Bayesian Level Set Method for Piecewise Geometry Reconstruction

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Uncertainty in Complex Computer Models

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Outline

1 Introduction

2 Bayesian Level Set Method

3 Numerical Experiments
Geometric Inverse Problems

1 Petroleum reservoir
2 Seismic imaging
3 EIT

1 http://www.coatsengineering.com
2 http://gmig.math.purdue.edu
3 http://www.siltanen-research.net
Model Problems: Source Problem

**Forward Model**
Given $f = 1_D$ with $D \subset \tilde{D}$, find $v \in H^1_0(\tilde{D})$ satisfying

$$-\Delta v = f \text{ in } \tilde{D}, \quad v = 0 \text{ on } \partial \tilde{D}$$

**Inverse Problem**
Given the noisy boundary measurement data

$$y = L(v) + \eta, \text{ with } L(v) = \left\{ \ell_j \left( \frac{\partial v}{\partial \nu} \bigg|_{\partial \tilde{D}} \right) \right\}^{N}_{j=1}$$

and noise $\eta$, find the domain $D$. 
Model Problems: Groundwater Flow Problem

**Darcy Flow Model**

\[-\nabla \cdot (\kappa \nabla p) = f, \quad x \in D\]
\[p = 0, \quad x \in \partial D\]

with permeability \( \kappa = \sum_{i=1}^{n} \kappa_i 1_{D_i} \). Given \( f \in H^{-1}(D) \) and \( \kappa \), find \( p \in H_0^1(D) \).

**Inverse Problem**

Given noisy data

\[y = L(p) + \eta, \text{ with } L(p) = \{\ell_j(p)\}_{j=1}^{N}\]

with noise \( \eta \), find the domains \( \{D_i\}_{i=1}^{N} \).
Explicit Geometry

- Boundaries by parameterization

\[(x, y) = (x(s), y(s))\]
Geometry Representations

**Explicit Geometry**
- Boundaries by parameterization

\[(x, y) = (x(s), y(s))\]

**Implicit Geometry**
- Boundaries as zero level set
Inversion Methods

1 Optimization method (shape derivative + regularization)
   - Geometry parameterization
   - Level set representation

Bayesian inference

- Geometry parameterization
- Level set representation
  Bayesian Level Set
Inversion Methods

1. Optimization method (shape derivative + regularization)
   - Geometry parameterization
   - Level set representation

2. Bayesian inference
   - Geometry parameterization
Inversion Methods

1. **Optimization method (shape derivative + regularization)**
   - **Geometry parameterization**
   - **Level set representation**

2. **Bayesian inference**
   - **Geometry parameterization**
   - **Level set representation**

Bayesian Level Set!
Given constants $-\infty = c_0 < c_1 < \cdots < c_n = \infty$, let $D = \bigcup_{i=1}^{n} D_i$ where

$$D_i = \{ x \in D \mid c_{i-1} \leq u(x) < c_i \}, \ i = 1, \cdots, n,$$

$$D_i^0 = \overline{D_i} \cap \overline{D_{i+1}} = \{ x \in D \mid u(x) = c_i \}$$

where $u$ is the level set function. The level set map $F$ is defined by

$$F : u(x) \rightarrow f(x) = \sum_{i=1}^{n} f_i 1_{D_i}(x)$$

which maps the level set function to the physical parameter in the model. Here $f_i, \ i = 1, \cdots, n$ are constants known a priori.
Geometric Inverse problem: determine $u$ from noisy data $y$

$$y = G(u) + \eta$$

where the observational operator $G : U \to Y := \mathbb{R}^J$ takes the form

$$G = L \circ G \circ F$$

with geometric operator $F : U \to X$, forward operator $G : X \to V$ and observation operator $L : V \to Y$.

Bayesian approach:

Prior measure $\mu_0$ on $u$ + the data likelihood $y|u$

Bayes $\implies$ Posterior measure $\mu^y$
Theoretical Foundations

- **Existence**
  
  Given the Gaussian prior $u \sim \mu_0$, noise $\eta \sim \mathcal{Q}_0 = \mathcal{N}(0, \Gamma)$, let
  
  $\Phi(u; y) = \frac{1}{2} |y - \mathcal{G}(u)|^2_\Gamma$, then the posterior

  $$\mu^y(du) \propto \exp(-\Phi(u; y)) \mu_0(du)$$

- **Well-posedness**
  
  $\mu^y$ is locally Lipschitz with respect to $y$, in Hellinger distance.
Minimization Versus Bayesian

- **Minimization Approach**

  Find $\bar{u}$ such that

  $$\bar{u} = \arg\min_u \{ \Phi(u; y) + \lambda R(u) \}$$

  Classical minimization technique fails to work because $u \mapsto \Phi(u; y)$ is discontinuous.

- **Bayesian Approach**

  Putting a Gaussian prior on $u$, the map $u \mapsto \Phi(u; y)$ is continuous almost surely, i.e. the discontinuity set of $\Phi(u; y)$ has prior measure zero.
Prior $\mu_0 = \mathcal{N}(0, C)$ where the covariance $C$ is given by

$$C \varphi(x) = \int_D K(x, y) \varphi(y) dy \text{ with } K(x, y) = \exp\left(-\frac{|x - y|^2}{2\varepsilon^2}\right)$$
Inverse Source Problem: Effect of Prior Length-Scale I

From left to right prior length-scale: 0.1, 0.15, 0.2, 0.3, 0.4.

Mean of level set functions

Map forward onto $f$

Mean of $f$

Variance of $f$
Inverse Source Problem: Effect of Prior Length-Scale II

From left to right prior length-scale: 0.1, 0.15, 0.2, 0.3, 0.4.

An arbitrary (level-set function) sample from the posterior

push forward of (level-set function) sample from the posterior
Numerical Experiments: Groundwater Flow

Gelman-Rubin inter-chain statistic:
Groundwater Flow: Effect of Prior Length-Scale I

From left to right prior length-scale: 0.2, 0.3, 0.4, 0.5.

Mean of level set functions

Map forward onto $\kappa$

Mean of $\kappa$

Variance of $\kappa$
Groundwater Flow: Effect of Prior Length-Scale II

From left to right prior length-scale: 0.2, 0.3, 0.4, 0.5.

Samples from the posterior on level-set function

and corresponding permeability functions
T. Bui-Thanh, and O. Ghattas.
An analysis of infinite dimensional Bayesian inverse shape acoustic scattering and its numerical approximation.

M.A. Iglesias, K. Lin and A. M. Stuart
Well-posed Bayesian geometric inverse problems arising in subsurface flow.

M. A. Iglesias, Y. Lu and A. M. Stuart
A level-set approach to Bayesian geometric inverse problems.
*In Preparation*, 2015.

F. Santosa
A level-set approach for inverse problems involving obstacles.