The universal shape of Bose-Einstein condensates

jt. work with V Betz, S Dereich, D Ueltschi

Condensation = a positive fraction of an observed quantity concentrate in a single state

Aim = describe how the distribution of the quantity behaves before it collapses to form the

condensate

2) Framework and examples

$p_n(x)$, $x>0$ density of the quantity at x

$\uparrow$ "time"

$$(*)\quad p_{n+1}(x) = p_n(x)\, \underbrace{b_n(x)} + \underbrace{c_n(x)} \qquad \forall x>0$$

depend on $p_n(y)$, $y>0$

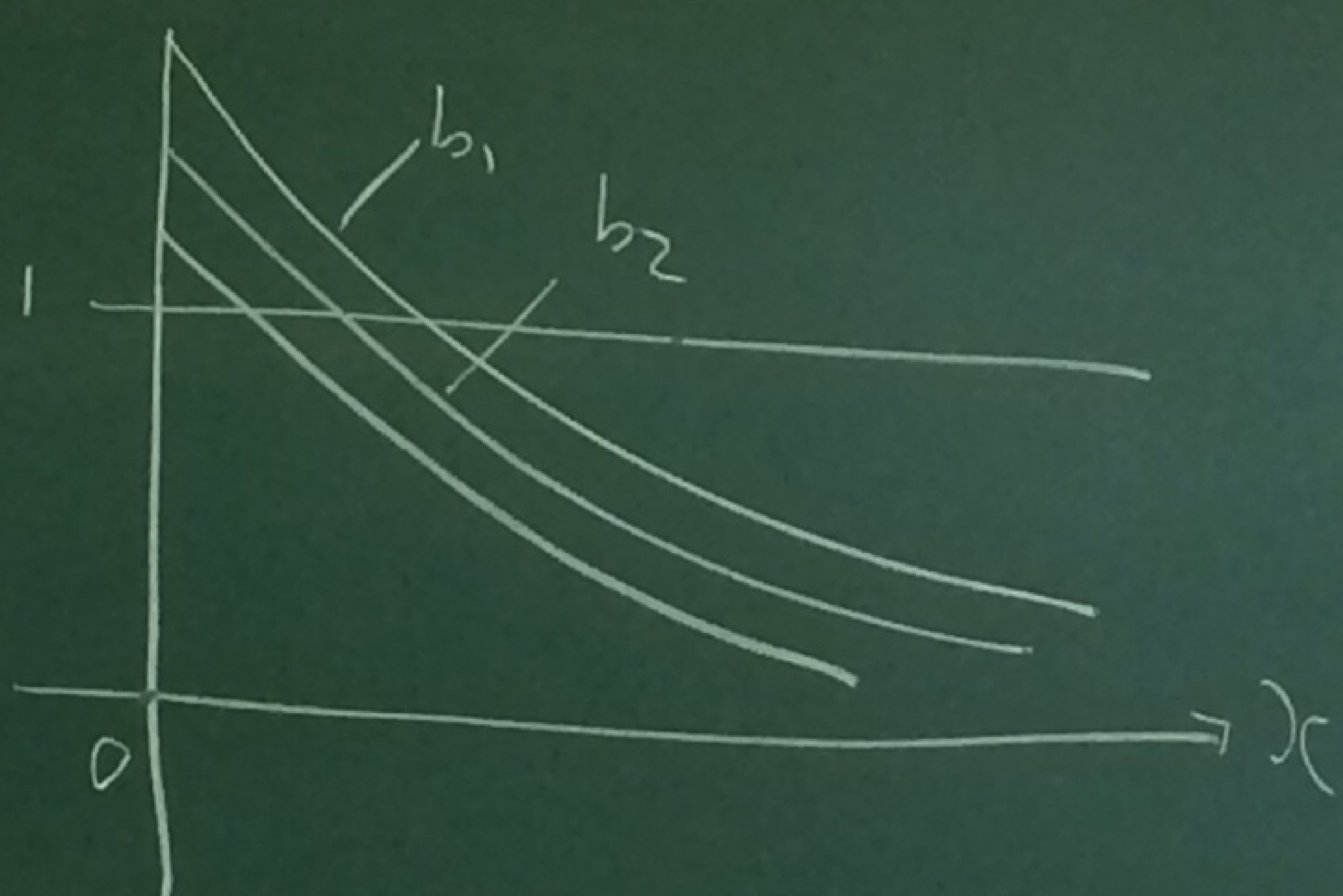
such that

- the total mass $\rho = \int p_n(x)\,dx$ is preserved over time

- $b_n(x) \longrightarrow b_\infty(x)$ for $x > 0$ with $b_\infty(x) < 1$ with

$$b_\infty(0) = \lim_{x \downarrow 0} b_\infty(x) = 1$$

- $c_n(x) \longrightarrow c_\infty(x)$



Examples:

(1) Kingman's model of selection and mutation ('78)

$0 < \beta < 1$ mutation "probability".

$q(x)$ mutant fitness probability density on $(0,1)$

$$p_{n+1}(x) = (1-\beta)\frac{(1-x)\,p_n(x)}{\int_0^1 (1-y)\,p_n(y)\,dy} + \beta\, q(x)$$

$$b_n(x) = \frac{(1-\beta)(1-x)}{\int_0^1 (1-y)\,p_n(y)\,dy} \qquad c_n(x) = \beta\, q(x)$$

...man's model of selection and mutation ('78)

(2) Energy distribution in a Bose gas interacting with a heat bath (Buffet, de Smedt, Pulé '84)

$$b_n(x) = 1 + \int_0^\infty \left(\hat{C}(x-y) - \hat{C}(y-x)\right) p_n(y)\,dy - \int_0^\infty \hat{C}(y-x) F(y)\,dy$$

$$c_n(x) = F(x) \int_0^\infty \hat{C}(x-y)\, p_n(y)\,dy$$

"volume form"

$F(x) \sim \sqrt{x}$ Bose gas in 3D box

$\sim x^2$ " in 3D-harmonic trap

KMS relation

$\hat{C}(x) = e^{-\beta x}\, \hat{C}(-x)$

Condensat

$$(**) \quad p_{n+1}(x) = \sum_{r=0}^{n} c_r(x)\, b_{r+1}(x) \cdots b_n(x)$$

where $c_0(x) = p_1(x)$

Easy to see that $p_n(x) \longrightarrow \dfrac{c_\infty(x)}{1 - b_\infty(x)} =: p_\infty(x)$

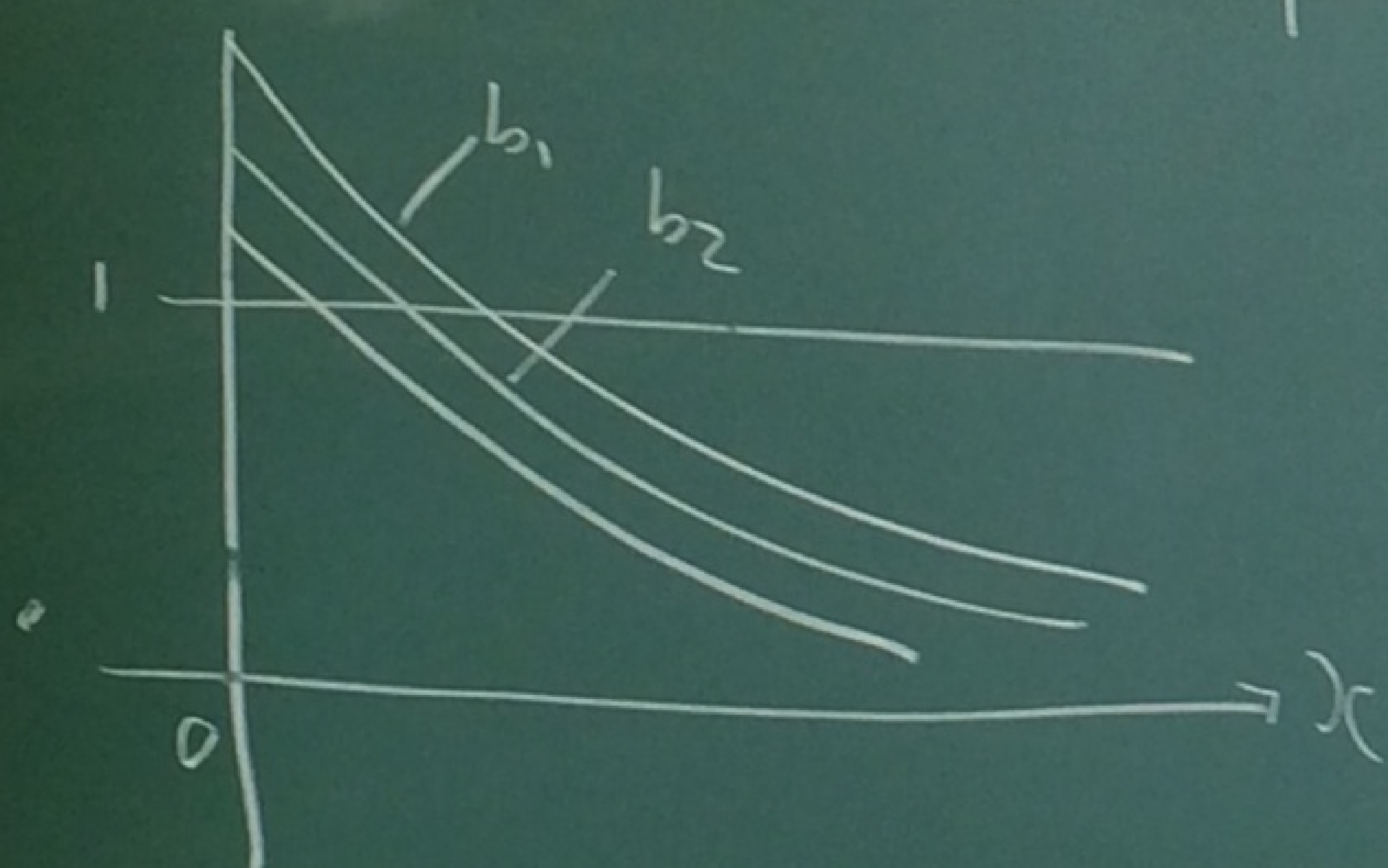
and only large indices $r = n-j$, $j = 1, 2, \ldots$ contribute.

Condensation occurs if $\rho > \rho^* = \int_0^\infty p_\infty(x)\, dx$

Examples: (1) $p_\infty(x) = \frac{\beta q(x)}{x}$ condensation occurs

iff $\beta \int_0^1 \frac{q(x)}{x} dx < 1$.

(2) $p_\infty(x) = \frac{F(x)}{e^{\beta x}-1}$



condensation occurs if

$$\rho > \rho^* = \int_0^\infty \frac{F(x)}{e^{\beta x}-1} dx.$$

$\frac{\ell(cx)}{\ell(x)} \xrightarrow{x \downarrow 0} 1$

(3) Shape of the condensate

Find $x_n \downarrow 0$ such that $x_n P_n'(x_n) \approx 1$

Assume:

- $\rho > \rho^*$ (condensation in zero)
- $C_n(x) \sim C_\infty(x) C_n$ as $x \downarrow 0$ unif in n
- $C_\infty(x)$ is regularly varying $\boxed{\text{index } \alpha}$
- $b_n(x) = b_n(0) + x\, b_n'(0) + o(x)$
- $b_n(0) \downarrow 1$, $b_n'(0) \to \boxed{-\gamma} < 0$

Theorem: $$\lim_{n\to\infty} \frac{1}{n} p_n\left(\frac{z}{n}\right) = \frac{\rho-\rho^*}{\mathcal{K}} \underbrace{z^\alpha e^{-\gamma z}}_{\text{gamma density}} \quad \forall z>0$$

for $\mathcal{K} = \gamma^{-1-\alpha}\,\Gamma(\alpha+1)$

Sketch of proof

$$p_{n+1}(x) = \sum_{r=0}^{n} c_r(x) \prod_{i=r+1}^{n} \left(b_i(0) + x\, b_i'(0) + o(x)\right)$$

$$\approx c_\infty(x) \sum_{r=0}^{n} c_r \underbrace{\left(\prod_{i=r+1}^{n} b_i(0)\right)}_{=\frac{W_r}{W_n}.} (1-\gamma x)^{n-r} \quad (***)$$

Investigate $W_n = \left(\prod_{i=1}^{n} b_i(0)\right)^{-1}$

<u>Lemma</u>: $W_n \sim a \frac{1}{n} c_\infty\left(\frac{1}{n}\right)$ where $a = \frac{\varkappa}{\rho - \rho^*} \sum_{r=0} c_r W_r$

<u>Idea</u>: Choose small $\varepsilon > 0$ and let

$$\mu_n = \int_0^\varepsilon (1-\gamma x)^n c_\infty(x)\, dx \sim \varkappa \frac{1}{n} c_\infty\left(\frac{1}{n}\right)$$

Integrate (***) over $(0, \varepsilon)$

$$\rho - \rho^* \approx \sum_{r=0}^{n} c_r \frac{W_r}{W_n} \mu_{n-r} \Rightarrow W_n = \frac{1}{\rho - \rho^* - c_n \mu_0} \sum_{r=0}^{n-1} c_r W_r \mu_{n-r}$$

for $\varkappa = \gamma^{-1-\alpha} \Gamma(\alpha+1)$ gamma density

<u>Sketch of proof</u> $p_{n+1}(x) = \sum_{r=0}^{n} c_r(x) \prod_{i=r+1}^{n} \left(b_i(0) + x b_i'(0) + o(x)\right)$

Completion of the argument

Put $x = \frac{z}{n}$ into (***)

$$\frac{1}{n} p_n\left(\frac{z}{n}\right) \approx \frac{1}{n} c_\infty\left(\frac{z}{n}\right) \sum_{r=0}^{n} c_r \frac{W_r}{W_n} \left(1 - \gamma \frac{z}{n}\right)^{n-r}$$

$$\sim z^\alpha e^{-\gamma z} \underbrace{\left(\frac{\frac{1}{n} c_\infty\left(\frac{1}{n}\right)}{W_n} \sum_{r=0}^{\infty} c_r W_r \right)}_{\sim \frac{\rho - \rho^k}{k}}$$