb{E}\left(\mathbb{E}(\xi_k \mid \underline{\eta})^2\right) = \infty$$

$\boxed{d \geq 3}$ 13

$$c\, u^{-(1+\frac{d}{2})} \leq \mathbb{P}(\mu_k > u) \leq C\, u^{-(1+\frac{d}{2})}$$

$$c\, \frac{1}{u^2} \leq \mathbb{P}(\alpha_k > u) \leq C\, \frac{1}{u^2}$$

$$c\, \frac{1}{u^2} \leq \mathbb{P}(\beta_k > u) \leq C\, \frac{1}{u^2}$$

$$\mathbb{E}(\mu_k) = \mathbb{E}(\xi_k) < \infty \qquad \mathbb{E}(\alpha_k^2) = \infty$$

$$\mathbb{E}(\mu_k^2) = \mathbb{E}\left(\mathbb{E}(\xi_k | \underline{\eta})^2\right) < \infty$$

With cutoff ($\ldots$ m_k)

$\boxed{d=2}$

$$\mathbb{P}(m_k > u) \leq C \frac{1}{u^2 \lg u} \mathbb{1}(u < \sqrt{k})$$

$$0 \leq \mathbb{E}(\mu_k) - \mathbb{E}(m_k) \leq (\sqrt{k})^{-1}$$

$$\mathbb{E}(m_k^2) = \mathcal{O}(\log \log k)$$

$\boxed{d \geq 3}$

$$\mathbb{P}(m_k > u) \leq C \cdot u^{-(1+\frac{d}{2})} \mathbb{1}(u < \sqrt{k})$$

$$0 \leq \mathbb{E}(\mu_k) - \mathbb{E}(m_k) \leq (\sqrt{k})^{-1}$$

$$\mathbb{E}(m_k^2) = \mathcal{O}(1)$$

With cutoff $(\; a_k)$

$\boxed{d=2}$ $$\mathbb{E}(a_k^2) = \frac{\Theta_2}{2} \log k + O(\log\log k)$$

$\boxed{d \geq 3}$ $$\mathbb{E}(a_k^2) = \frac{\Theta_d}{2} \log k + O(1)$$

$$\Theta_d = \frac{2^{2-d}}{d(d+1)\,\zeta(d)}$$

$\boxed{d \geq 2}$ $$\mathbb{E}(a_k^4) = O(r_k^2) = O\left(k (\log k)^r\right)$$

Spectral estimates

V: finite dimensional real Euclidean sp, with $\langle f, g\rangle$

$$\mathcal{H} := \mathcal{L}^2\left(\mathcal{B}_1^{d-1} \to V, \frac{dw}{|\mathcal{B}_1^{d-1}|}\right)$$ $\left[\left(\mathcal{B}_1^{d-1}, \frac{dw}{|\mathcal{B}_1^{d-1}|}\right)\right.$: the state sp of the reversible chain $\left.(\eta_t)\right]$

$$\mathcal{H}_0 = \{ f \in \mathcal{H} : f(w) \equiv v \in V \}$$ constants

$$\mathcal{H}_1 = \{ f \in \mathcal{H} : \int f \, dw = 0 \} = \mathcal{H}_0^\perp$$

$$\Pi f = \int f(w) \, dw$$ – orth. proj to $\mathcal{H}_0$

$$Pf(w) = \int K_0(w, z) f(z) \, dz = \mathbb{E}\left(f(\eta_1) \mid \eta_0 = w\right)$$

$$\Pi P = P \Pi = \Pi, \quad I - \Pi = (I - \Pi) P (I - \Pi)$$

Let $\rho : SO(d) \to O(V)$ be a non-trivial representation (17)

[not necessarily irreducible]

then $\left\| \int \rho(S(\omega))\, d\omega \right\|_V < 1.$

define the operator

$$U_\rho f(\omega) = \rho(S(\omega)) f(\omega)$$

follows : $\| \Pi U_\rho \Pi \|_{\mathcal{H}} < 1.$

Spectral gap for P

$$\omega_0 := \| P - \Pi \|_{\mathcal{H}} = \| (1-\Pi) P (1-\Pi) \|_{\mathcal{H}} < 1$$

(since $K_0(w,z)$ is bdd from below)

Spectral radius of UP

$$\omega_s := \lim_{n \to \infty} \| (U_s P)^n \|_{\mathcal{H}}^{1/n} < 1.$$

Exponential decay of correlations

$$\varrho_1 : SO(d) \to O(\underbrace{\mathbb{R}^d}_{V_1}) \ : \ \varrho_1(R) = R \quad (\text{id-representation})$$

$$\varrho_2 : SO(d) \to O(\underbrace{M_0^{d\times d}(\mathbb{R})}_{V_2})$$

V_2 = real, symmetric, traceless $d\times d$ matrices with H-Sch norm

$$\varrho_2(R) : \quad M \mapsto RMR^t$$

$$\left(\begin{array}{l} \forall\, \omega \in (\min(\omega_0, \omega_{\rho_1}, \omega_{\rho_2}), 1) \\ \forall\, m \in \mathbb{N} \end{array} \right) \cdot \left(\exists\, C = C(\omega, m) < \infty \right) :$$

$$g, f : (\mathcal{B}_r^{d-1})^{m+1} \to \mathbb{R} :$$

$$\left| \mathrm{Cov}\left((e\cdot v_0)^p f(\eta_0, \dots \eta_m), (e\cdot v_n)^p g(\eta_n, \dots, \eta_{n+m}) \right) \right| \leq$$

$$C \cdot \omega^n \cdot \sqrt{\mathbb{E}\left(f(\eta_0, \dots, \eta_m)^2 \right)} \sqrt{\mathbb{E}\left(g(\eta_0, \dots, \eta_m)^2 \right)},$$

Proofs (see page 11 for the decomposition)

III $\mathbb{P}(\xi''_k \neq 0) = \mathbb{P}(\xi_k > r_k) \leq \frac{C}{r_k^2} = \frac{C}{k(\log k)^\gamma}$

use : $\boxed{\mathbb{P}(\xi \geq u) \leq \frac{C}{u^2}}$ $\boxed{\gamma > 1}$

II $\mathbb{E}\left(\left(\sum_{k=1}^n r_k m_k\right)^2\right) \leq \begin{cases} C\, n \log\log n & d=2 \\ C\, n & d \geq 3 \end{cases}$

use $\boxed{\mathbb{E}(m_k^2) \leq \begin{matrix} C \log\log k & d=2 \\ C & d \geq 3 \end{matrix}}$ $+$ $\boxed{\text{expo mixing}}$

I Conditional $(|\eta)$ Lindeberg for $\sum_{k=1}^{n} v_k \tilde{\xi}_k$

I_1 $\dfrac{\sum_{k=1}^{n} (v_k \wedge v_k) a_k^2}{n \log n} \xrightarrow{\mathbb{P}} \sigma_d^2 I_d$

use $\boxed{\mathbb{E}(a_k^2) = \frac{\Theta_d}{2} \log k + \begin{cases} O(\log\log k) & d=2 \\ O(1) & d \geq 3 \end{cases}}$

$\oplus$ $\boxed{\mathbb{E}(a_k^4) = O(k (\log k)^{\gamma})}$

$\oplus$ $\boxed{\text{expo mixing}}$ $\boxed{\gamma < 2}$

I_2 Lindeberg condition – lower tail

$$\mathbb{E}\left((\xi_k^1 - m_k)^2 \mathbb{1}(\xi_k^1 - m_k \leq -\varepsilon\sqrt{n\log n} \mid \eta) \right) \leq$$

$$m_k^2 \; \mathbb{1}(m_k \geq \varepsilon\sqrt{n\log n}) \leq$$

$$m_k^2 \; \mathbb{1}(m_k \geq \varepsilon\sqrt{k\log k})$$

a.s. $\boxed{=0}$ for all but finitely many k-s

use
$$\boxed{P(m_k \geq u) \leq \begin{cases} C/u^2\log u & d=2 \\ C/u^{1+\frac{d}{2}} & d\geq 3 \end{cases}}$$

I_3 Lindeberg condition – upper tail

$$\mathbb{E}\left(\left(\xi_k' - m_k\right)^2 \mathbb{1}\left(\xi_k' - m_k \geq \varepsilon\sqrt{n\log n}\right) \Big| \underline{\eta}\right) \leq$$

$$\mathbb{E}\left(\left(\xi_k'\right)^2 \mathbb{1}\left(\xi_k' \geq \varepsilon\sqrt{n\log n}\right) \Big| \underline{\eta}\right) =$$

$$\mathbb{E}\left(\left(\xi_k\right)^2 \mathbb{1}\left(\varepsilon\sqrt{n\log n} \leq \xi_k \leq \sqrt{k\log k}^{\gamma}\right) \Big| \underline{\eta}\right) =$$

$$=: \xi_{k,\varepsilon} \quad ; \quad \mathbb{E}\left(\xi_{k,\varepsilon}\right) \leq C\left(\frac{\gamma-1}{2}\log\log k - \log\varepsilon\right)$$

use $\boxed{P(\xi > u) \leq \frac{c}{u^2}}$

follows:

$$\frac{\sum_{k=1}^{n} \xi_{k,\varepsilon}}{n\log n} \xrightarrow[\mathbb{P}]{\mathcal{L}^1} 0$$

Conditional Lindeberg conditions – concluded.

$$\Sigma_n^2 := \mathrm{Cov}\left(\sum_{k=1}^{n} v_k \tilde{\xi}_k \,\middle|\, \underline{\eta}\right); \quad A_n^2 = \mathrm{tr}\,\Sigma_n^2$$

$$\frac{\Sigma_n^2}{n \log n} \xrightarrow{\mathbb{P}} \sigma_d^2 I_d ; \qquad A_n^2 \xrightarrow{\mathbb{P}} d\,\sigma_d^2 I_d$$

$$\frac{1}{A_n^2} \sum_{k=1}^{n} \mathbb{E}\left((v_k \tilde{\xi}_k)^2 \,\mathbb{1}\left((v_k \tilde{\xi}_k)^2 > \varepsilon^2 A_n^2 \right)\right) \xrightarrow{P} 0$$

End of Proof – discrete time:

$\boxed{\dfrac{Q_n}{\sqrt{n \log n}} \text{ is tight}}$ → $\boxed{\text{choose weakly cvg subsequence}}$ → $\boxed{\text{choose subsubsequence so that the above convergences are a.s.}}$ → $\boxed{\text{done}}$

From discrete to continuous time:

recall

$\tau_n := \sum_{k=1}^{n} \xi_k$ time to n-th collision

$\nu_t := \max\{n : \tau_n < t\}$ no. of collitions till t

$n_t := \left\lfloor \frac{t}{\mathbb{E}(\xi)} \right\rfloor$ (asymptotic) expected no of collisions till t — deterministic

$Q_n := \sum_{k=1}^{n} v_k \xi_k$ discrete time displacement

$X_t := Q_{\nu_t} + (t - \tau_{\nu_t}) v_{\nu_t + 1}$ continuous time displacement

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Done so far:

$$\frac{Q_n}{\sigma \sqrt{n \log n}} \Rightarrow \mathcal{N}(0, I); \quad \frac{\tau_n - n\,\mathbb{E}(\xi)}{\sqrt{n \log n}} \Rightarrow \mathcal{N}(0, d\cdot\sigma^2)$$

We prove: $\dfrac{|X_t - Q_{n_t}|}{t^{5/12+\varepsilon}} \xrightarrow{\mathbb{P}} 0.$

Ingredients:

$\lim_{t\to\infty} \mathbb{P}(|\nu_t - n_t| > \delta t^{1/2+\varepsilon}) = 0$, use $\boxed{\text{tightness of } \dfrac{\tau_n - n\mathbb{E}(\xi)}{\sqrt{n\log n}}}$ ✓

$\lim_{n\to\infty} \mathbb{P}(\max_{0\le k\le n} \xi_k > \delta n^{1/2+\varepsilon}) = 0$, use $\boxed{\mathbb{E}(\xi_k^{2-\varepsilon}) < \infty}$ ✓

$\lim_{n\to 0} \mathbb{P}(\max_{0\le k\le n} |Q_k| \ge \delta n^{5/6+\varepsilon}) = 0$ use $\boxed{\mathbb{P}(|Q_n| > u) \le C\dfrac{n\log u}{u} \text{ uniformly in } n \text{ and } u}$

(write: $\max_{0\le k\le n} |Q_k| \le \max_{0\le k\le n^{2/3}} |Q_{kn^{1/3}}| + n^{1/3}\max_{0\le k\le n} \xi_k$)

...