Cycle Structure of Random Mallows Permutation

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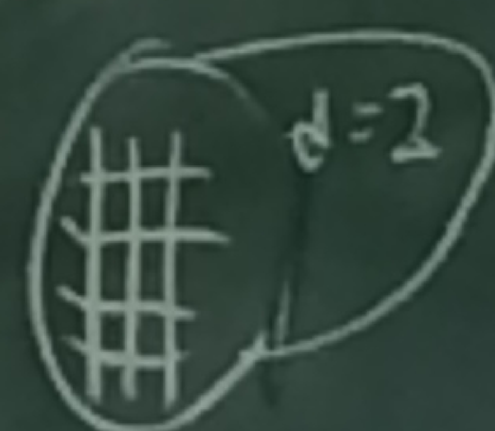
Spatial Random Permutations

Ex. Stirring Model

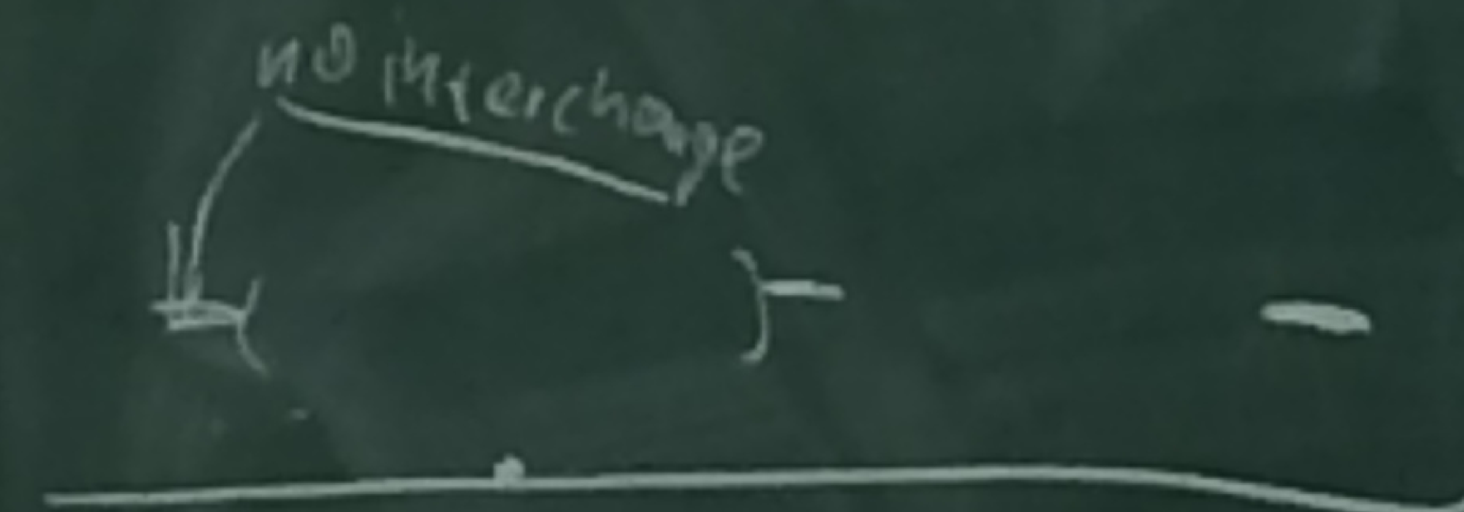
G on n vertices
We place particle i on vertex i.
On each edge we place exponential clock (exp(1)). Whenever clock rings we swap particles on endvertices.

Question: Are there cycles of length $\Theta(n)$?

$G = \mathbb{Z}^d$



$$\mathbb{P}\left[\begin{matrix}\exists \text{ inf cycle} \\ \text{at time } t\end{matrix}\right] = \begin{cases} 0, & d=1 \\ 0 \overset{\text{conj}}{,} & d=2 \\ \overset{\text{conj}}{1}\ t > t_c, & d \geq 3 \\ 0\ \ t \leq t_c & \end{cases}$$



Main known Results

① G = Clique, O. Shramm

② G = inf. regular tree. A Hammond
O. Angel

③ $G = \mathbb{Z}^1$, $\mathbb{E}(\mathrm{diam}\, C_s) \asymp t$ G. Kozma

On endvertices.

Ex: $\mathbb{P}[\pi] \sim e^{-\beta d(\pi, id)}$

1. $d = \frac{1}{n}\sum_i |x_i - \pi_{x_i}|^2$

2. $d = \left.\begin{cases}\text{min number} \\ \text{of transpositions}\end{cases}\right\}$ Ewens distribution

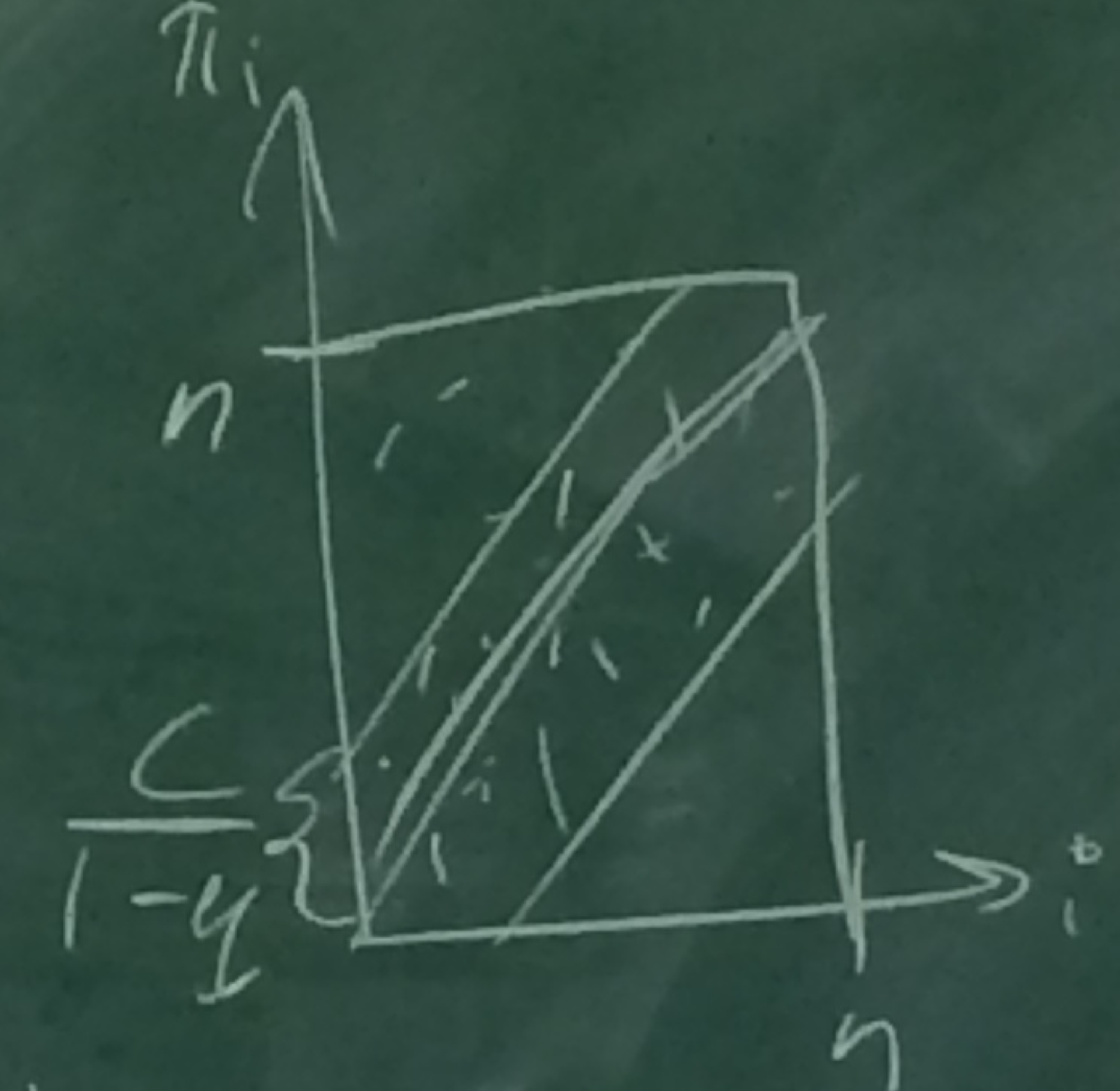
   $\mathbb{P}[\pi] \sim e^{\beta \#\text{cycles}}$

3. $d = \{\text{adj. transpositions}\}$ Mallows Measure

Mallows Distribution

$\mathbb{P}[\pi] \sim q^{inv(\pi)} \quad 0<q<1$

$inv(\pi) = \{ i<j \mid \pi_i > \pi_j \}$



$|i - \pi_i| \leq \frac{c}{1-q}$

① $\pi \longleftrightarrow \pi^{-1}$

② Sampling Formula

Main Result: $\mathbb{E}|C_s| \asymp \min\left(n, \frac{1}{(1-q)^2}\right) \asymp \mathbb{E}(\mathrm{diam}(C_s))$

length $\quad 1 \le s \le n$

C_s - cycle of s $\qquad\qquad$ $1 + \max C_s - \min C_s$

$f \asymp g$

$g \cdot c_1 \le f \le g \cdot c_2$

Sampling Algorithm

$1 \overset{\pi}{\longmapsto} \overset{a_1=2}{\underline{\ }_1\ \underline{\ }_2\ \underline{1}_3} \quad \underline{\ }_n$

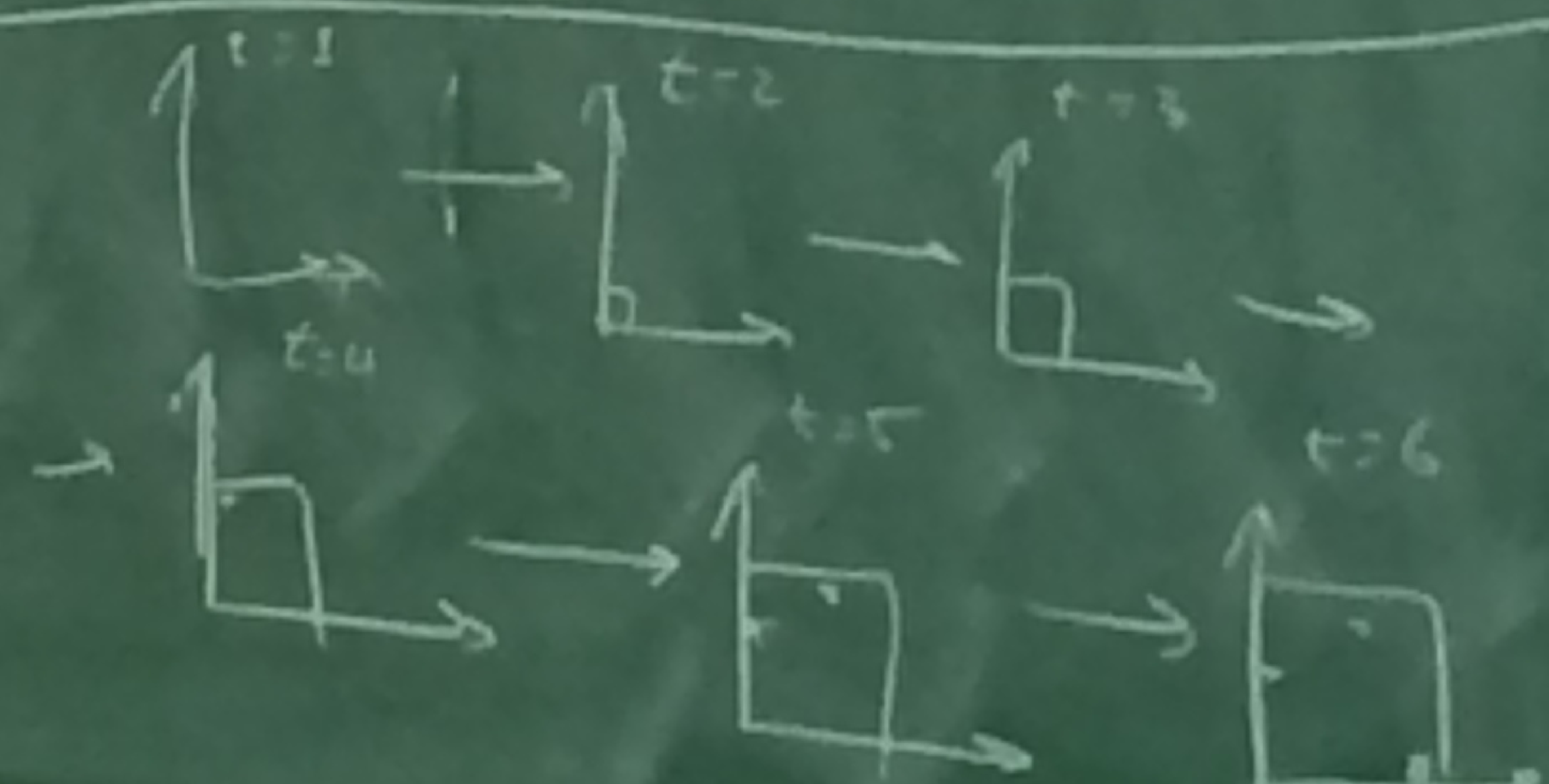
$1 : q : q^2 \quad : q^{n-1}$

$2 \longmapsto \overbrace{\underline{\ }_1\ \underline{\ }_2\ \underline{2}_3\ \underline{\ }_4}^{a_2=3}\ \underline{2}_5 \ \cdots \ \underline{\ }_n$

$1 : q : q^2 \quad : q^{n-2}$

$$\sum \alpha_i = \mathrm{inv}(\pi)$$

$$\mathbb{P}[\pi_1 = i] \sim q^i$$



Sample

N

On each vertex $v \in \mathbb{N}\times\mathbb{N}$ we place random variable

$$X_v \sim \begin{cases} 0, & q \\ 1, & 1-q \end{cases}$$

Theorem $\mathbb{E}[\max C_1] \gtrsim \frac{1}{(1-q)^2}$ (on $\mathbb{N}$)

$$\mathbb{P}_t[\max C_1 = t \mid \max C_1 \geq t] \leq (1-q)^2$$

[illegible]

$$\mathbb{P}[\max C_1 \geq t] \geq q(1-(1-q)^2)^t \approx e^{-(1-q)^2 t}$$

$$t = \frac{1}{(1-q)^2} \qquad \frac{1}{e}$$

$? \to 5 \to 1 \to 3 \to 4 \to ?$

we expose $\boxed{\begin{matrix} 6 \to ? \\ ? \to 6 \end{matrix}}$

$\boxed{\begin{matrix} 6 \to 5 \\ 4 \to 6 \end{matrix}} \Leftrightarrow \max C_1 = 6$

$\mathbb{P}_{t=6}[6 \to 5 \text{ and } 4 \to 6] = q^2(1-q)^2$

$\mathbb{E}|C_1| = \sum \mathbb{P}[i \in C_1]$

<u>Theorem:</u> $\mathbb{P}[i \in C_1] \geq c$

$i \leq \frac{1}{(1-q)^2}$ ...

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