Take the vector space with basis $\{ v_k : k \in \mathbb{Z} + 1/2 \}$

There is an action of $U_q(sl(2))$ given by

$$E\, v_k = [z + k + 1/2]\, v_{k+1}$$

$$F\, v_k = [z' + k - 1/2]\, v_{k-1}$$

$$K\, v_k = q^{z+z'+2k}\, v_k .$$

Put $q=1$. The Clifford algebra has generators ψ_k, ψ_k^*

~~$\psi_k \psi_k$~~ These mostly commute, with the exception of

$$\psi_k \psi_k^* + \psi_k^* \psi_k = 1 .$$

$$E \longmapsto \sum_k (z + k + 1/2)\, \psi_{k+1} \psi_k^*$$

$$F \longmapsto \sum_k (z' + k + 1/2)\, \psi_k \psi_{k+1}^*$$

This leads to Kerov operators, basis $\{ \delta_\lambda , \lambda \text{ a partition} \}$.

$$E\, \delta_\lambda = \sum_{\mu = \lambda + \square} (z + c(\square))\, \delta_\mu$$

$$F\, \delta_\lambda = \sum_{\mu = \lambda - \square} (z' + c(\square))\, \delta_\mu$$

$$H\, \delta_\lambda = (z z' + 2|\lambda|)\, \delta_\lambda .$$

Temperley-Lieb.

A_L^P has generators $u_1, \ldots, u_{L-1}$ and defining relations

$$u_i^2 = \delta u_i \qquad \text{for } 1 \leq i \leq L-1.$$

$$u_i u_{i\pm 1} u_i = u_i$$

$$u_i u_j = u_j u_i \qquad \text{for } |i-j| > 1 .$$

A_L^D has basis of diagrams with multiplication given by stacking diagrams

There is ~~an isomorphism~~ a homomorphism

$$A_L^P \longrightarrow A_L^D \qquad u_1 \longmapsto \overset{\cup}{\cap}|\cdots|$$

$$u_2 \longmapsto |\overset{\cup}{\cap}|\cdots|$$

$$u_3 \longmapsto ||\overset{\cup}{\cap}|\cdots|$$

which is an isomorphism.

Integrable models.

in general S (finite dimensional) state space of a site

$\otimes^L S$ state space.

then we have a linear operator on state space

- $T(\beta)$ β inverse temperature, transfer matrix [statistical mechanics]
- H, Hamiltonian [quantum field theory]
- L, generator [point process].

Then we are interested in computing correlation functions.
This leads to the problem of diagonalising.

Take S to be two dimensional representation of $U_q(sl(2))$

$$E \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad F \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad K \mapsto \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}.$$

Then $\otimes^L S$ has an action of $U_q(sl(2))$.

If E, F, K commute with $T(\beta) / H / L$
then the model has $U_q(sl(2))$ as "symmetry group".

This means each eigenspace is a representation of $U_q(sl(2))$.

Put $u = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & q & -1 & 0 \\ 0 & -1 & q^{-1} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in \operatorname{End}(S \otimes S)$

and $u_1 = u \otimes 1 \otimes \cdots \otimes 1 \quad \in \operatorname{End}(\otimes^L S)$.

$u_2 = 1 \otimes u \otimes 1 \otimes \cdots \otimes 1$

$\vdots$

Then these operators have two crucial properties:

- u_i commutes with E, F, K for $1 \leq i \leq L-1$.
- they satisfy Temperley-Lieb relations.

This means that if $T/H/L$ is a linear combination of words in these operators then the model has $U_q(\mathfrak{sl}(2))$ symmetry.

six vertex model.
Heisenberg XXZ spin ½ model
ASEP

The matrix in the last two models is the same and is

$$H = \sum_{i=1}^{L-1} u_i .$$

A model with next-to-nearest interactions is

$$H = \sum_{i=1}^{L-2} (1+q^2)\, u_{i+1} - q\, u_i u_{i+1} - q^{-1} u_{i+1} u_i .$$